Algorithmic Lie Symmetry Analysis and Group Classification for Ordinary Differential Equations

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Kolchin Seminar in Differential Algebra

Introduction

Symmetry group analysis is classical and the most fundamental method for solving differential equations and constructing solutions with particular properties.

Modern Lie theory has a lot of applications in natural sciences:

- automatic (analytical) solvers of ordinary differential equations in computer algebra systems
- construction of conservation laws, equivalence mappings for partial differential systems
- numerical (Lie) integrators (which preserve symmetries) reveal much better numerical behavior
- pattern recognition in computer vision (based on group equivalence of boundary curves)

Two equations

1. Constant coefficient equation

$$\frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + ... + a_1\frac{dy}{dx} + a_0 y = 0.$$

2. Euler equation

$$\frac{d^n y}{dx^n} + a_{n-1}x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + ... + a_1 x\frac{dy}{dx} + a_0 y = 0.$$

Two equations

1. Constant coefficient equation

$$\frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + ... + a_1\frac{dy}{dx} + a_0 y = 0$$

admits shifts for x

$$\bar{x} = x + a$$

and dilations for y

$$\bar{y} = C \cdot y$$

Two equations

2. Euler equation

$$\frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + ... + a_1 x \frac{dy}{dx} + a_0 y = 0$$

admits dilations for x

$$\bar{x} = A \cdot x$$

and dilations for y

$$\bar{y} = C \cdot y$$

Two equations

Transformation of independent variable, which maps dilation into shift

$$\bar{x} = A \cdot x \to ln(\bar{x}) = ln(A) + ln(x)$$

defines change of variables

$$t = exp(x), u = y$$

which maps Euler equation into Constant coefficient equation.

Remark.

Two equivalent equations have isomorphic symmetry groups.

Symmetry

Consider differential system of general form

$$\triangle(x, y^{(n)}) = 0 \tag{1}$$

where $x = (x^1, ..., x^p) \in X$ and $y = (y_1, ..., y_q) \in Y$ are vectors of independent and dependent variables correspondingly.

Definition.

The point transformation $g : (x, y) \mapsto (\bar{x}, \bar{y})$ of phase space $E = X \times Y$ is called symmetry if it transforms any solution $y(x)$ of (1) into new function defined by $\bar{y}(\bar{x})$, which is also solution.

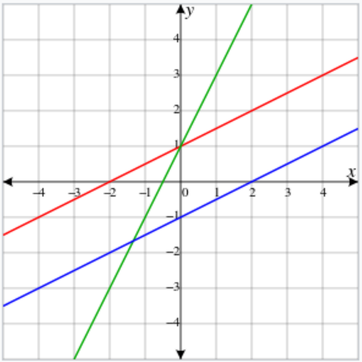
Symmetry (example)

Trivial second-order ODE

$$y''(x) = 0$$

Symmetry (example)

General solution: $\boxed{y = C_1 x + C_2}$



Symmetry condition means to transform solutions into solutions, which implies to map straight lines $\mapsto$ straight lines on plane.

Symmetry (example, continue)

Trivial second-order ODE

$$y''(x) = 0$$

by substitution

$$[u = f(x, y), \quad t = g(x, y)]$$

is transformed into

$$u''(t) + A_3 \cdot (u')^3 + A_2 \cdot (u')^2 + A_1 \cdot u' + A_0 = 0.$$

Symmetry condition implies

$$A_3 = 0, A_2 = 0, A_1 = 0, A_0 = 0.$$

Symmetry (example, continue)

$$A_3 = -\frac{\partial^2 g}{\partial y^2}\frac{\partial f}{\partial y} + \frac{\partial^2 f}{\partial y^2}\frac{\partial g}{\partial y} = 0,$$

$$A_2 = -\frac{\partial^2 g}{\partial y^2}\frac{\partial f}{\partial x} + \frac{\partial^2 f}{\partial x \partial y}\frac{\partial g}{\partial y} + 2\frac{\partial^2 f}{\partial y \partial y}\frac{\partial g}{\partial x} - 2\frac{\partial^2 g}{\partial x \partial y}\frac{\partial f}{\partial y} = 0,$$

$$A_1 = -\frac{\partial^2 g}{\partial x^2}\frac{\partial f}{\partial y} + \frac{\partial^2 f}{\partial x^2}\frac{\partial g}{\partial y} + 2\frac{\partial^2 f}{\partial x \partial y}\frac{\partial g}{\partial x} - 2\frac{\partial^2 g}{\partial x \partial y}\frac{\partial f}{\partial x} = 0,$$

$$A_0 = -\frac{\partial^2 g}{\partial x^2}\frac{\partial f}{\partial x} + \frac{\partial^2 f}{\partial x^2}\frac{\partial g}{\partial x} = 0\,.$$

Lie symmetry

One of the prominent idea of Sophus Lie is to study symmetry properties under one-parameter group of transformation.

