Effective Integration of Polynomial Differential Equations

Jorge Vitório Pereira

Kolchin Seminar 2019

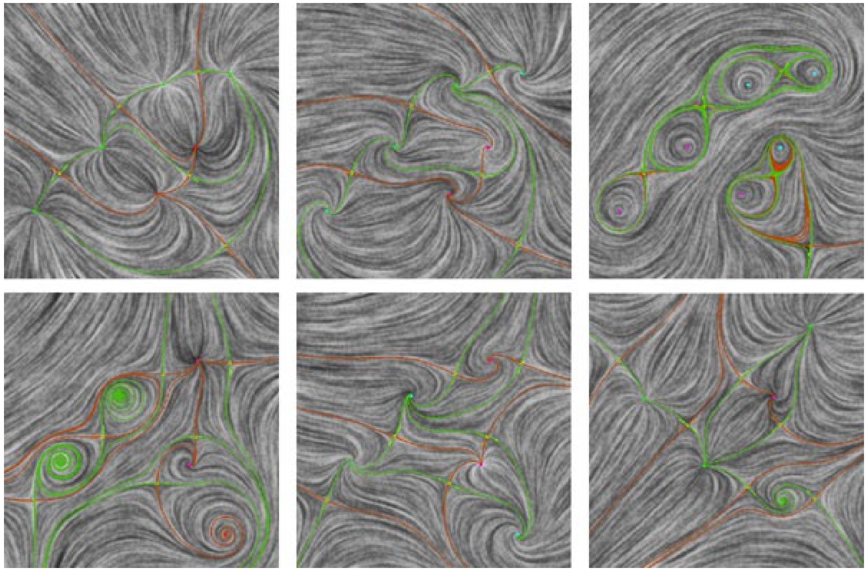
The Problem

Given a polynomial differential equation

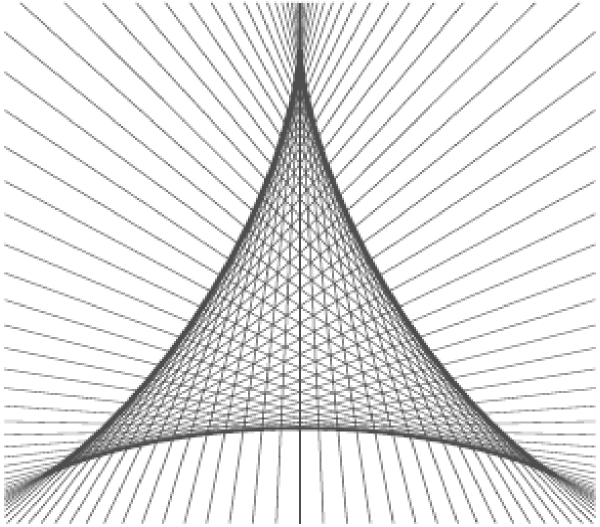
$$F(x,y,y',y'',y''',..) = 0$$

determine (algorithmically) if all its solutions are algebraic.

We are mainly interested on ordinary equations linear on y' (first order and of first degree).

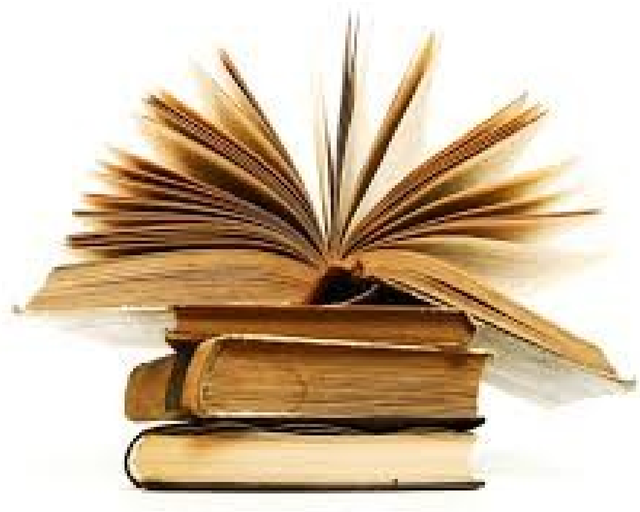


Pencil sketches of phase flows of vector fields by Eugene Zhang
(http://web.engr.oregonstate.edu/~zhange/vecfld_design.html)



Cubic in y'.

History



$$\text{(A.)} \qquad \frac{d^2 y}{dx^2} + \frac{\gamma - (\alpha + \beta + 1)x}{x(1-x)} \cdot \frac{dy}{dx} - \frac{\alpha \cdot \beta}{x(1-x)} \cdot y = 0.$$

Gauss hypergeometric equation

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Gauss hypergeometric equation

Ueber diejenigen Fälle, in welchen die *Gauss*ische hypergeometrische Reihe eine *algebraische* Function ihres vierten Elementes darstellt.

Nebst zwei Figurentafeln.

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Gauss hypergeometric equation

Schwarz's List

List of parameters for which **all** solutions of Gauss hypergeometric equation are algebraic.

No.	λ''	μ''	ν''	Inhalt / π	Polyeder
I.	$\frac{1}{2}$	$\frac{1}{2}$	ν	ν	Regelmässige Doppelpyramide
II.	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}=A$	Tetraeder
III.	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}=2A$	
IV.	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{12}=B$	Würfel und Oktaeder
V.	$\frac{2}{3}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}=2B$	
VI.	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{1}{30}=C$	Dodekaeder und Ikosaeder
VII.	$\frac{2}{5}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{15}=2C$	
VIII.	$\frac{2}{3}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{15}=2C$	
IX.	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{1}{10}=3C$	
X.	$\frac{3}{5}$	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{2}{15}=4C$	
XI.	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{1}{5}=6C$	
XII.	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{1}{5}=6C$	
XIII.	$\frac{4}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}=6C$	
XIV.	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{1}{3}$	$\frac{7}{30}=7C$	
XV.	$\frac{3}{5}$	$\frac{2}{5}$	$\frac{1}{3}$	$\frac{1}{3}=10C$	

Why Schwarz succeeded ?

A Gauss hypergeometric equation is a linear differential equation over the Riemann sphere with 3 poles.

Its monodromy is a representation of the punctured sphere in GL(2).

If all the solutions are algebraic then the local monodromies are of finite order.

