Algorithmic Approach to Strong Consistency Analysis of Finite Difference Approximations to PDE Systems

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Outline

- Strong consistency of finite difference approximations
- Thomas decomposition for nonlinear PDE systems
- Decomposition of nonlinear difference systems

1. Strong consistency of finite difference approximations

Strong vs. weak consistency

Approximate PDE system (differential polynomials in $u^{(1)}, \ldots, u^{(m)}$)

by a difference system (difference polynomials in $\widetilde{u}^{(1)}, \ldots, \widetilde{u}^{(m)}$)

on Cartesian grid $\{ (x_1 + k_1 h, \ldots, x_n + k_n h) \mid k_1, \ldots, k_n \in \mathbb{Z} \}$, $h > 0$

e.g., $\partial_j \, u^{(\alpha)}(\mathbf{x}) \,=\, \frac{\widetilde{u}^{(\alpha)}_{k_1,\ldots,k_j+1,\ldots,k_n} - \widetilde{u}^{(\alpha)}_{k_1,\ldots,k_j-1,\ldots,k_n}}{2\,h} + \mathcal{O}(h^2)$

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Def. $\widetilde{f} \rhd f$ if Taylor expansion of $\widetilde{f}$ about grid point $\mathbf{x}$ yields

$$\widetilde{f}(\widetilde{\mathbf{u}}) \, = \, h^d \, f(\mathbf{u}) + \mathcal{O}(h^{d+1}) \, , \qquad d \in \mathbb{Z}_{\geq 0} \, ,$$

after clearing denominators containing h.

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after clearing denominators containing h.

Def. FDA $\widetilde{F}$ is *strongly consistent* (s-consistent) with PDEs F if

$$(\,\forall \tilde{f} \in [\![\tilde{F}]\!]\,)\;(\,\exists\, f \in [\![F]\!]\,)\;\;[\,\tilde{f} \rhd f\,]\,.$$

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Consistency of Finite Difference Approximations for Linear PDE Systems and its Algorithmic Verification,
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$\rightsquigarrow$ new difference scheme for 2D Navier-Stokes equations:

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Applied Mathematics and Computation 314:408–421, 2017.

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$$
\begin{cases}
\frac{u(n,j+1,k)-u(n,j-1,k)}{2h} + \frac{v(n,j,k+1)-v(n,j,k-1)}{2h} = 0, \\
\frac{u(n+1,j,k)-u(n,j,k)}{\tau} + \frac{u(n,j+1,k)^2-u(n,j-1,k)^2}{2h} + \\
\qquad \frac{u(n,j,k+1)v(n,j,k+1)-u(n,j,k-1)v(n,j,k-1)}{2h} + \frac{p(n,j+1,k)-p(n,j-1,k)}{2h} - \\
\qquad \frac{1}{re} \frac{(u(n,j+1,k)-2u(n,j,k)+u(n,j-1,k))+(u(n,j,k+1)-2u(n,j,k)+u(n,j,k-1))}{h^2} = 0, \\
\frac{v(n+1,j,k)-v(n,j,k)}{\tau} + \frac{u(n,j+1,k)v(n,j+1,k)-u(n,j-1,k)v(n,j-1,k)}{2h} + \\
\qquad \frac{v(n,j,k+1)^2-v(n,j,k-1)^2}{2h} + \frac{p(n,j,k+1)-p(n,j,k-1)}{2h} - \\
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\frac{u(n,j+2,k)^2-2u(n,j,k)^2+u(n,j-2,k)^2}{4h^2} + \frac{v(n,j,k+2)^2-2v(n,j,k)^2+v(n,j,k-2)^2}{4h^2} + \\
\qquad 2\,\frac{u(n,j+1,k+1)v(n,j+1,k+1)-u(n,j+1,k-1)v(n,j+1,k-1)}{4h^2} - \\
\qquad 2\,\frac{u(n,j-1,k+1)v(n,j-1,k+1)+u(n,j-1,k-1)v(n,j-1,k-1)}{4h^2} + \\
\qquad \frac{p(n,j+2,k)-2p(n,j,k)+p(n,j-2,k)}{4h^2} + \frac{p(n,j,k+2)-2p(n,j,k)+p(n,j,k-2)}{4h^2} = 0
\end{cases}
$$

Consistency, stability, convergence

Def. FDA to PDEs is *stable* if the error caused by a small perturbation in the numerical solution of the difference equations stays bounded.

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Thm. (Lax-Richtmyer, 1956).
A consistent finite difference scheme for a linear PDE, for which the initial value problem is well-posed, is convergent if it is stable.

Strong vs. weak consistency

Example.

$$(*) \qquad \begin{cases} \dfrac{\partial u}{\partial x} - u^2 & = & 0 \\ \dfrac{\partial u}{\partial y} + u^2 & = & 0 \end{cases} \qquad u = u(x, y)$$

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$$\begin{cases} D_1^+ \widetilde{u} - \widetilde{u}^2 & = & 0 & (A) \\ D_2^+ \widetilde{u} + \widetilde{u}^2 & = & 0 & (B) \end{cases}$$

D_1^+, D_2^+ forward differences

$$\rightsquigarrow \quad \sigma_2 A - \sigma_1 B + (\ldots)A + (\ldots)B \; = \; -2h^3 u_{i,j}^4$$

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D_1^+, D_2^+ forward differences

$$\rightsquigarrow \quad \sigma_2 A - \sigma_1 B + (\dots)A + (\dots)B \; = \; -2h^3 u_{i,j}^4$$

$$\begin{cases} D_1^+ \widetilde{u} - \widetilde{u}^2 & = & 0 \qquad (A') \\ D_2^- \widetilde{u} + \widetilde{u}^2 & = & 0 \qquad (B') \end{cases}$$

D_2^- backward difference

is s-consistent with $(*)$.

2. Thomas decomposition for nonlinear PDE systems

Systems of linear PDEs

$$\begin{cases} \dfrac{\partial^2 u}{\partial x \partial y} - \dfrac{\partial u}{\partial y} & = & 0 \\ \dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial u}{\partial y} & = & 0 \end{cases}$$

find: $u = u(x, y)$ analytic

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$$u(x, y) = a_{0,0} + a_{1,0}\, x + a_{0,1}\, y + a_{2,0}\, \tfrac{x^2}{2!} + a_{1,1}\, \tfrac{xy}{1!1!} + a_{0,2}\, \tfrac{y^2}{2!} + \ldots$$

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$$\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y}\right) \rightarrow \frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial y} = 0$$

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Janet's algorithm computes a vector space basis for power series solutions

(Maurice Janet, $\sim$ 1920)

Janet's Algorithm

$$\begin{cases} u_{y,y} = 0 \\ u_{x,x} - y u_{z,z} = 0 \end{cases}$$

is equivalent to

$$\begin{cases} u_{y,y} = 0 \\ u_{x,x} - y u_{z,z} = 0 \\ u_{y,z,z} = 0 \\ u_{x,y,y} = 0 \\ u_{z,z,z,z} = 0 \\ u_{x,y,z,z} = 0 \\ u_{x,z,z,z,z} = 0 \end{cases}$$

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$$\begin{cases} u_{y,y} = 0 & A \\ u_{x,x} - y u_{z,z} = 0 & B \end{cases}$$

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$$\begin{cases} u_{y,y} = 0 & A \\ u_{x,x} - y u_{z,z} = 0 & B \\ u_{y,z,z} = 0 & \frac{1}{2}(\partial_x^2 - y\partial_z^2)A - \frac{1}{2}\partial_y^2 B \\ u_{x,y,y} = 0 & \partial_x A \\ u_{z,z,z,z} = 0 & \frac{1}{2}(\partial_x^4 - 2y\partial_x^2\partial_z^2 + y^2\partial_z^4)A - \frac{1}{2}(\partial_x^2\partial_y^2 - y\partial_y^2\partial_z^2 + 2\partial_y\partial_z^2)B \\ u_{x,y,z,z} = 0 & \frac{1}{2}(\partial_x^3 - y\partial_x\partial_z^2)A - \frac{1}{2}\partial_x\partial_y^2 B \\ u_{x,z,z,z,z} = 0 & \frac{1}{2}(\partial_x^5 - 2y\partial_x^3\partial_z^2 + y^2\partial_x\partial_z^4)A - \frac{1}{2}(\partial_x^3\partial_y^2 + y\partial_x\partial_y^2\partial_z^2 - 2\partial_x\partial_y\partial_z^2)B \end{cases}$$

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Taylor coeff's for $1, z, y, x, z^2, yz, xz, xy, z^3, xz^2, xyz, xz^3$ *arbitrary*,

all other coeff's determined by linear equations

Differential algebraic geometry

Differential algebra (Ritt, Kolchin, Seidenberg, ...)