Definition.

Set of transformation

$$g_a : (x, y) \mapsto (\bar{x}, \bar{y})$$

is called one-parameter group of transformation of differential system

$$\triangle(x, y^{(n)}) = 0$$

if
1) g_a is symmetry transformation for $\forall a$,
2) it is a local group: $g_a g_b = g_{a+b}$, where (a – group parameter).

Typically it leads to overdetermined system of linear partial differential equations of finite type, which could be efficiently analyzed symbolically by means of computer algebra for further reduction and explicit solving.

Lie symmetry (examples)

1. Shift
$$\bar{x} = x + a, \bar{y} = y$$
2. Dilation
$$\bar{x} = e^{a} \cdot x, \bar{y} = y$$
3. Rotation
$$\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = \begin{bmatrix} cos(a) & sin(a) \\ -sin(a) & cos(a) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
4. Shear
$$\bar{x} = x + ay, \bar{y} = y$$
5. Projection
$$\bar{x} = \frac{x}{1 - ax}, \bar{y} = \frac{y}{1 - ax}$$

Infinitesimal generators

The important role plays the first term in Taylor expansion of one-parameter group of transformation

$$\bar{x} = x + \varepsilon \, \underline{\xi(x, y)} + \mathcal{O}(\varepsilon^2) \, , \quad \bar{y} = y + \varepsilon \, \underline{\eta(x, y)} + \mathcal{O}(\varepsilon^2) \, .$$

It is equivalent to

$$\xi(x, y) = \left. \frac{\partial \bar{x}(x, y, a)}{\partial a} \right|_{a=0}, \; \eta(x, y) = \left. \frac{\partial \bar{y}(x, y, a)}{\partial a} \right|_{a=0}$$

And

$$\bar{x}'(a + b) = \frac{\partial \bar{x}(a + b)}{\partial b} = \frac{\partial \bar{x}(\bar{x}(x, y, a), \bar{y}(x, y, a), b)}{\partial a}, \bar{y}'(a + b) = \ldots$$

Thus assuming $b = 0$:

$$\boxed{\bar{x}'(a) = \xi(\bar{x}, \bar{y}), \bar{y}'(a) = \eta(\bar{x}, \bar{y}).}$$

Examples

1. Shift

$$\tilde{x} = x + a, \tilde{y} = y \to \xi = 1, \eta = 0$$

2. Dilation

$$\tilde{x} = e^a \cdot x, \tilde{y} = y \to \xi = x, \eta = 0$$

3. Rotation

$$\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} cos(a) & sin(a) \\ -sin(a) & cos(a) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \to \xi = y, \eta = -x$$

4. Shear

$$\tilde{x} = x + ay, \tilde{y} = y \to \xi = y, \eta = 0$$

5. Projection

$$\tilde{x} = \frac{x}{1 - ax}, \tilde{y} = \frac{y}{1 - ax} \to \xi = x^2, \eta = xy$$

Symmetry Analysis (example, continue)

$$g(x,y) := x + a\cdot\xi(x,y) + o(a),$$
$$f(x,y) := y + a\cdot\eta(x,y) + o(a)\,.$$

Substitution to symmetry condition

$$A_3 = 0, A_2 = 0, A_1 = 0, A_0 = 0$$

implies

$$\frac{\partial^2\eta}{\partial x^2} = 0, -\frac{\partial^2\xi}{\partial x^2} + 2\frac{\partial^2\eta}{\partial x\partial y} = 0, \frac{\partial^2\xi}{\partial y^2} = 0, -\frac{\partial^2\eta}{\partial y^2} + 2\frac{\partial^2\xi}{\partial x\partial y} = 0.$$

General solution

$$\xi(x,y) = (C_7\cdot x + C_8)\cdot y + C_5\cdot x^2 + C_3\cdot x + C_4,$$
$$\eta(x,y) = (C_5\cdot y + C_6)\cdot x + C_7\cdot y^2 + C_1\cdot y + C_2\,.$$

Consider ODE ($n \geq 2$) solved with respect to the highest order derivative

$$y^{(n)} + f(x, y, y', \ldots, y^{(n-1)}) = 0\,, \quad y^{(k)} := \frac{d^k y}{dx^k} \tag{2}$$

where f is rational function.

Algorithms

What can we do algorithmically?