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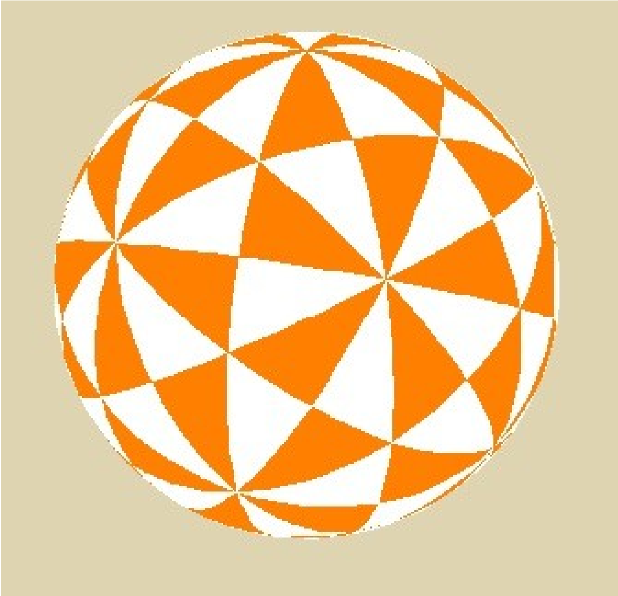

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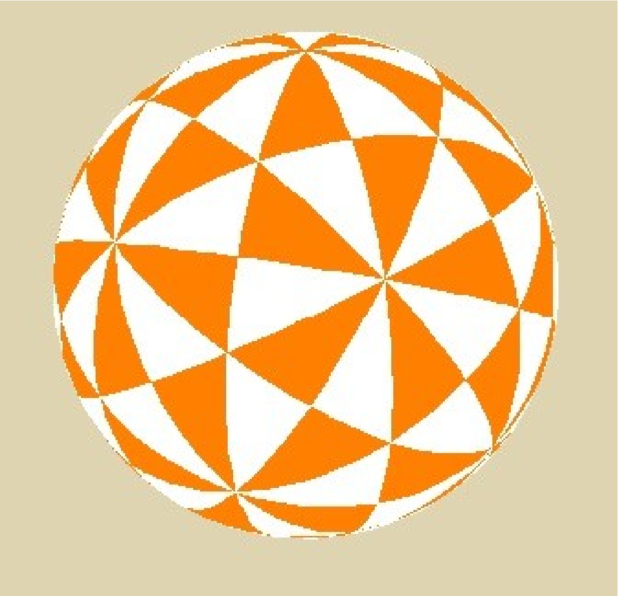

H. A. Schwarz [15] determined all such operators with three singular points whose kernel consists of algebraic functions. His method was to show that if B and C lie in $\mathbf{R}(x)$ then the monodromy group can be calculated from the group generated by the reflections relative to three circles which meet at angles determined by the exponent differences of L. He used this to show that the solutions of L are all algebraic if and only if these angles coincide with the angles of a spherical triangle whose vertices are fixed points of three rotations any pair of which generates a finite rotation group.

ON SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS WITH ALGEBRAIC SOLUTIONS

By F. Baldassarri *and* B. Dwork

Generalizations

A lot of activity on the problem followed in the course of subsequent years. There are contributions on the subject by Fuchs, Jordan, Poincaré, Painlevé, Boulanger, Pépin, Frobenius, Halphen and others.

In particular, motivated by this problem, Jordan proved that any finite subgroup of GL(n) has an abelian normal subgroup of index bounded by a computable function J(n).

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Felix Klein at the beginning of Section 3 of Chapter V of his Lectures on the Icosahedron writes:

" (…) we now concern ourselves (…) with the problem:
to present all linear homogenous differential equations of the second order with rational coefficients:

$$y'' + p\,y' + q\,y = 0$$

which possess altogether algebraic solutions."

LECTURES

THE IKOSAHEDRON,

EQUATIONS OF THE FIFTH DEGREE

FELIX KLEIN

LUDGATE HILL

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By the **end of XIXth** century the problem for linear differential equations was considered **completely solved.**

What's next ?

Liste des attributions du Grand Prix des Sciences Mathématiques depuis 1881 jusqu'à 1915.

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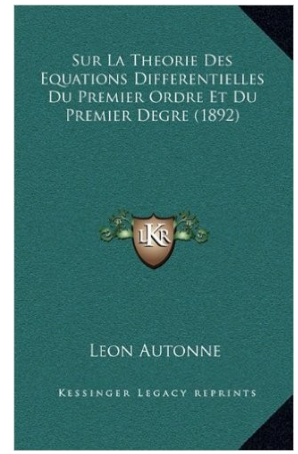

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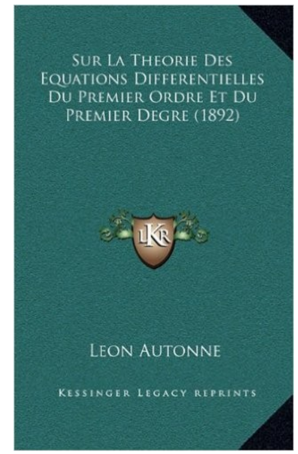

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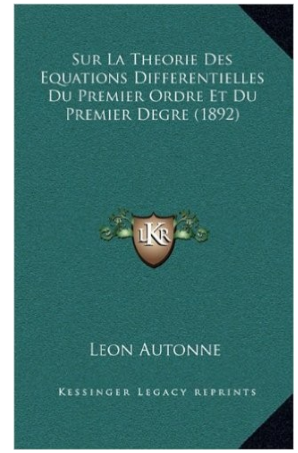
Pour reconnaitre si une équation différentielle du 1er ordre et du 1er degré est intégrable algébriquement, il suffit évidemment de trouver une limite supérieure du degré de l'intégrale; il ne reste plus ensuite qu'à effectuer des calculs purement algébriques.

C'est là un problème qui, semble-t-il, aurait dû tenter les géomètres, et cependant il s'en sont fort peu occupés. Depuis l'œuvre magistrale de M. Darboux, publiée dans le *Bulletin des Sciences Mathématiques,* la question a été négligée pendant vingt ans et il a fallu, pour attirer de nouveau sur elle l'attention qu'elle méritait, que l'Académie des Sciences la proposât comme sujet du concours pour le Grand Prix des Sciences Mathématiques. Deux mémoires furent récompensés, M. Painlevé obtint le prix et M. Autonne une mention honorable: l'un de ces deux mémoires a été publié dans les *Annales de l'École Normale Supérieure* et l'autre dans le *Journal de l'École Polytechnique.*



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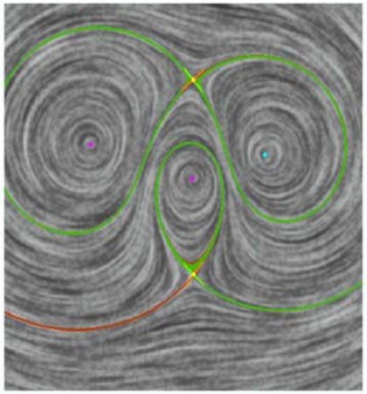

Assuming that all singularities are either centers or radials singularities, Poincaré provided an explicit bound for the degree of the general algebraic leaf.

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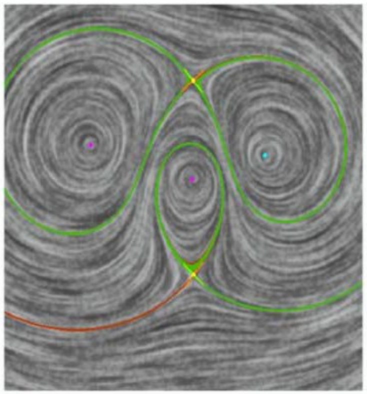

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Limitation du degré.