$\mathbb{Q} \subseteq K$ a differential field with commuting derivations ∂_1, ..., ∂_n

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Differential polynomial ring with derivations ∂_1, ..., ∂_n

$$K\{u\} := K[\partial_1^{i_1} \cdots \partial_n^{i_n} u \mid i \in (\mathbb{Z}_{\geq 0})^n] = K[u, u_{z_1}, ..., u_{z_n}, u_{z_1,z_1}, ...]$$

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Thm. (Differential Nullstellensatz).
Every radical diff. ideal $I \subsetneq K\{u^{(1)}, ..., u^{(m)}\}$ has a zero in a diff. field ext. of K. If $f \in K\{u^{(1)}, ..., u^{(m)}\}$ vanishes for all zeros of I, then $f \in I$.

Thomas decomposition for nonlinear PDE systems

$K\{u\} = K[u, u_x, u_y, \ldots, u_{x,x}, u_{x,y}, u_{y,y}, \ldots]$ diff. polynomial ring

$u < \ldots < u_y < u_x < \ldots < u_{y,y} < u_{x,y} < u_{x,x} < \ldots$ (ranking)

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algebraic reduction:

$$p = u_{x,x,y}^3 + \ldots$$

$$q = c\, u_{x,x,y}^2 + \ldots$$

$$p \rightarrow r = c \cdot p - u_{x,x,y} \cdot q$$

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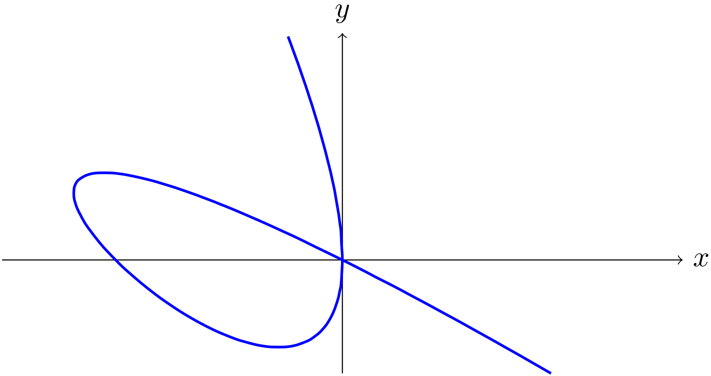
$$p \;\rightarrow\; r = \frac{\partial q}{\partial u_{x,x,y}} \cdot p - u_{x,x,y,y}^2 \cdot \partial_y\, q$$

reduction requires: initial $c \neq 0$ and separant $\frac{\partial q}{\partial u_{x,x,y}} \neq 0$

Thomas decomposition

$$p = x^3 + (3y+1)x^2 + (3y^2+2y)x + y^3 = 0$$



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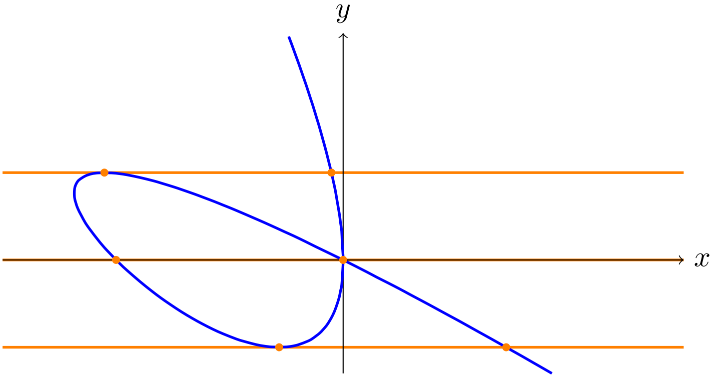

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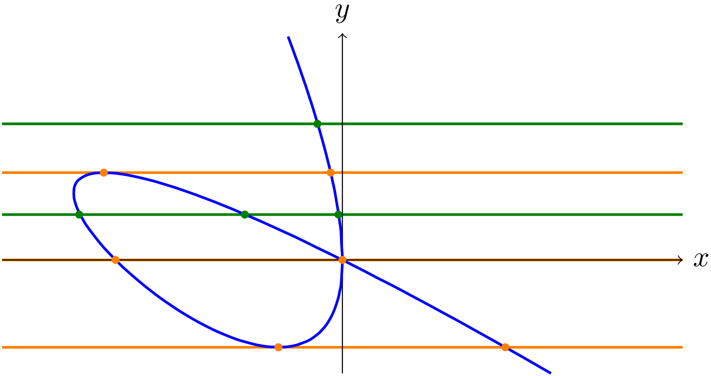

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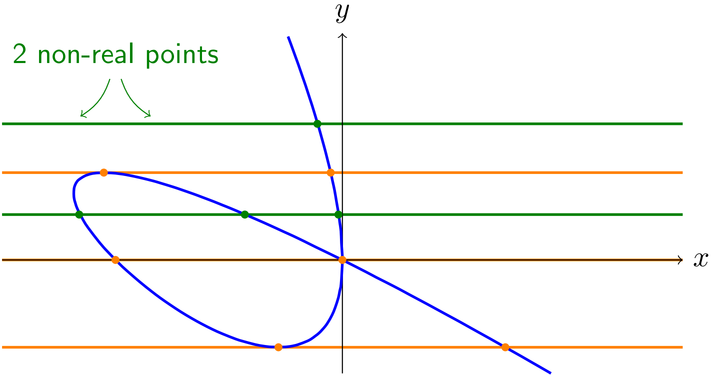

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Thomas decomposition for nonlinear PDE systems

$$S = \{ p_1 = 0, \ldots, p_s = 0, q_1 \neq 0, \ldots, q_t \neq 0 \}$$

Def. *Thomas decomposition* of differential system S (or $\mathrm{Sol}(S)$):

$\mathrm{Sol}(S) = \mathrm{Sol}(S_1) \uplus \ldots \uplus \mathrm{Sol}(S_r)$, S_i simple differential system

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Def. S is *simple* if

(a) $p_1, \ldots, p_s, q_1, \ldots, q_t$ have pairwise distinct leaders,

(b) initials and discriminants of p_i and q_j do not vanish,

(c) $p_1, \ldots, p_s$ form a passive PDE system,

(d) $q_1, \ldots, q_t$ are reduced modulo $p_1, \ldots, p_s$.

set of *admissible derivations* $\mu_i \subseteq \{\partial_1, \ldots, \partial_n\}$ for p_i, $i = 1, \ldots, s$

Thomas decomposition for nonlinear PDE systems

$R = K\{u^{(1)}, \ldots, u^{(m)}\}$

Def. *Thomas decomposition* of differential system S (or $\mathrm{Sol}(S)$):

$\mathrm{Sol}(S) = \mathrm{Sol}(S_1) \uplus \ldots \uplus \mathrm{Sol}(S_r)$, S_i simple differential system

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Thm. $S = \{p_1 = 0, ..., p_s = 0, q_1 \neq 0, ..., q_t \neq 0\}$ simple diff. system

E differential ideal generated by $p_1, \ldots, p_s$

q product of initials and separants of all p_i

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Then

$$E : q^\infty := \{\, p \in R \mid q^r \cdot p \in E \text{ for some } r \in \mathbb{Z}_{\geq 0} \,\} = \mathcal{I}_R(\mathrm{Sol}(S))$$

consists of all differential polynomials in R vanishing on $\mathrm{Sol}(S)$.

Example

Navier-Stokes equations for an incompressible flow of a constant viscosity fluid:

$$\left\{ \begin{array}{rcl} \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\,\mathbf{u} + \nabla p - \mu\,\Delta \mathbf{u} & = & 0 \\ \nabla \cdot \mathbf{u} & = & 0 \end{array} \right.$$

In cartesian coordinates x, y, z of $\mathbb{R}^3$ we have, equivalently,

$$\left\{ \begin{array}{rcl} u_t + u\,u_x + v\,u_y + w\,u_z + p_x - \mu\,(u_{x,x} + u_{y,y} + u_{z,z}) & = & 0 \\ v_t + u\,v_x + v\,v_y + w\,v_z + p_y - \mu\,(v_{x,x} + v_{y,y} + v_{z,z}) & = & 0 \\ w_t + u\,w_x + v\,w_y + w\,w_z + p_z - \mu\,(w_{x,x} + w_{y,y} + w_{z,z}) & = & 0 \\ u_x + v_y + w_z & = & 0 \end{array} \right.$$

where u, v, w denote the components of the velocity vector $\mathbf{u}$.

Poisson pressure equation

$$\Delta p + \nabla \cdot ((\mathbf{u} \cdot \nabla)\,\mathbf{u}) \;=\; 0$$

is a consequence of the Navier-Stokes equations.