- generation of determining equations
- reduction by integrability conditions
- dimension of solution space
- structure constants of Lie algebra

Initial value problem are given by finite number of values:

$$\left\{ \frac{\partial^{l_1+m_1}\xi}{\partial x^{l_1}\partial y^{m_1}}(x_0,y_0), \frac{\partial^{l_2+m_2}\eta}{\partial x^{l_2}\partial y^{m_2}}(x_0,y_0) \right\}, \begin{bmatrix}\xi\\ \eta\end{bmatrix} = \sum_{i,j=0}^{n} \begin{bmatrix}a_{i,j}\\ b_{i,j}\end{bmatrix} x^i y^j$$

Since Lie algebra is closed under Lie bracket, then

$$\mathcal{X} = \textit{truncated Taylor series} \rightarrow [\mathcal{X}_i, \mathcal{X}_j] = \sum_{k=1}^{m} C^k_{i,j}\mathcal{X}_k\,, \quad 1 \le i < j \le m\,.$$

Remark.

Beautiful point of construction that you actually need only truncated Taylor series (approximate solution) to obtain exact values of structure constant.

Linearizability

Theorem.

Eq. (2) with $n \geq 2$ is linearizable by a point transformation if and only if one of the following conditions is fulfilled:

1. $n = 2,\ m = 8$;
2. $n \geq 3,\ m = n + 4$;
3. $n \geq 3,\ m \in \{n + 1, n + 2\}$ and derived algebra is abelian and has dimension n.

Proof. According to group classification of linear equations:
1. Trivial equations $y^{(n)}(x) = 0$ have maximal possible dimension $n + 4$
2. Constant coefficients equations have Lie algebra span by operators

$$\left\{ f_1(x)\frac{\partial}{\partial y}, f_2(x)\frac{\partial}{\partial y}, ..., f_n(x)\frac{\partial}{\partial y}, \frac{\partial}{\partial x}, y\frac{\partial}{\partial y} \right\}$$

3. Generic case corresponds to Lie algebra

$$\left\{ f_1(x)\frac{\partial}{\partial y}, f_2(x)\frac{\partial}{\partial y}, ..., f_n(x)\frac{\partial}{\partial y}, y\frac{\partial}{\partial y} \right\}$$

Group classification

The problem of group classification of differential equations was first posed by Sophus Lie

He also began to solve the problem of group classification of the second-order ordinary equation of general form

$$y'' + f(x, y, y') = 0$$

Group classification

Lev Ovsyannikov considered simpler case of

$$y'' + f(x, y) = 0$$

and solved the problem of group classification by admissible operators

f	X_1	X_2	X_3
$f(y)*$	∂_x	0	0
e^y	∂_x	$x\partial_x - 2\partial_y$	0
$y^k, k \neq -3$	∂_x	$(k-1)x\partial_x - 2y\partial_y$	0
$\pm y^{-3}$	∂_x	$2x\partial_x + y\partial_y$	$x^2\partial_x + xy\partial_y$
$x^{-2}g(y)*$	$\mathrm{x}\partial_x$	0	0

Group classification

$$\frac{\partial^2\eta}{\partial x^2} - \frac{\partial f}{\partial x}\cdot\xi - \frac{\partial f}{\partial y}\cdot\eta + f\cdot(\frac{\partial\eta}{\partial y} - 2\frac{\partial\xi}{\partial x}) = 0,$$

$$-\frac{\partial^2\xi}{\partial x^2} + 2\frac{\partial^2\eta}{\partial x\partial y} - 3f\cdot\frac{\partial\xi}{\partial y} = 0,$$

$$\frac{\partial^2\xi}{\partial y^2} = 0,$$

$$-\frac{\partial^2\eta}{\partial y^2} + 2\frac{\partial^2\xi}{\partial x\partial y} = 0\,.$$

Group classification (invariant form)

The form of second-order ODE

$$y'' + f(x, y, y') = 0$$

given by

$$f = F_3(x, y)(y')^3 + F_2(x, y)(y')^2 + F_1(x, y)\, y' + F_0(x, y)\,. \tag{3}$$

is invariant under point transformation.

Group classification (linearizable branch)

Sophus Lie showed that only equations of the following form are linearizable by point transformations if and only if

Theorem.

$$
\begin{aligned}
&3(F_3)_{xx} - 2(F_2)_{xy} + (F_1)_{yy} - 3F_1(F_3)_x + 2F_2(F_2)_x \\
&-3F_3(F_1)_x + 3F_0(F_3)_y + 6F_3(F_0)_y - F_2(F_1)_y = 0\,, \\
&(F_2)_{xx} - 2(F_1)_{xy} + 3(F_0)_{yy} - 6F_0(F_3)_x + F_1(F_2)_x \\
&-3F_3(F_0)_x + 3F_0(F_2)_y + 3F_2(F_0)_y - 2F_1(F_1)_y = 0\,.
\end{aligned}
\tag{4}
$$

Thank you!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