Dans le cas où tous les cols sont du 1er ou du 2d genre, il est possible de trouver une limite supérieure du degré p et par conséquent de reconnaître si l'équation est intégrable algébriquement.

Nous venons de trouver, en effet, sans avoir besoin de supposer que tous les nœuds soient dicritiques :

$$(m+2) = p\left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2}\right),$$

d'où :

$$\alpha_1 \alpha_2 (m+2) = p(\alpha_1 + \alpha_2).$$

Or, α_1 et α_2 sont premiers entre eux et par conséquent chacun d'eux est premier avec $\alpha_1 + \alpha_2$. Donc $\alpha_1 + \alpha_2$ divise $m + 2$.

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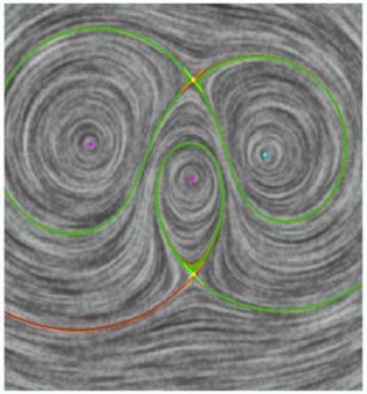

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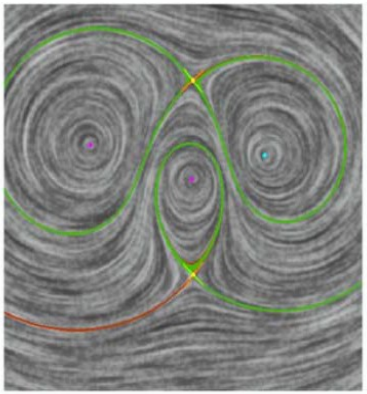

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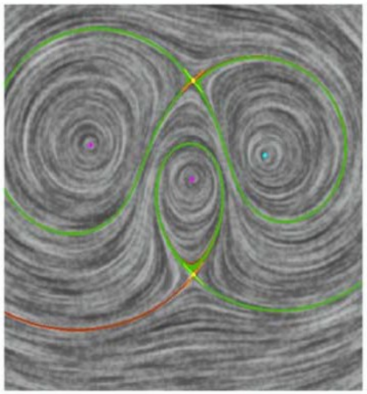

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Application d'un théorème d'Halphen.

$$f,\ \varphi \text{ et } -(f+\varphi)$$

seraient des puissances parfaites. Soient :

$$X^{\alpha_1},\ Y^{\alpha_2},\ Z^{\alpha_3}$$

ces trois puissances parfaites; on devrait avoir identiquement :

$$X^{\alpha_1} + Y^{\alpha_2} + Z^{\alpha_3} = 0,$$

X, Y et Z étant des polynômes homogènes de degré

$$\frac{p}{\alpha_1},\ \frac{p}{\alpha_2},\ \frac{p}{\alpha_3}$$

en x, y et z.

Or Halphen, au début de son mémoire couronné sur les équations linéaires, a étudié les identités de cette forme. Il a montré d'abord que les nombres α_1, α_2 et α_3 devaient avoir certaines valeurs particulières $(\alpha_1, 2, 2)$, $(2, 3, 3)$, $(2, 3, 4)$, $(2, 3, 5)$; il a fait voir ensuite qu'on devait avoir :

$$X = P_1(\eta_1, \eta_2),$$

$$Y = P_2(\eta_1, \eta_2),$$

$$Z = P_3(\eta_1, \eta_2),$$

P_1, P_2 et P_3 étant des polynômes homogènes en η_1 et η_2 qu'Halphen a complètement formés et qu'il est inutile de transcrire ici, pendant que η_1 et η_2 sont deux polynômes homogènes de même degré en x, y et z.

Alors la courbe

$$f + C\varphi = X^{\alpha_1} + CY^{\alpha_2} = 0$$

est décomposable quel que soit C en un certain nombre de courbes appartenant au réseau :

$$\frac{\eta_1}{\eta_2} = \text{const.}$$

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Bound the degree of algebraic curves invariant by foliations of the projective plane in function of the degree of the foliation and combinatorial data attached to the singularities of the foliation.

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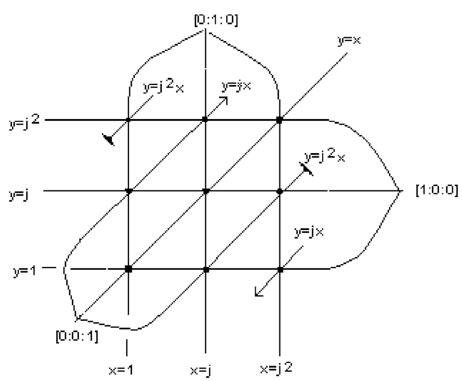

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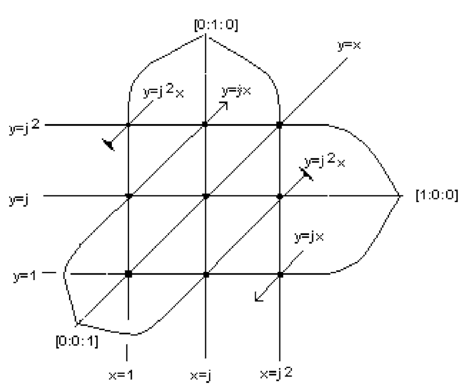

Later shown by McQuillan to be quotients of linear flows on abelian surfaces

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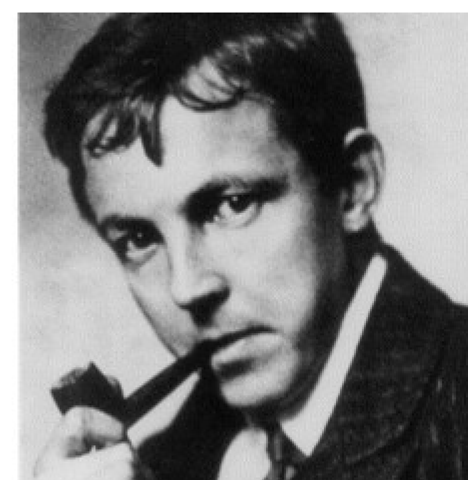
Cambridge Tracts in Mathematics
and Mathematical Physics
GENERAL EDITORS
P. HALL, F.R.S. AND F. SMITHIES, PH.D.

No. 2

THE
INTEGRATION OF FUNCTIONS
OF A SINGLE VARIABLE
BY
G. H. HARDY

CAMBRIDGE UNIVERSITY PRESS



Revisiting the linear case

The results of the XIXth century did not actually solved the problem, but instead reduced it to dimension one.

They show that in order to decide if all the solutions of linear differential equation of arbitrary rank are algebraic it suffices to be able to decide if the solutions of a rank one differential equation with **algebraic** coefficients

y' = a(x) y

are all algebraic.