Example

degree-reverse lexicographic ranking $>$ on the set of partial derivatives

$$\left\{ \frac{\partial^{i+j+k+l} f}{\partial t^i \, \partial x^j \, \partial y^k \, \partial z^l} \;\middle|\; f \in \{u, v, w, p\}, \; i, j, k, l \in \mathbb{Z}_{\geq 0} \right\}$$

$\rightsquigarrow$ one simple differential system:

$$\left\{ \begin{array}{rcl} \underline{u_x} + v_y + w_z & = & 0 \\ \mu \, \underline{v_{x,x}} + \mu \, v_{y,y} + \mu \, v_{z,z} - u v_x - v v_y - w v_z - p_y - v_t & = & 0 \\ \mu \, u_{y,y} + \mu \, u_{z,z} - \mu \, \underline{v_{x,y}} - \mu \, w_{x,z} + u v_y + u w_z - v u_y - w u_z - p_x - u_t & = & 0 \\ \mu \, \underline{w_{x,x}} + \mu \, w_{y,y} + \mu \, w_{z,z} - u w_x - v w_y - w w_z - p_z - w_t & = & 0 \\ \underline{p_{x,x}} + p_{y,y} + p_{z,z} + 2 \, u_y v_x + 2 \, u_z w_x + 2 \, v_y^2 + 2 \, v_y w_z + 2 \, v_z w_y + 2 \, w_z^2 & = & 0 \end{array} \right.$$

Example

degree-reverse lexicographic ranking $>$ on the set of partial derivatives

$$\left\{ \frac{\partial^{i+j+k+l} f}{\partial t^i \, \partial x^j \, \partial y^k \, \partial z^l} \;\middle|\; f \in \{u, v, w, p\}, \; i, j, k, l \in \mathbb{Z}_{\geq 0} \right\}$$

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The last equation was obtained as

$$\frac{\partial}{\partial x} A_1 + \frac{\partial}{\partial y} A_2 + \frac{\partial}{\partial z} A_3 + \left[v \, \Delta - \frac{\partial}{\partial t} - u \frac{\partial}{\partial x} - v \frac{\partial}{\partial y} - w \frac{\partial}{\partial z} + 2 \, v_y + 2 \, w_z \right] A_4 \, ,$$

where A_1, A_2, A_3, A_4 are the left hand sides of the Navier-Stokes equations.

Modulo the other equations in the system, last equation $\Leftrightarrow$ Poisson pressure equation.

Example

Taylor coefficients of $u(t,x,y,z)$, $v(t,x,y,z)$, $w(t,x,y,z)$, $p(t,x,y,z)$ whose values can be chosen arbitrarily:

$u(t,x,t,z)$:

$$\frac{1}{(1-\partial_t)\,(1-\partial_y)\,(1-\partial_z)}$$

$v(t,x,y,z)$:

$$\frac{1}{(1-\partial_t)\,(1-\partial_y)\,(1-\partial_z)} + \frac{\partial_x}{(1-\partial_t)\,(1-\partial_z)}$$

$w(t,x,y,z)$:

$$\frac{1}{(1-\partial_t)\,(1-\partial_y)\,(1-\partial_z)} + \frac{\partial_x}{(1-\partial_t)\,(1-\partial_y)\,(1-\partial_z)}$$

$p(t,x,y,z)$:

$$\frac{1}{(1-\partial_t)\,(1-\partial_y)\,(1-\partial_z)} + \frac{\partial_x}{(1-\partial_t)\,(1-\partial_y)\,(1-\partial_z)}$$

generalized Hilbert series

Example

Extending the Cauchy-Kovaleskaya Theorem we may pose the Cauchy problem for the Navier-Stokes equations around an arbitrary point (t_0, x_0, y_0, z_0) as follows:

$$\begin{cases} u(t, x_0, y, z) &= f_1(t, y, z) \\ v(t, x_0, y, z) &= f_2(t, y, z) \\ \dfrac{\partial v}{\partial x}(t, x_0, y_0, z) &= f_3(t, z) \\ w(t, x_0, y, z) &= f_4(t, y, z) \\ \dfrac{\partial w}{\partial x}(t, x_0, y, z) &= f_5(t, y, z) \\ p(t, x_0, y, z) &= f_6(t, y, z) \\ \dfrac{\partial p}{\partial x}(t, x_0, y, z) &= f_7(t, y, z) \end{cases}$$

where f_1, f_2, ..., f_7 are arbitrary functions of their arguments which are analytic around the point (t_0, x_0, y_0, z_0). The arbitrariness of analytic solutions to the Navier-Stokes equations is determined by f_1, f_2, ..., f_7.

Singular solutions

$$p = \dot{y}^2 - 4t\dot{y} - 4y + 8t^2 = 0 \qquad p \in \mathbb{Q}(t)\{y\}$$

Separant of p: $\frac{\partial p}{\partial \dot{y}} = 2\dot{y} - 4t$

Singular solutions

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Separant of p: $\frac{\partial p}{\partial \dot{y}} = 2\dot{y} - 4t$

$\mathrm{res}(p, \frac{\partial p}{\partial \dot{y}}, \dot{y}) = -16y + 16t^2$

Thomas decomposition:

$p = 0$ $y - t^2 \neq 0$	$y - t^2 = 0$

Singular solutions

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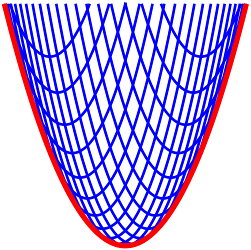
$\mathrm{res}(p, \frac{\partial p}{\partial \dot{y}}, \dot{y}) = -16y + 16t^2$



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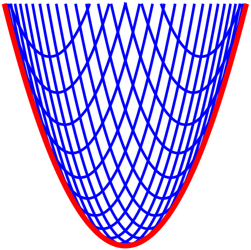
Thomas decomposition:

$p = 0$ $y - t^2 \neq 0$	$y - t^2 = 0$



general solution: $\quad y(t) = 2((t+c)^2 + c^2), \qquad c \in \mathbb{R}$

essential singular solution: $\quad y(t) = t^2$

Thomas decomposition

Example.

Thomas decomposition of $\{u_t - 6uu_x + \underline{u_{x,x,x}} = 0,\ u\underline{u_{t,x}} - u_t u_x = 0\}$:

Thomas decomposition

Example.

Thomas decomposition of $\{u_t - 6uu_x + \underline{u_{x,x,x}} = 0,\ u\underline{u_{t,x}} - u_t u_x = 0\}$:

$u = 0 \quad \{\partial_t, \partial_x\}$	$\underline{u_t} - 6uu_x = 0 \quad \{\partial_t, \partial_x\}$ $u_{x,x} = 0 \quad \{*, \partial_x\}$ $u \neq 0$	$u_t = 0 \quad \{\partial_t, \partial_x\}$ $\underline{u_{x,x,x}} - 6uu_x = 0 \quad \{*, \partial_x\}$ $u_{x,x} \neq 0$ $u \neq 0$

Thomas Decomposition

- 1937: J. M. Thomas: “Differential Systems”.
- 1998: D. Wang: implementation for algebraic systems
- 2007: V. Gerdt: “On decomposition of algebraic PDE systems into simple subsystems”
- 2009: W. Plesken: “Counting solutions of polynomial systems via iterated fibrations”
- since 2009: implementations in Maple for
  - algebraic systems (T. Bächler)
  - systems of PDEs (M. Lange-Hegermann)

T. Bächler, V. P. Gerdt, M. Lange-Hegermann, D. R.,
Algorithmic Thomas decomposition of algebraic and differential systems,
J. Symbolic Computation 47(10):1233–1266, 2012.

Implementation

Maple package `DifferentialThomas` (M. Lange-Hegermann)

`http://www.mathb.rwth-aachen.de/go/id/rnab/lidx/1`

GNU LPGL license

V. P. Gerdt, M. Lange-Hegermann, D. R.

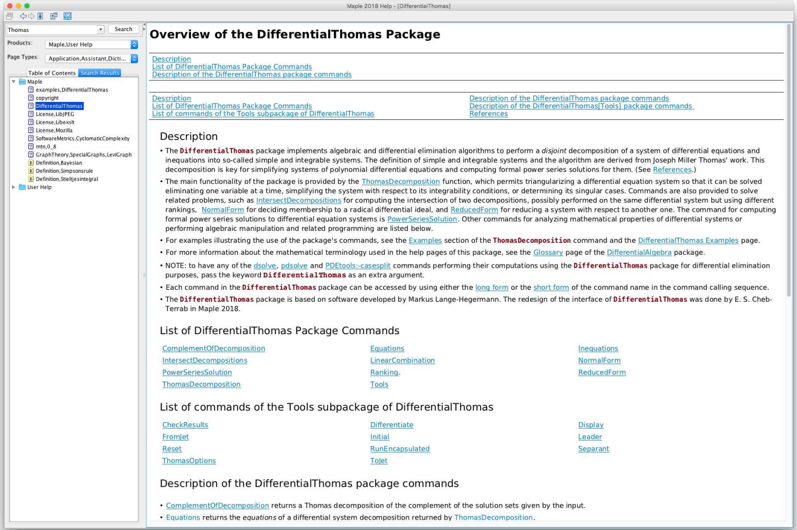
The MAPLE package TDDS for computing Thomas decompositions of systems of nonlinear PDEs

Computer Physics Communications 234:202–215, 2019

`arXiv:1801.09942`

`DifferentialThomas` in Maple 2018 (interface by E. S. Cheb-Terrab)

Maple 2018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Overview of the DifferentialThomas Package

Description
List of DifferentialThomas Package Commands
Description of the DifferentialThomas package commands

Description
List of DifferentialThomas Package Commands
List of commands of the Tools subpackage of DifferentialThomas
Description of the DifferentialThomas package commands
Description of the DifferentialThomas[Tools] package commands
References

Description

- The **DifferentialThomas** package implements algebraic and differential elimination algorithms to perform a *disjoint* decomposition of a system of differential equations and inequations into so-called simple and integrable systems. The definition of simple and integrable systems and the algorithm are derived from Joseph Miller Thomas' work. This decomposition is key for simplifying systems of polynomial differential equations and computing formal power series solutions for them. (See References.)
- The main functionality of the package is provided by the ThomasDecomposition function, which permits triangularizing a differential equation system so that it can be solved eliminating one variable at a time, simplifying the system with respect to its integrability conditions, or determining its singular cases. Commands are also provided to solve related problems, such as IntersectDecompositions for computing the intersection of two decompositions, possibly performed on the same differential system but using different rankings, NormalForm for deciding membership to a radical differential ideal, and ReducedForm for reducing a system with respect to another one. The command for computing formal power series solutions to differential equation systems is PowerSeriesSolution. Other commands for analyzing mathematical properties of differential systems or performing algebraic manipulation and related programming are listed below.
- For examples illustrating the use of the package's commands, see the Examples section of the **ThomasDecomposition** command and the DifferentialThomas Examples page.
- For more information about the mathematical terminology used in the help pages of this package, see the Glossary page of the DifferentialAlgebra package.
- NOTE: to have any of the dsolve, pdsolve and PDEtools:-casesplit commands performing their computations using the **DifferentialThomas** package for differential elimination purposes, pass the keyword **DifferentialThomas** as an extra argument.
- Each command in the **DifferentialThomas** package can be accessed by using either the long form or the short form of the command name in the command calling sequence.
- The **DifferentialThomas** package is based on software developed by Markus Lange-Hegermann. The redesign of the interface of **DifferentialThomas** was done by E. S. Cheb-Terrab in Maple 2018.

List of DifferentialThomas Package Commands

ComplementOfDecomposition	Equations	Inequations
IntersectDecompositions	LinearCombination	NormalForm
PowerSeriesSolution	Ranking.	ReducedForm
ThomasDecomposition	Tools	

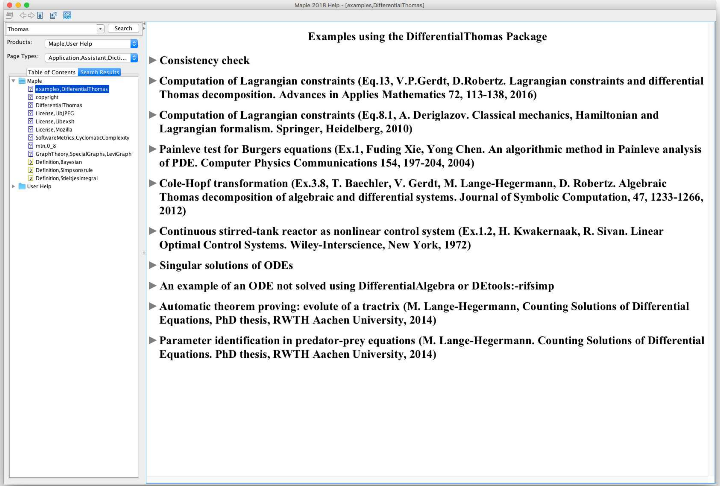
List of commands of the Tools subpackage of DifferentialThomas

CheckResults	Differentiate	Display
FromJet	Initial	Leader
Reset	RunEncapsulated	Separant
ThomasOptions	ToJet	

Description of the DifferentialThomas package commands

- ComplementOfDecomposition returns a Thomas decomposition of the complement of the solution sets given by the input.
- Equations returns the *equations* of a differential system decomposition returned by ThomasDecomposition.

Maple 2018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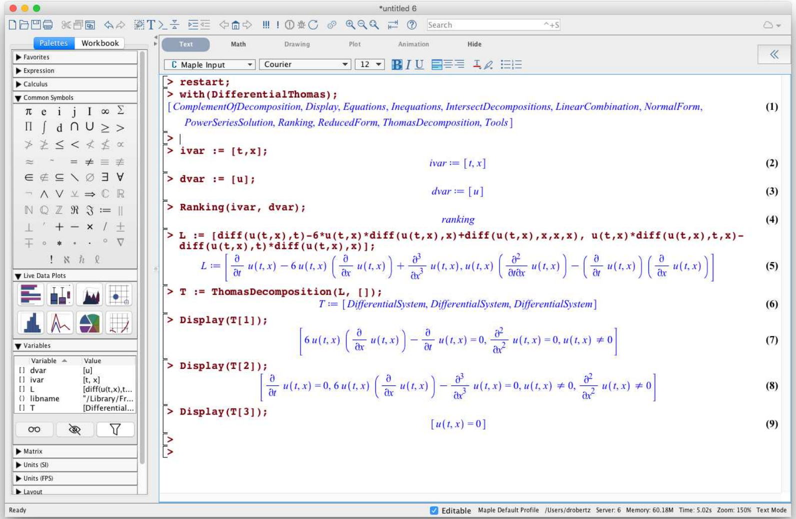
Examples using the DifferentialThomas Package
Consistency check
Computation of Lagrangian constraints (Eq.13, V.P.Gerdt, D.Robertz. Lagrangian constraints and differential Thomas decomposition. Advances in Applies Mathematics 72, 113-138, 2016)
Computation of Lagrangian constraints (Eq.8.1, A. Deriglazov. Classical mechanics, Hamiltonian and Lagrangian formalism. Springer, Heidelberg, 2010)
Painleve test for Burgers equations (Ex.1, Fuding Xie, Yong Chen. An algorithmic method in Painleve analysis of PDE. Computer Physics Communications 154, 197-204, 2004)
Cole-Hopf transformation (Ex.3.8, T. Baechler, V. Gerdt, M. Lange-Hegermann, D. Robertz. Algebraic Thomas decomposition of algebraic and differential systems. Journal of Symbolic Computation, 47, 1233-1266, 2012)
Continuous stirred-tank reactor as nonlinear control system (Ex.1.2, H. Kwakernaak, R. Sivan. Linear Optimal Control Systems. Wiley-Interscience, New York, 1972)
Singular solutions of ODEs
An example of an ODE not solved using DifferentialAlgebra or DEtools:-rifsimp
Automatic theorem proving: evolute of a tractrix (M. Lange-Hegermann, Counting Solutions of Differential Equations, PhD thesis, RWTH Aachen University, 2014)
Parameter identification in predator-prey equations (M. Lange-Hegermann. Counting Solutions of Differential Equations. PhD thesis, RWTH Aachen University, 2014)

Maple 2018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dvar
[u]
ivar
[t, x]
L
[diff(u(t,x),t...
libname
"/Library/Fr...
T
[Differential...
Matrix
Units (SI)
Units (FPS)
Layout
Text
Math
Drawing
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Animation
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Courier
12
> restart;
> with(DifferentialThomas);
[ComplementOfDecomposition, Display, Equations, Inequations, IntersectDecompositions, LinearCombination, NormalForm, PowerSeriesSolution, Ranking, ReducedForm, ThomasDecomposition, Tools] (1)
> ivar := [t,x];
ivar := [t, x] (2)
> dvar := [u];
dvar := [u] (3)
> Ranking(ivar, dvar);
ranking (4)
> L := [diff(u(t,x),t)-6*u(t,x)*diff(u(t,x),x)+diff(u(t,x),x,x,x), u(t,x)*diff(u(t,x),t,x)-diff(u(t,x),t)*diff(u(t,x),x)];
(5)
> T := ThomasDecomposition(L, []);
T := [DifferentialSystem, DifferentialSystem, DifferentialSystem] (6)
> Display(T[1]);
(7)
> Display(T[2]);
(8)
> Display(T[3]);
[u(t, x) = 0] (9)
Ready
Editable
Maple Default Profile
/Users/drobertz
Server: 6
Memory: 60.18M
Time: 5.02s
Zoom: 150%
Text Mode

Example

$$\det \begin{pmatrix} u_{x,x} & u_{x,y} & u_{y,y} \\ u_{x,y} & u_{y,y} & u_{y,z} \\ u_{x,z} & u_{y,z} & u_{z,z} \end{pmatrix} = 0,$$

$\mathbb{Q}\{u\}$ with degrevlex ranking

Example

$$\det \begin{pmatrix} u_{x,x} & u_{x,y} & u_{y,y} \\ u_{x,y} & u_{y,y} & u_{y,z} \\ u_{x,z} & u_{y,z} & u_{z,z} \end{pmatrix} = 0,$$