The general questions of this nature which arise in connection with integrals of the form

$$\int \frac{Q}{\sqrt{X}}\, dx,$$

or, more generally,

$$\int \frac{Q}{\sqrt[m]{X}}\, dx,$$

are of extreme interest and difficulty. The case which has received most attention is that in which $m = 2$ and X is of the third or fourth degree, in which case the integral is said to be *elliptic*. An integral of this kind is called *pseudo-elliptic* if it is expressible in terms of algebraical and logarithmic functions.

But no method has been devised as yet by which we can always determine in a finite number of steps whether a *given* elliptic integral is pseudo-elliptic, and integrate it if it is, and there is reason to suppose that no such method can be given.

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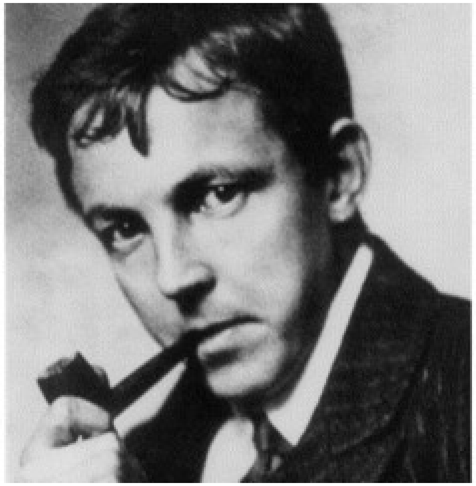

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In modern language, this problem reduces to decide if a holomorphic connection on a line-bundle over an algebraic curve has all its flat sections algebraic.

In order to solve it (algorithmically) it suffices to be able to bound a priori the order of a degree 0 divisor on an algebraic curve.

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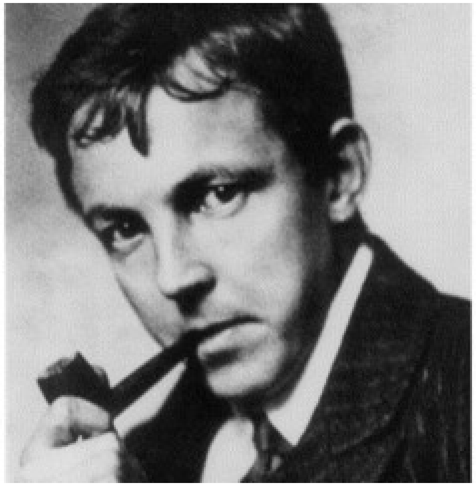

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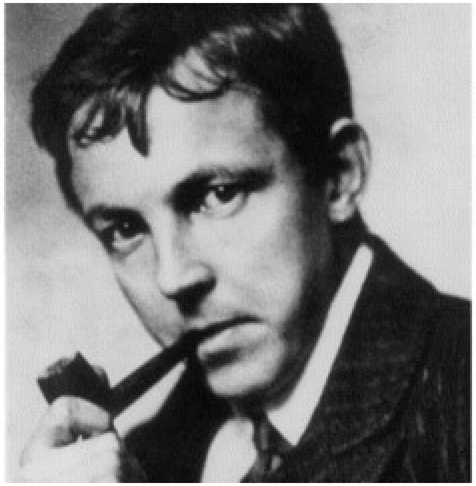

Arithmetic comes to rescue

THE SOLUTION OF THE PROBLEM OF INTEGRATION IN FINITE TERMS

BY ROBERT H. RISCH

Communicated by M. H. Protter, October 22, 1969

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A bound for the torsion. It has been conjectured that there is a universal bound, depending only on the genus and the ground field for torsion on the Jacobian variety of an algebraic curve defined over a finitely generated field k. (See [**2**, p. 264] for some discussion of the elliptic curve case.) When integrating a given elementary function, one needs only to be able to find the bound for an explicitly given curve. This can be done using things now in the repertory of arithmetical algebraic geometry. One method is outlined below.

Risch showed how to (algorithmically) decide wheter the solutions of y'=a(x)y are algebraic or not by considering reduction to positive characteristic of the equations involved.

This completes the solution of our problem for linear differential equations.

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A conjecture by **Grothendieck-Katz** predicts that all solutions of a linear differential equation over a field of characteristic zero are algebraic if and only if the same holds true for almost every reduction modulo p of the equation.

Risch showed how to (algorithmically) decide wheter the solutions of y'=a(x)y are algebraic or not by considering reduction to positive characteristic of the equations involved.

This completes the solution of our problem for linear differential equations.

New Developments

Effective algebraic integration

Conjecture 1.1. *The Zariski closure in* $\mathbb{P}H^0(\mathbb{P}^2, T_{\mathbb{P}^2}(d-1))$ *of the set of foliations of degree* d *on* $\mathbb{P}^2$ *which admit a rational integral consists in transversely projective foliations.*

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Algebraically integrable foliations

Transversely projective foliations

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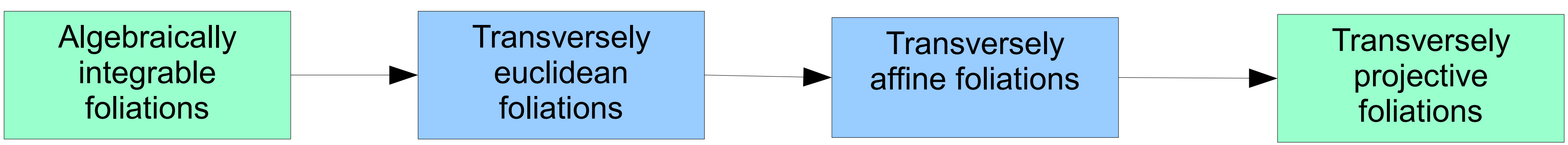

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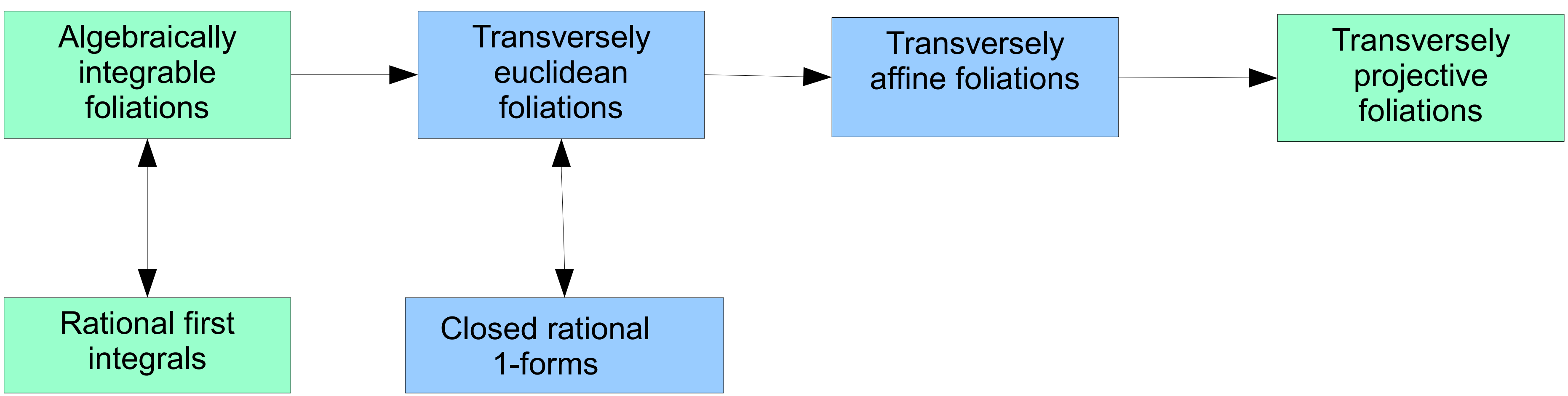