$\mathbb{Q}\{u\}$ with degrevlex ranking

Thomas decomposition:

$$\begin{array}{rcl} \det(\ldots) & = & 0 \\ u_{z,z}u_{y,y} - u_{y,z}^2 & \neq & 0 \\ u_{z,z} & \neq & 0 \end{array}$$

$$\begin{array}{rcl} -u_{y,z}^2 u_{x,x} + 2u_{y,z}u_{x,z}u_{x,y} - u_{y,y}u_{x,z}^2 & = & 0 \\ u_{y,z} & \neq & 0 \\ u_{z,z} & = & 0 \end{array}$$

$$\begin{array}{rcl} u_{z,z}u_{x,y} - u_{y,z}u_{x,z} & = & 0 \\ u_{z,z}u_{y,y} - u_{y,z}^2 & = & 0 \\ u_{z,z} & \neq & 0 \end{array}$$

$$\begin{array}{rcl} u_{x,z} & = & 0 \\ u_{y,z} & = & 0 \\ u_{z,z} & = & 0 \\ u_{y,y} & \neq & 0 \end{array}$$

$$\begin{array}{rcl} u_{y,y} & = & 0 \\ u_{y,z} & = & 0 \\ u_{z,z} & = & 0 \end{array}$$

Example

$$\det\begin{pmatrix} u_{x,x} & u_{x,y} & u_{y,y} \\ u_{x,y} & u_{y,y} & u_{y,z} \\ u_{x,z} & u_{y,z} & u_{z,z} \end{pmatrix} = 0,$$

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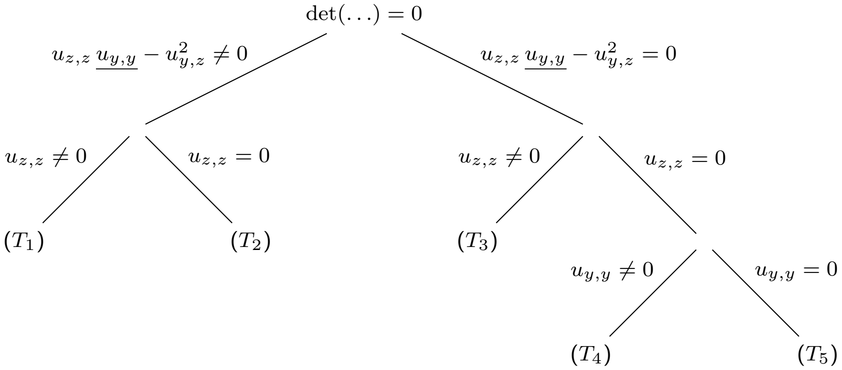
Example

$$\det\begin{pmatrix} u_{x,x} & u_{x,y} & u_{y,y} \\ u_{x,y} & u_{y,y} & u_{y,z} \\ u_{x,z} & u_{y,z} & u_{z,z} \end{pmatrix} = 0,$$

$\mathbb{Q}\{u\}$ with degrevlex ranking



Example

$$\det \begin{pmatrix} u_{x,x} & u_{x,y} & u_{y,y} \\ u_{x,y} & u_{y,y} & u_{y,z} \\ u_{x,z} & u_{y,z} & u_{z,z} \end{pmatrix} = 0,$$

$\mathbb{Q}\{u\}$ with degrevlex ranking

Example

$$\det\begin{pmatrix} u_{x,x} & u_{x,y} & u_{y,y} \\ u_{x,y} & u_{y,y} & u_{y,z} \\ u_{x,z} & u_{y,z} & u_{z,z} \end{pmatrix} = 0,$$

$\mathbb{Q}\{u\}$ with degrevlex ranking

DifferentialAlgebra (Maple 17):

$$\begin{array}{|rcl|} \hline \det(\ldots) & = & 0 \\ u_{z,z}u_{y,y} - u_{y,z}^2 & \neq & 0 \\ \hline \end{array} \qquad \begin{array}{|rcl|} \hline u_{z,z}u_{x,y} - u_{y,z}u_{x,z} & = & 0 \\ u_{z,z}u_{y,y} - u_{y,z}^2 & = & 0 \\ u_{z,z} & \neq & 0 \\ \hline \end{array}$$

$$\begin{array}{|rcl|} \hline u_{x,z} & = & 0 \\ u_{y,z} & = & 0 \\ u_{z,z} & = & 0 \\ \hline \end{array} \qquad \begin{array}{|rcl|} \hline u_{y,y} & = & 0 \\ u_{y,z} & = & 0 \\ u_{z,z} & = & 0 \\ \hline \end{array}$$

3. Decomposition of nonlinear difference systems

Difference algebra

Difference algebra (Ritt, Cohn, Levin, ...)

$\mathbb{Q} \subseteq \widetilde{K}$ a difference field with commuting automorphisms σ_1, ..., σ_n

Difference algebra

Difference algebra (Ritt, Cohn, Levin, ...)

$\mathbb{Q} \subseteq \widetilde{K}$ a difference field with commuting automorphisms σ_1, ..., σ_n

Difference polynomial ring with comm. endomorphisms σ_1, ..., σ_n

$\widetilde{K}\{\widetilde{u}\} := \widetilde{K}[\sigma_1^{i_1} \cdots \sigma_n^{i_n} \widetilde{u} \mid i \in (\mathbb{Z}_{\geq 0})^n]$, $\qquad \widetilde{u}_{(i_1,\ldots,i_n)} := \sigma_1^{i_1} \cdots \sigma_n^{i_n} \widetilde{u}$

Difference algebra

Difference algebra (Ritt, Cohn, Levin, . . .)

$\mathbb{Q} \subseteq \widetilde{K}$ a difference field with commuting automorphisms σ_1, ..., σ_n

Difference polynomial ring with comm. endomorphisms σ_1, ..., σ_n

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$\widetilde{K}\{\widetilde{u}\}$ not Noetherian (e.g., $[\widetilde{u}\widetilde{u}_1, \widetilde{u}\widetilde{u}_2, \widetilde{u}\widetilde{u}_3, \ldots] \subseteq \widetilde{K}\{\widetilde{u}\}$ not fin. gen.)

Difference algebra

Difference algebra (Ritt, Cohn, Levin, ...)

$\mathbb{Q} \subseteq \widetilde{K}$ a difference field with commuting automorphisms σ_1, ..., σ_n

Difference polynomial ring with comm. endomorphisms σ_1, ..., σ_n

$\widetilde{K}\{\widetilde{u}\} := \widetilde{K}[\sigma_1^{i_1} \cdots \sigma_n^{i_n} \widetilde{u} \mid i \in (\mathbb{Z}_{\geq 0})^n]$, $\qquad \widetilde{u}_{(i_1,\ldots,i_n)} := \sigma_1^{i_1} \cdots \sigma_n^{i_n} \widetilde{u}$

$\widetilde{K}\{\widetilde{u}\}$ not Noetherian (e.g., $[\widetilde{u}\widetilde{u}_1, \widetilde{u}\widetilde{u}_2, \widetilde{u}\widetilde{u}_3, \ldots] \subseteq \widetilde{K}\{\widetilde{u}\}$ not fin. gen.)

A difference ideal $\widetilde{I}$ of $\widetilde{K}\{\widetilde{u}\}$ is said to be

- *reflexive* if $\sigma_i(\widetilde{f}) \in \widetilde{I}$ implies $\widetilde{f} \in \widetilde{I}$ $\qquad (\widetilde{f} \in \widetilde{K}\{\widetilde{u}\})$
- *perfect* if the iteration of the process of forming the difference ideal generated by $\widetilde{f}$ such that $\sigma^{i_1}(\widetilde{f})^{k_1} \ldots \sigma^{i_r}(\widetilde{f})^{k_r} \in \widetilde{I}$ yields $\widetilde{I}$

Difference algebra

Thm. (Ritt-Raudenbush).
Every perfect difference ideal of $\widetilde{K}\{\widetilde{u}^{(1)}, \ldots, \widetilde{u}^{(m)}\}$ is finitely generated and is intersection of finitely many prime difference ideals.

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Every difference field embeds into a difference closed field.

Thm. (Cohn)
Existence of generic zeros of reflexive prime difference ideals in piecewise analytic functions on $\mathbb{R}_+$

Decomposition of nonlinear difference systems

Notation.