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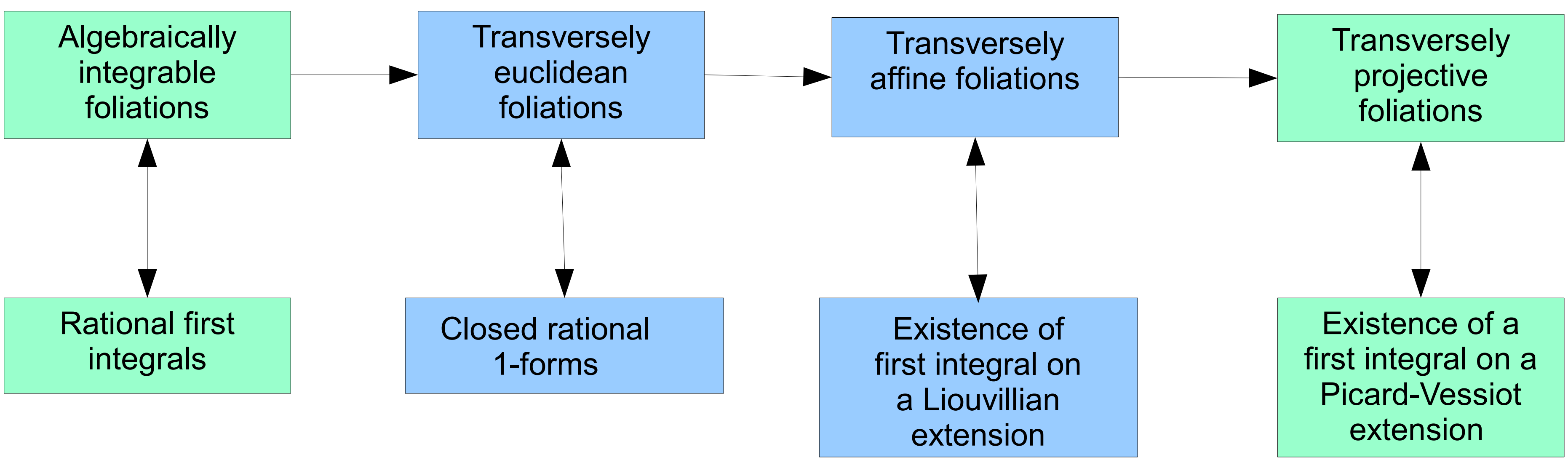

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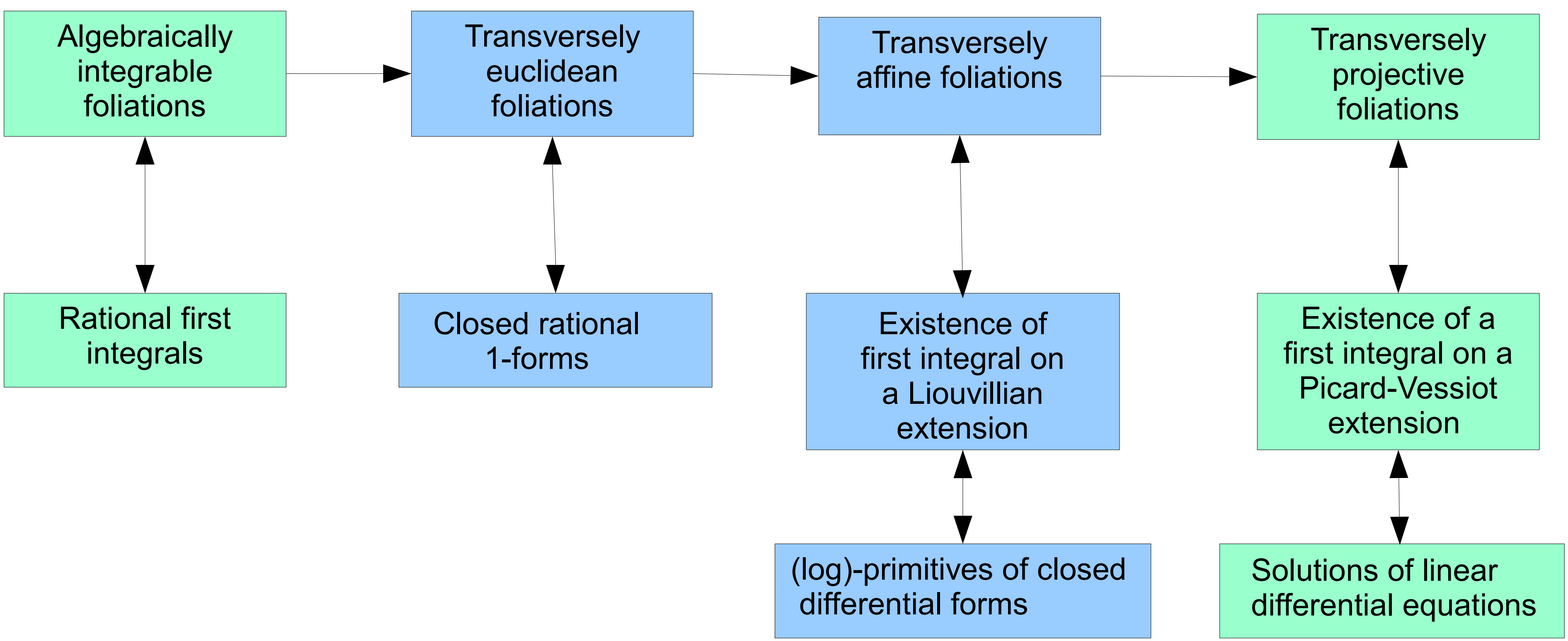