Recall $\quad \partial_j\, u^{(\alpha)}(\mathbf{x}) \,=\, \frac{\widetilde{u}^{(\alpha)}_{k_1,\ldots,k_j+1,\ldots,k_n} - \widetilde{u}^{(\alpha)}_{k_1,\ldots,k_j-1,\ldots,k_n}}{2\,h} + \mathcal{O}(h^2)$

grid functions $\widetilde{u}^{(\alpha)}_{k_1,\ldots,k_n}$

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$\widetilde{K} = \mathbb{Q}(\mathbf{a}, h)\,,$ a difference field of constants

$\widetilde{R} = \widetilde{K}\{\widetilde{u}^{(1)}, \ldots, \widetilde{u}^{(m)}\}$ difference polynomial ring over $\widetilde{K}$,

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$\mathrm{Mon}(\Sigma) =$ monomials in σ_1, ..., σ_n

leader, initial, discriminant, ... are defined accordingly w.r.t. a ranking

Decomposition of nonlinear difference systems

$$\widetilde{S} = \{\, \widetilde{f}_1 = 0,\, \ldots,\, \widetilde{f}_s = 0,\, \widetilde{f}_{s+1} \neq 0,\, \ldots,\, \widetilde{f}_{s+t} \neq 0 \,\}$$ difference system

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$\Omega \subseteq \mathbb{R}^n$ open and connected, $\mathbf{x} \in \Omega$

$\Gamma_{\mathbf{x},h} := \{\, (x_1 + k_1 h, \ldots, x_n + k_n h) \mid k_1, \ldots, k_n \in \mathbb{Z} \,\}, \quad h > 0$

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$$\mathrm{Sol}_{\Omega,\mathbf{x},h} := \{ (\widetilde{u}_1, \ldots, \widetilde{u}_m) \in \mathcal{F}^m_{\Omega,\mathbf{x},h} \mid \widetilde{f}_1(\widetilde{u}) = 0, \ldots, \widetilde{f}_s(\widetilde{u}) = 0, \widetilde{f}_{s+1}(\widetilde{u}) \neq 0, \ldots, \widetilde{f}_{s+t}(\widetilde{u}) \neq 0 \}$$

Decomposition of nonlinear difference systems

$$\widetilde{S} = \{\, \widetilde{f}_1 = 0,\, \ldots,\, \widetilde{f}_s = 0,\, \widetilde{f}_{s+1} \neq 0,\, \ldots,\, \widetilde{f}_{s+t} \neq 0 \,\}$$

Def. *Difference decomposition* of difference system $\widetilde{S}$

$$\mathrm{Sol}_{\Omega,\mathbf{x},h}(\widetilde{S}) \;=\; \mathrm{Sol}_{\Omega,\mathbf{x},h}(\widetilde{S}_1) \uplus \ldots \uplus \mathrm{Sol}_{\Omega,\mathbf{x},h}(\widetilde{S}_r), \qquad \widetilde{S}_i \quad \text{quasi-simple}$$

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Def. $\widetilde{S}$ is *(quasi-) simple* if

(a) $\widetilde{f}_1, \ldots, \widetilde{f}_s, \widetilde{f}_{s+1}, \ldots, \widetilde{f}_{s+t}$ have pairwise distinct leaders,

(b) initials and (not nec.) discriminants of $\widetilde{f}_i$ do not vanish, $1 \le i \le s+t$,

(c) $\widetilde{f}_1, \ldots, \widetilde{f}_s$ form a passive difference system,

(d) $\widetilde{f}_{s+1}, \ldots, \widetilde{f}_{s+t}$ are reduced modulo $\widetilde{f}_1, \ldots, \widetilde{f}_s$.

set of *admissible automorphisms* $\mu_i \subseteq \{\sigma_1, ..., \sigma_n\}$ for $\widetilde{f}_i$, $i = 1, \ldots, s$

Decomposition of nonlinear difference systems

$\widetilde{R} = \widetilde{K}\{\widetilde{u}^{(1)}, \ldots, \widetilde{u}^{(m)}\}$

Prop. 6.3 $\widetilde{S} = \{\, \widetilde{f}_1 = 0, ..., \widetilde{f}_s = 0, \widetilde{f}_{s+1} \neq 0, ..., \widetilde{f}_{s+t} \neq 0 \,\}$ quasi-simple

E difference ideal generated by $\widetilde{f}_1$, ..., $\widetilde{f}_s$

Q smallest subset of $\widetilde{R}$ containing $q_1 := \mathrm{init}(\widetilde{f}_1)$, ..., $q_s := \mathrm{init}(\widetilde{f}_s)$, which is multiplicatively closed and closed under σ_1, ..., σ_n

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Then the elements of

$$E : Q \quad := \quad \{\, \widetilde{f} \in \widetilde{R} \mid (\theta_1(q_1))^{r_1} \ldots (\theta_s(q_s))^{r_s}\, \widetilde{f} \in E \text{ for some } \theta_1, \ldots, \theta_s \in \mathrm{Mon}(\Sigma),\, r_1, \ldots, r_s \in \mathbb{Z}_{\geq 0} \,\}$$

are the difference polynomials reducing to zero modulo $\widetilde{f}_1$, ..., $\widetilde{f}_s$.

Input: $L \subset \widetilde{R} \setminus \widetilde{K}$ finite and a ranking $\succ$ on $\widetilde{R}$ such that $L = \widetilde{S}^{=}$ for some finite difference system $\widetilde{S}$ which is quasi-simple as an algebraic system (in the finitely many indeterminates $(u_k)_J$ which occur in it, totally ordered by $\succ$, not necessarily square-free)

Output: $a \in \{\textbf{true}, \textbf{false}\}$ and $L' \subset \widetilde{R} \setminus \widetilde{K}$ finite such that

$$\langle\, L' \,\rangle : Q \;=\; \langle\, L \,\rangle : Q$$

where Q is the smallest multiplicatively closed subset of $\widetilde{R}$ containing all $\mathrm{init}(\theta \widetilde{f})$, where $\widetilde{f} \in L$ and $\theta \in \mathrm{ld}(L \setminus \{\widetilde{f}\}) : \mathrm{ld}(\widetilde{f})$, and which is closed under σ_1, ..., σ_s; and, in case $a = \textbf{true}$, there exist no $\widetilde{f}_1$, $\widetilde{f}_2 \in L'$, $\widetilde{f}_1 \neq \widetilde{f}_2$, such that we have $v := \mathrm{ld}(\widetilde{f}_1) = \theta\, \mathrm{ld}(\widetilde{f}_2)$ for some $\theta \in \mathrm{Mon}(\Sigma)$ and $\deg_v(\widetilde{f}_1) \geq \deg_v(\theta \widetilde{f}_2)$

Algorithm:

$L' \leftarrow L$
while there exist $\widetilde{f}_1, \widetilde{f}_2 \in L'$, $\widetilde{f}_1 \neq \widetilde{f}_2$ and $\theta \in \mathrm{Mon}(\Sigma)$ such that we have $v := \mathrm{ld}(\widetilde{f}_1) = \theta\, \mathrm{ld}(\widetilde{f}_2)$ and $\deg_v(\widetilde{f}_1) \geq \deg_v(\theta \widetilde{f}_2)$ **do**
    $L' \leftarrow L' \setminus \{\widetilde{f}_1\}$; $v \leftarrow \mathrm{ld}(\widetilde{f}_1)$
    $\widetilde{r} \leftarrow \mathrm{init}(\theta \widetilde{f}_2) \cdot \widetilde{f}_1 - \mathrm{init}(\widetilde{f}_1) \cdot v^d \cdot \theta \widetilde{f}_2$, where $d := \deg_v(\widetilde{f}_1) - \deg_v(\theta \widetilde{f}_2)$
    if $\widetilde{r} \neq 0$ **then**
        return $(\textbf{false}, L' \cup \{\widetilde{r}\})$
    end if
end while
return $(\textbf{true}, L')$

Input: $\widetilde{r} \in \widetilde{R}$, $T = \{\, (\widetilde{f}_1, \mu_1), (\widetilde{f}_2, \mu_2), \ldots, (\widetilde{f}_s, \mu_s) \,\}$, and a ranking $\succ$ on $\widetilde{R}$, where T is Janet complete (with respect to $\succ$)

Output: $\widetilde{r}' \in \widetilde{R}$ and an element b of the multiplicatively closed set generated by

$$\bigcup_{i=1}^{s} \{\, \theta \operatorname{init}(\widetilde{f}_i) \mid \theta \in \operatorname{Mon}(\Sigma),\ \operatorname{ld}(\widetilde{r}) \succ \theta \operatorname{ld}(\widetilde{f}_i) \,\} \cup \{1\}$$

such that $\widetilde{r}'$ is Janet reduced modulo T, and such that $\widetilde{r}' = \widetilde{r}$, $b = 1$ if $T = \emptyset$, and $\widetilde{r}' + \langle\, \widetilde{f}_1, \ldots, \widetilde{f}_s \,\rangle = b \cdot \widetilde{r} + \langle\, \widetilde{f}_1, \ldots, \widetilde{f}_s \,\rangle$ otherwise

Algorithm:

$\widetilde{r}' \leftarrow \widetilde{r}$; $\quad b \leftarrow 1$
if $\widetilde{r}' \notin \widetilde{K}$ **then**
    $v \leftarrow \operatorname{ld}(\widetilde{r}')$
    while $\widetilde{r}' \notin \widetilde{K}$, $\exists\, (\widetilde{f}, \mu) \in T$, $\theta \in \operatorname{Mon}(\mu)$: $v = \theta \operatorname{ld}(\widetilde{f})$, $\deg_v(\widetilde{r}') \geq \deg_v(\theta \widetilde{f})$ **do**
        $\widetilde{r}' \leftarrow \operatorname{init}(\theta \widetilde{f}) \cdot \widetilde{r}' - \operatorname{init}(\widetilde{r}') \cdot v^d \cdot \theta \widetilde{f}$, where $d := \deg_v(\widetilde{r}') - \deg_v(\theta \widetilde{f})$
        $b \leftarrow \operatorname{init}(\theta \widetilde{f}) \cdot b$
    end while
    for each coefficient $\widetilde{c}$ of $\widetilde{r}'$ (as a polynomial in v) **do**
        $(\widetilde{r}'', b') \leftarrow$ `Janet-reduce`($\widetilde{c}$, T, $\succ$)
        replace the coefficient $b' \cdot \widetilde{c}$ in $b' \cdot \widetilde{r}'$ with $\widetilde{r}''$ and replace $\widetilde{r}'$ with this result
        $b \leftarrow b' \cdot b$
    end for
end if
return $(\widetilde{r}', b)$

Input: A finite difference system $\widetilde{S}$ over $\widetilde{R}$, a ranking $\succ$ on $\widetilde{R}$, and a total ordering on Σ (used by `Decompose`)

Output: A difference decomposition of $\widetilde{S}$

Algorithm:

$Q \leftarrow \{\widetilde{S}\}$; $T \leftarrow \emptyset$
repeat
    choose $L \in Q$ and remove L from Q
    compute a decomposition $\{ A_1, \ldots, A_r \}$ of L, considered as an algebraic system, into quasi-simple systems
    for $i = 1, \ldots, r$ **do**
        if $A_i = \emptyset$ **then** // *no equation and no inequation*
            return $\{\emptyset\}$
        else
            $(a, G) \leftarrow$ `Auto-reduce`$(A_i^{=}, \succ)$
            if $a =$ **true then**
                ...
            else
                insert $\{ \widetilde{f} = 0 \mid \widetilde{f} \in G \} \cup \{ \widetilde{g} \neq 0 \mid \widetilde{g} \in A_i^{\neq} \}$ into Q
            end if
        end if
    end for
until $Q = \emptyset$
return T

Input: A finite difference system $\widetilde{S}$ over $\widetilde{R}$, a ranking $\succ$ on $\widetilde{R}$, and a total ordering on Σ (used by `Decompose`)

Output: A decomposition of $\widetilde{S}$

Algorithm:

...
$(a, G) \leftarrow$ `Auto-reduce`$(A_i^{=}, \succ)$
if $a =$ **true then**
    $J \leftarrow$ `Decompose`(G)
    $P \leftarrow \{\, \mathrm{NF}(\sigma \widetilde{f}, J, \succ) \mid (\widetilde{f}, \mu) \in J,\ \sigma \in \overline{\mu} \,\}$
    if $P \subseteq \{0\}$ **then** *// J is passive*
        replace each inequation $\widetilde{g} \neq 0$ in A_i with $\mathrm{NF}(\widetilde{g}, J, \succ) \neq 0$
        if $0 \notin A_i^{\neq}$ **then**
            insert $\{\, \widetilde{f} = 0 \mid (\widetilde{f}, \mu) \in J \,\} \cup \{\, \widetilde{g} \neq 0 \mid \widetilde{g} \in A_i^{\neq} \,\}$ into T
        end if
    else if $P \cap \widetilde{K} \subseteq \{0\}$ **then**
        insert $\{\, \widetilde{f} = 0 \mid (\widetilde{f}, \mu) \in J \,\} \cup \{\, \widetilde{f} = 0 \mid \widetilde{f} \in P \setminus \{0\} \,\} \cup \{\, \widetilde{g} \neq 0 \mid \widetilde{g} \in A_i^{\neq} \,\}$ into Q
    end if
else
    insert $\{\, \widetilde{f} = 0 \mid \widetilde{f} \in G \,\} \cup \{\, \widetilde{g} \neq 0 \mid \widetilde{g} \in A_i^{\neq} \,\}$ into Q
end if
...

S-ConsistencyCheck

Input: A simple differential system S over R, a ranking $>$ on R, a ranking $\succ$ on $\widetilde{R}$, a total ordering on Σ (used by `Decompose`) and a difference system $\widetilde{S}$ consisting of equations that are w-consistent with S

Output: $\widetilde{L} = \{\,(\widetilde{L}_1, b_1), \ldots, (\widetilde{L}_r, b_r)\,\}$, where $\widetilde{L}_i$ is s-consistent (resp. w-consistent) with $L_i \xleftarrow[h \to 0]{} \tilde{L}_i$ if $b_i =$ **true** (resp. **false**)

Algorithm:

$\tilde{L} = \{\tilde{L}_1, \ldots, \tilde{L}_k\} \leftarrow$ `DifferenceDecomposition`$(\widetilde{S}, \succ)$
for $i = 1, \ldots, k$ **do**
  if $\exists \widetilde{f} \in \tilde{L}_i^{\neq}$ such that $\widetilde{f} \rhd f \in [\![S^{=}]\!]$ **then**
    $\tilde{L} \leftarrow \tilde{L} \setminus \{\tilde{L}_i\}$
  else
    $b_i \leftarrow$ **true**
    for $\widetilde{f} \in \tilde{L}^{=}$ **do**
      compute $f \in R$ such that $\widetilde{f} \rhd f$
      if $\mathrm{NF}(f, S^{=}, >) \neq 0$ **then**
        $b_i \leftarrow$ **false**; **break**
      end if
    end for
  end if
end for
return $\{\,(\tilde{L}_i, b_i) \mid \tilde{L}_i \in \tilde{L}\,\}$

Example

Example.

$$(*) \qquad \begin{cases} \dfrac{\partial u}{\partial x} - u^2 &= 0 \\ \dfrac{\partial u}{\partial y} + u^2 &= 0 \end{cases} \qquad u = u(x,y)$$

$$\begin{cases} D_1^+\widetilde{u} - \widetilde{u}^2 &= 0 \qquad (A) \\ D_2^+\widetilde{u} + \widetilde{u}^2 &= 0 \qquad (B) \end{cases}$$

D_1^+, D_2^+ forward differences

$$\rightsquigarrow \quad \sigma_2 A - \sigma_1 B + (\ldots)A + (\ldots)B \;=\; -2h^3 u_{i,j}^4$$

$$\begin{cases} D_1^+\widetilde{u} - \widetilde{u}^2 &= 0 \qquad (A') \\ D_2^-\widetilde{u} + \widetilde{u}^2 &= 0 \qquad (B') \end{cases}$$

D_2^- backward difference

is s-consistent with $(*)$.

Example

Navier-Stokes equations for an incompressible flow of a constant viscosity fluid:

$$\begin{cases} \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\,\mathbf{u} + \nabla p - \mu\,\Delta \mathbf{u} = 0 \\ \nabla \cdot \mathbf{u} = 0 \end{cases}$$

Example

Navier-Stokes equations for an incompressible flow of a constant viscosity fluid:

$$\left\{ \begin{array}{rcl} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\,\mathbf{u} + \nabla p - \mu\,\Delta\mathbf{u} & = & 0 \\ \nabla\cdot\mathbf{u} & = & 0 \end{array} \right.$$

ranking TOP-lex with $\partial_t > \partial_1 > \partial_2 > \partial_3$, $p > u > v > w$; passivity:

$$(*) \qquad \Delta p + \nabla\cdot(\mathbf{u}\cdot\nabla)\,\mathbf{u} \;=\; 0$$

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FDA:

$$D_t\,\widetilde{\mathbf{u}} + (\widetilde{\mathbf{u}}\cdot\mathbf{D})\,\widetilde{\mathbf{u}} + \mathbf{D}\,\widetilde{p} - \mu\,\widetilde{\Delta}\,\widetilde{\mathbf{u}} \;=\; 0\,,$$

where D_t approx. ∂_t, $\mathbf{D} = (D_1, D_2, D_3)$ approx. ∇, $\widetilde{\Delta}$ approx. Δ.

Example

Navier-Stokes equations for an incompressible flow of a constant viscosity fluid:

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ranking TOP-lex with $\sigma_t \succ \sigma_1 \succ \sigma_2 \succ \sigma_3$, $\widetilde{u} \succ \widetilde{u} \succ \widetilde{v} \succ \widetilde{w}$; passivity:

$$(**) \qquad (\mathbf{D}\cdot\mathbf{D})\,\widetilde{p} + \mathbf{D}\cdot(\widetilde{\mathbf{u}}\cdot\mathbf{D})\,\widetilde{\mathbf{u}} \;=\; 0$$

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Navier-Stokes equations for an incompressible flow of a constant viscosity fluid:

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where D_t approx. ∂_t, $\mathbf{D} = (D_1, D_2, D_3)$ approx. ∇, $\widetilde{\Delta}$ approx. Δ.