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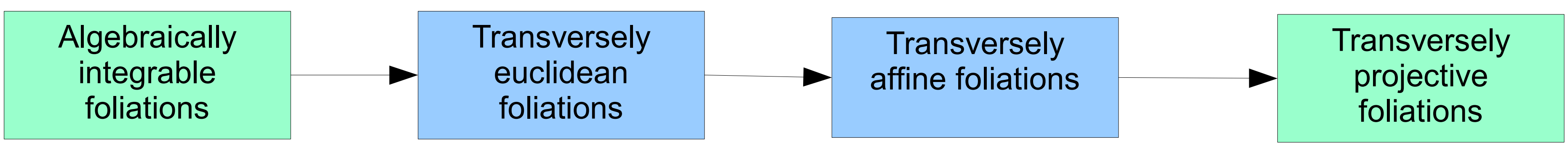

J'ajoute qu'on ne peut espérer résoudre d'un coup le problème qui consiste à limiter n. L'énoncé vers lequel il faut tendre doit avoir la forme suivante : « On sait reconnaître si l'intégrale « d'une équation (1) donnée est algébrique ou ramener l'équation aux « quadratures. » Dans ce dernier cas, la question reviendrait à reconnaître

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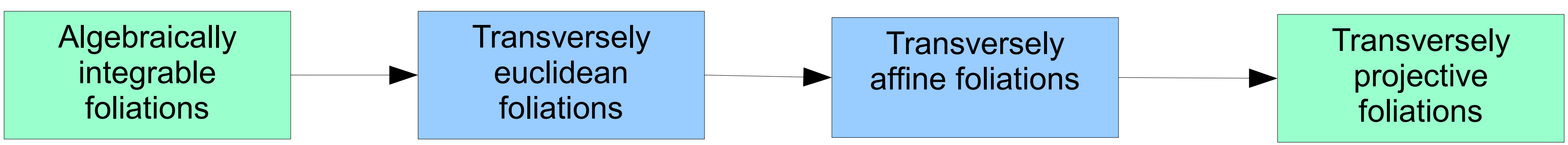

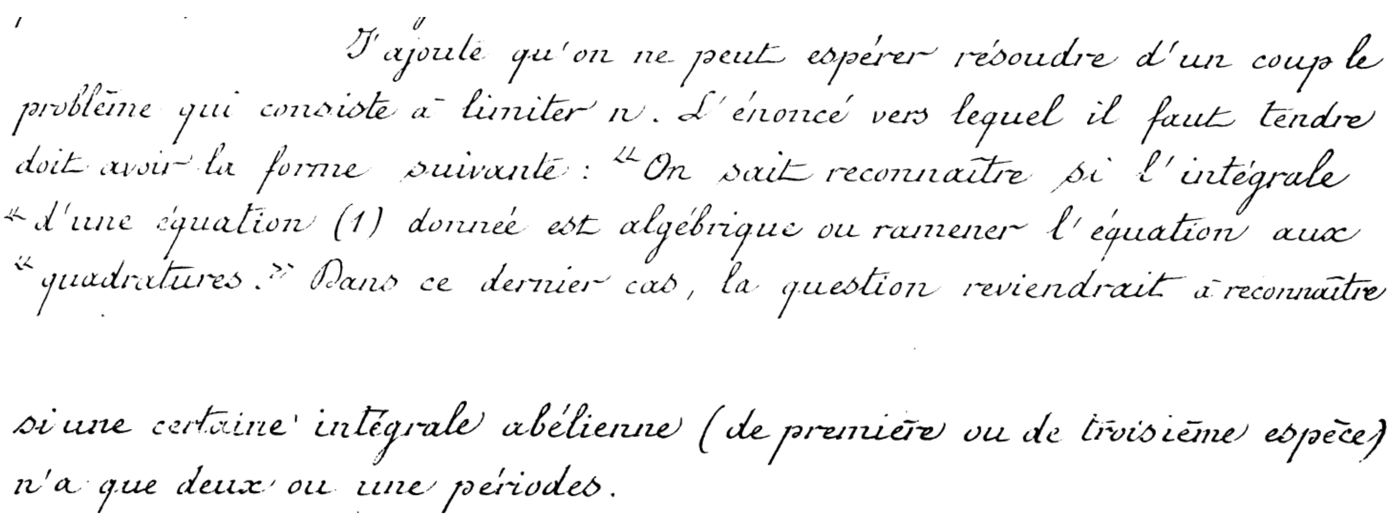

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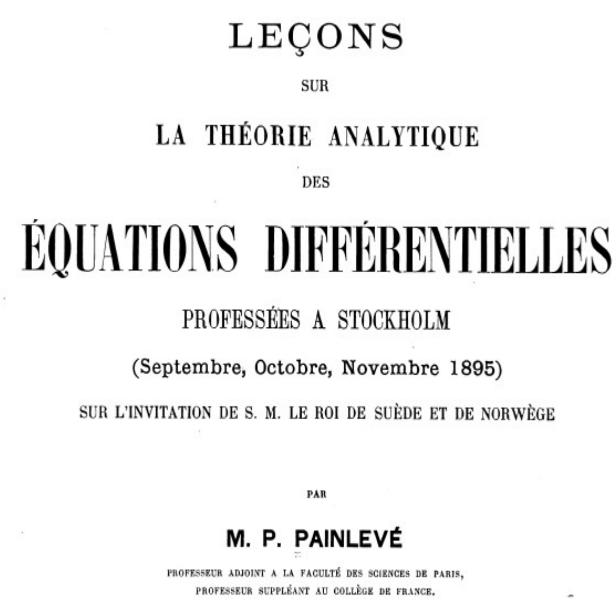

LEÇONS
SUR
LA THÉORIE ANALYTIQUE
DES
ÉQUATIONS DIFFÉRENTIELLES
PROFESSÉES A STOCKHOLM
(Septembre, Octobre, Novembre 1895)
SUR L'INVITATION DE S. M. LE ROI DE SUÈDE ET DE NORWÈGE
PAR
M. P. PAINLEVÉ
PROFESSEUR ADJOINT A LA FACULTÉ DES SCIENCES DE PARIS,
PROFESSEUR SUPPLÉANT AU COLLÈGE DE FRANCE.

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TOWARD EFFECTIVE LIOUVILLIAN INTEGRATION

Transversely affine foliations

GAËL COUSIN, ALCIDES LINS NETO, AND JORGE VITÓRIO PEREIRA

arXiv.org

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Theorem A. *Let $\mathcal{F}$ be a foliation of degree $d \geq 2$ on the projective plane $\mathbb{P}^2$. Assume that $\mathcal{F}$ admits a Liouvillian first integral but does not admit a rational first integral. Then $\mathcal{F}$ admits an algebraic invariant curve of degree at most $12(d-1)$.*

Structure of transversely affine foliations (Cousin,P.)

Transversely affine foliations

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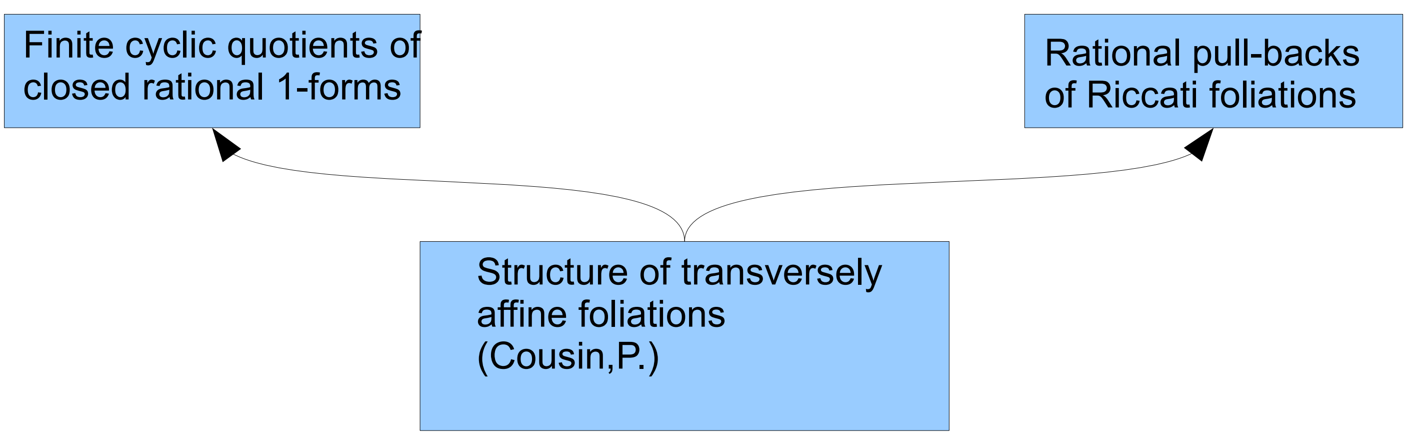

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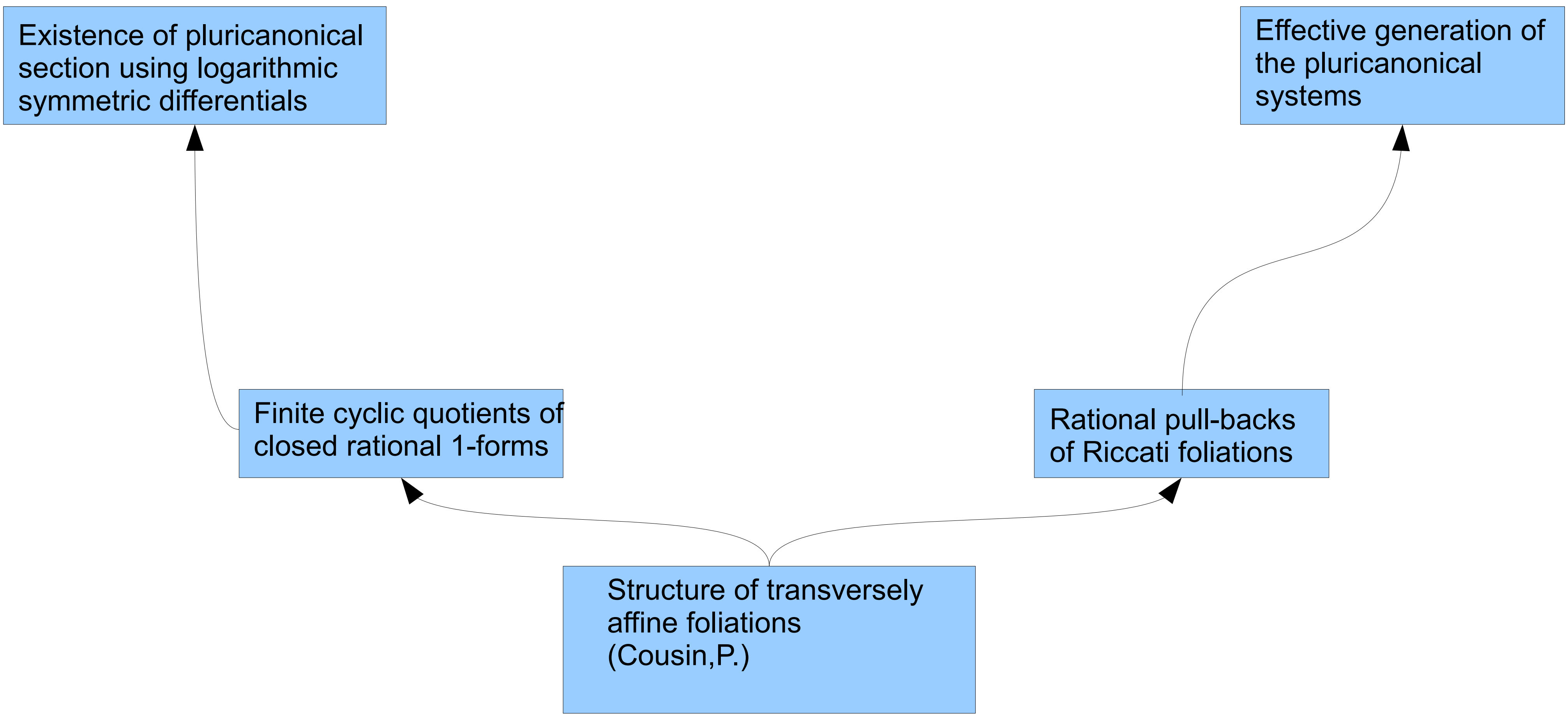

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Existence of pluricanonical section using logarithmic symmetric differentials

Effective generation of the pluricanonical systems

Theorem 3.10. *Let $2 \leq m \in \mathbb{N}$ be a natural number and Φ be a subset of the set $P(m) \subset \mathbb{C}^*$ of primitive m-th roots of the unity. If $P(m)$ is the disjoint union of Φ and Φ^{-1} then*

$$1 \in \Phi^N = \underbrace{\Phi \cdot \cdots \cdot \Phi}_{N \text{ times}}$$

for some $N \leq 6$.

Finite cyclic quotients of closed rational 1-forms

Rational pull-backs of Riccati foliations

Structure of transversely affine foliations (Cousin,P.)