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$$(**) \qquad (\mathbf{D}\cdot\mathbf{D})\,\widetilde{p} + \mathbf{D}\cdot(\widetilde{\mathbf{u}}\cdot\mathbf{D})\,\widetilde{\mathbf{u}} \;=\; 0$$

We have $(**) \rhd (*)$ modulo PDE system; FDA is s-consistent.

References

S. Kovalevskaya, *Zur Theorie der partiellen Differentialgleichung*,
J. Reine & Angewandte Mathematik, 80:1–32, 1875.

C. Méray,
Démonstration générale de l'existence des intégrales des équations aux dérivées partielles, Journal de mathématiques pures et appliquées, 3e série, tome VI:235–265, 1880.

C. Riquier,
Les systèmes d'équations aux dérivées partielles,
Gauthiers-Villars, Paris, 1910.

M. Janet,
Sur les systèmes d'équations aux dérivées partielles,
Gauthiers-Villars, Paris, 1920.

M. Janet,
Leçons sur les systèmes d'équations aux dérivées partielles,
Cahiers Scientifiques IV. Gauthiers-Villars, Paris, 1929.

References

J. M. Thomas,
Differential Systems,
AMS Colloquium Publications, vol. XXI, 1937.

J. M. Thomas,
Riquier's existence theorems,
Ann. of Math. (2) 30(1–4):285–310, 1928/29.

J. F. Ritt.
Differential Algebra,
American Mathematical Society, New York, N. Y., 1950.

E. R. Kolchin,
Differential algebra and algebraic groups,
Academic Press, 1973.

A. Seidenberg,
An elimination theory for differential algebra,
Univ. California Publ. Math. (N.S.), 3:31–65, 1956.

References

F. Boulier, D. Lazard, F. Ollivier, M. Petitot,
Representation for the radical of a finitely generated differential ideal,
ISSAC 1995, pp. 158–166.

C. J. Rust, G. J. Reid, A. D. Wittkopf,
Existence and uniqueness theorems for formal power series solutions of analytic differential systems,
ISSAC 1999, pp. 105–112.

E. Hubert,
Notes on triangular sets and triangulation-decomposition algorithms.
in: LNCS, Vol. 2630, 2003, pp. 1–39 and 40–87.

F. Lemaire, M. Moreno Maza, Y. Xie,
The RegularChains library in Maple, SIGSAM Bulletin 39(3):96–97, 2005.

P. Aubry, D. Lazard, M. Moreno Maza,
On the theories of triangular sets,
J. Symbolic Comp. 28(1–2):105–124, 1999.

References

Wu Wen-Tsun,
Mathematics mechanization,
Mechanical geometry theorem-proving, mechanical geometry problem-solving and polynomial equtaions-solving,
Kluwer, 2000.

Dongming Wang,
Decomposing polynomial systems into simple systems,
J. Symbolic Comput., 25(3):295–314, 1998.

Ziming Li, Dongming Wang,
Coherent, regular and simple systems in zero decompositions of partial differential systems,
Syst. Sci. Math. Sci. 12(Suppl.):43–60, 1999.

Dongming Wang,
Elimination practice. Software tools and applications,
Imperial College Press, London, 2004.

References

J. F. Ritt, J. L. Doob,
Systems of Algebraic Difference Equations,
Amer. J. Math., 55(1–4):505-514, 1933.

J. F. Ritt, H. W. Raudenbush, Jr.,
Ideal theory and algebraic difference equations,
Trans. Amer. Math. Soc., 46:445–452, 1939.

J. F. Ritt,
Complete difference ideals,
Amer. J. Math., 63:681–690, 1941.

R. M. Cohn,
Difference algebra, John Wiley & Sons, 1955.

R. M. Cohn,
An existence theorem for difference polynomials,
Proc. Amer. Math. Soc. 17:254–261, 1966.

References

A. Levin,
Difference algebra,
vol. 8 of Algebra and Applications, Springer, 2008.

D. V. Trushin,
Difference Nullstellensatz,
preprint, cf. `arXiv:0908.3865v2`.

Xiao-Shan Gao, Yong Luo, Chunming Yuan,
A characteristic set method for ordinary difference polynomial systems,
J. Symbolic Comput. 44(3):242–260, 2009.

X. S. Gao, J. van der Hoeven, C. M. Yuan, G. L. Zhang,
Characteristic set method for differential-difference polynomial systems,
J. Symbolic Comput. 44(9):1137–1163, 2009.

Xiao-Shan Gao, Zhang Huang, Chun-Ming Yuan,
Binomial Difference Ideals,
J. Symbolic Comput., 80:665–706, 2017.

References

V. P. Gerdt, Y. A. Blinkov, V. V. Mozzhilkin,
Gröbner Bases and Generation of Difference Schemes for Partial Differential Equations,
Symmetry, Integrability and Geometry: Methods and Applications, 2006.

V. P. Gerdt, R. La Scala,
Noetherian quotients of the algebra of partial difference polynomials and Gröbner bases of symmetric ideals, J. Algebra, 423:1233–1261, 2015.

V. P. Gerdt,
Consistency Analysis of Finite Difference Approximations to PDE Systems,
in: G. Adam, J. Buša, M. Hnatič (eds.), *Proceedings of MMCP 2011* (July 3–8, 2011, Stará Lesná, High Tatra Mountains, Slovakia), LNCS 7125, Springer-Verlag, Berlin, 2012, pp. 28–42.

P. Amodio, Y. Blinkov, V. Gerdt, R. La Scala,
Algebraic construction and numerical behavior of a new s-consistent difference scheme for the 2D Navier-Stokes equations,
Applied Mathematics and Computation, 314:408–421, 2017.

References

D. Robertz,
Formal Algorithmic Elimination for PDEs,
Lecture Notes in Mathematics, Vol. 2121, Springer, 2014.

V. P. Gerdt and D. Robertz,
Consistency of Finite Difference Approximations for Linear PDE Systems and its Algorithmic Verification,
in: S. M. Watt (ed.), Proceedings of ISSAC 2010, TU München, Germany, pp. 53–59.

V. P. Gerdt and D. Robertz,
Computation of Difference Gröbner Bases,
Computer Science Journal of Moldova, 20 (2), 2012, pp. 203–226.

V. P. Gerdt and D. Robertz,
A Maple Package for Computing Gröbner Bases for Linear Recurrence Relations,
Nucl. Instrum. Methods Phys. Res. A, 559 (1), 2006, pp. 215–219.

References

A. A. Samarskii,
Theory of Difference Schemes,
Marcel Dekker, New York, 2001.

S. H. Christiansen, H. Z. Munthe-Kaas, B. Owren,
Topics in structure-preserving discretization,
Acta Numerica, 11:1–119, 2011.

J. C. Strikwerda,
Finite Difference Schemes and Partial Differential Equations,
SIAM, Philadelphia, second edition, 2004.

D. Rempfer,
On Boundary Conditions for Incompressible Navier-Stokes Problems,
Appl. Mech. Rev., 59:107–125, 2006.

References

M. Wibmer. *Algebraic Difference Equations*. Lecture Notes, 2013.

O. Golubitsky, M. Kondratieva, A. Ovchinnikov, A. Szanto,
A bound for orders in differential Nullstellensatz,
J. Algebra, 322(11):3852–3877, 2009.

B. Martin, V. Levandovskyy,
Symbolic Approach to Generation and Analysis of Finite Difference Schemes of Partial Differential Equations, in: U. Langer and P. Paule (eds.), *Numerical and Symbolic Scientific Computing: Progress and Prospects*, Springer, Vienna, 2012, pp. 123–156.

D. Yu. Grigoriev,
Complexity of quantifier elimination in the theory of ordinary differentially closed fields,
J. Soviet Math. 59(3):814–822, 1992.

References

V. P. Gerdt, M. Lange-Hegermann, D. Robertz,
The MAPLE package TDDS for computing Thomas decompositions of systems of nonlinear PDEs,
Computer Physics Communications 234:202–215, 2019.

T. Bächler, V. P. Gerdt, M. Lange-Hegermann, D. Robertz,
Algorithmic Thomas Decomposition of Algebraic and Differential Systems, J. Symbolic Comp. 47(10):1233–1266, 2012.

Yu. A. Blinkov, C. F. Cid, V. P. Gerdt, W. Plesken, D. Robertz,
The MAPLE Package Janet: II. Linear Partial Differential Equations,
in: V. G. Ganzha, E. W. Mayr, E. V. Vorozhtsov (eds.) CASC 2003. Proc. 6th Int. Workshop on Computer Algebra in Scientific Computing, pp. 41–54. TU München (2003). Package JANET is freely available at
`http:///algebra.data.rwth-aachen.de/software/Janet`