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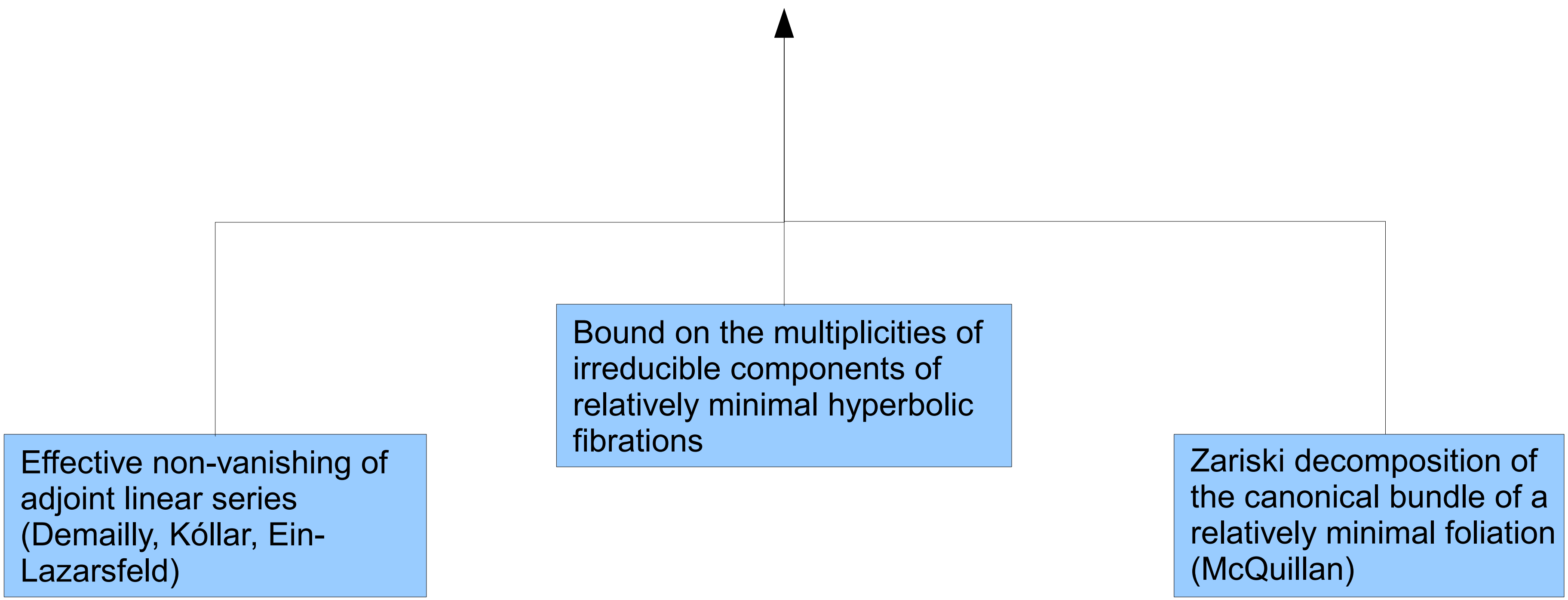

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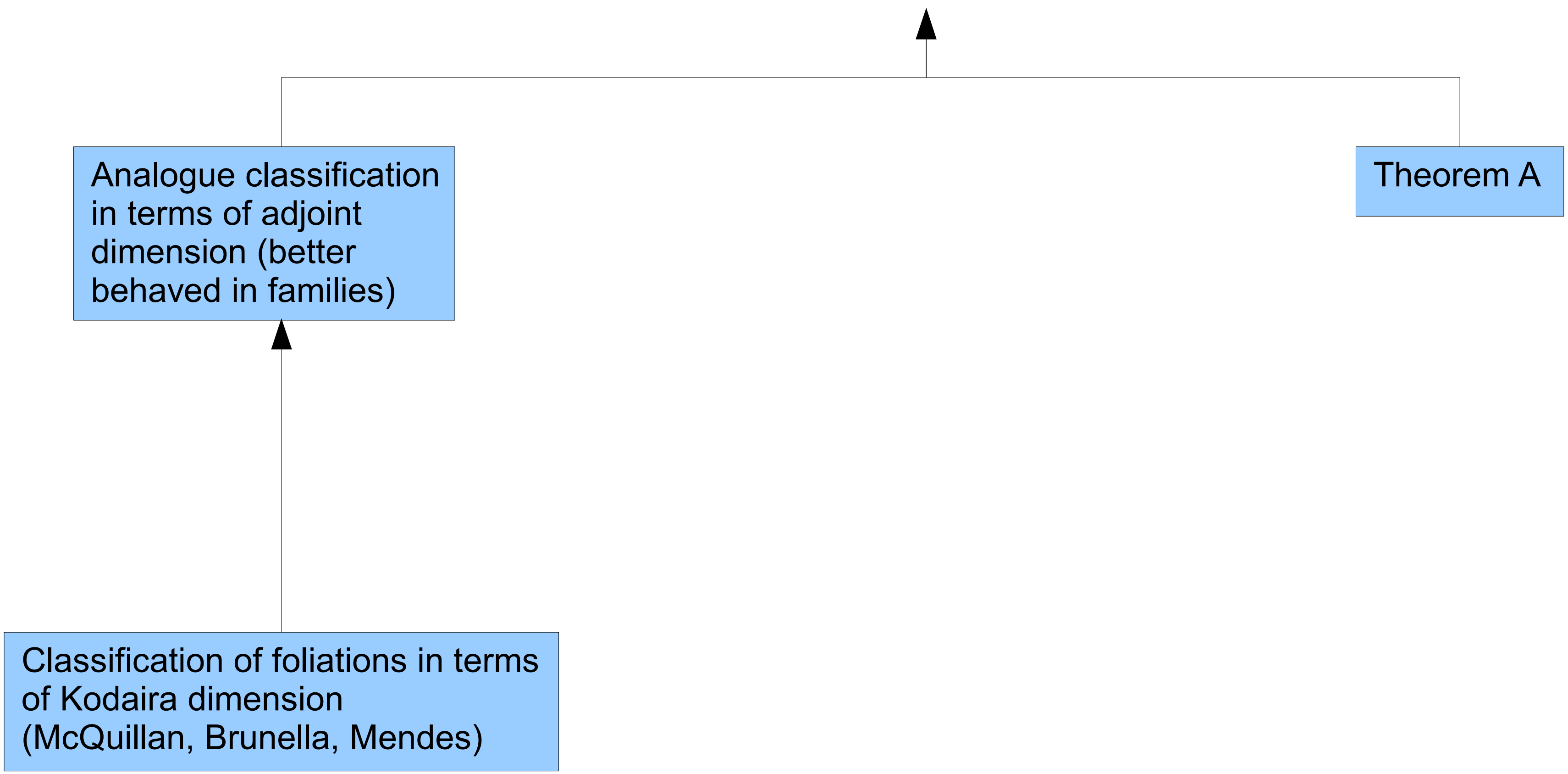

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Analogue classification in terms of adjoint dimension (better behaved in families)

Theorem A

Classification of foliations in terms of Kodaira dimension (McQuillan, Brunella, Mendes)

adj	kod	Description
$-\infty$	$-\infty$	Rational fibration
	0	Finite quotient of Riccati foliation generated by global vector field
	1	Riccati foliation
0	0	Finite quotient of linear foliation on a torus
1	0	Finite quotient of $E \times C \to C$, $g(C) \geq 2$
	1	Finite quotient of $E \times C \to E$, $g(C) \geq 2$
	1	Turbulent foliation
	1	Non-isotrivial elliptic fibration
2	$-\infty$	Irreducible quotient of $\mathbb{H} \times \mathbb{H} \to \mathbb{H}$
	1	Finite quotient of $C_1 \times C_2 \to C_1$, $g(C_i) \geq 2$
	2	General type

TABLE 1. Classification of foliations according to their adjoint/Kodaira dimensions.