Strengthenings of isotriviality in differentially closed fields of characteristic zero.

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Setup

For this talk, I will fix a saturated differentially closed field of characteristic zero (K, δ). This means:

- (K, δ) is a differential field of characteristic zero
- differentially closed : any finite system of δ-polynomial equations that has a solution in some differential field of characteristic zero has a solution in K
- saturated : for any δ-subfield k, and any collection $\mathcal{F}$ of Kolchin constructible subsets over k, if any finite subcollection of $\mathcal{F}$ has a nonempty intersection, then $\mathcal{F}$ has a nonempty intersection.

We denote $\mathcal{C}$ its field of constants, it is algebraically closed.
Analogy with fields: taking an algebraically closed field of large transcendence degree.

Isotriviality

We are interested in the following property of δ-varieties:

Definition

An irreducible δ-variety X is isotrivial if there exists an irreducible variety V over $\mathcal{C}$ and a δ-birational map $f : V(\mathcal{C}) \to X$ (potentially over some extra parameters).
An irreducible δ-variety X is almost isotrivial if there exists an irreducible variety V over $\mathcal{C}$ and an irreducible δ-variety $\Gamma \subset X \times V(\mathcal{C})$ (potentially over extra parameters) projecting generically finite-to-one and δ-dominantly onto X and $V(\mathcal{C})$.

So almost isotrivial corresponds to isotrivial "up to some finite noise".

A Few Properties

- if X is (almost) isotrivial and $Y \subset X$ is irreducible, then Y is (almost) isotrivial.
- if X and Y are (almost) isotrivial, so is $X \times Y$.
- if X is (almost) isotrivial and $f : X \to Y$ is a dominant δ-rational map, then Y is (almost) isotrivial.
- if Y is (almost) isotrivial and $f : X \to Y$ is a finite-to-one δ-rational map, then X is almost isotrivial (but not isotrivial).

Remark

If X is almost isotrivial, there exists a dominant finite-to-one δ-rational map $\pi : X \to Y$, where Y is isotrivial.

Examples

Example (Linear Equation)

Let $L(Y) = Y^{(n)} + \sum_{i=0}^{n-1} a_i Y^{(i)}$ be a linear homogeneous equation, with $a_i \in K$. Then the solution set $X = \{x \in K, L(x) = 0\}$ is isotrivial.

Indeed, in that case, X is a finite dimensional $\mathcal{C}$-vector space. If we pick a basis $\{b_1, \cdots b_n\}$ for X, then we define:

$$\begin{aligned} f : & \mathcal{C}^n \to X \\ & (c_1, \cdots, c_n) \to b_1 c_1 + \cdots + b_n c_n \end{aligned}$$

Example (Ricatti Equation)

Let $\alpha_1, \alpha_2, \alpha_3$ be independent differentially transcendental. Then $X = \{x' = \alpha_1 x^2 + \alpha_2 x + \alpha_3\}$ is isotrivial.

Fundamental systems

If X is isotrivial, then there are $b_1, \cdots, b_n \in X$ such that the map $f : X \to V(\mathcal{C})$ is defined over $b_1, \cdots b_n$. If it is almost isotrivial, it is the variety $\Gamma \subset X \times V(\mathcal{C})$ that is defined over $b_1, \cdots, b_n$. This tuple $(b_1, \cdots b_n)$ is called a fundamental system of solutions.
In the case of linear differential equations, this fundamental system is simply a $\mathcal{C}$-basis of the $\mathcal{C}$-vector space X.

Binding Groups

Theorem

Let X be an isotrivial irreducible δ-variety, defined over A. Then the group $\mathrm{Aut}_A(X/\mathcal{C})$ of permutations of $X(K)$ induced by automorphisms of K fixing $A \cup \mathcal{C}$ pointwise is isomorphic to an isotrivial δ-algebraic-group, defined over A. It is called the binding group of X over $\mathcal{C}$.

Idea of proof.

Pick a fundamental system $\overline{b} = (b_1, \cdots, b_n)$ and $\sigma \in \mathrm{Aut}_A(X/\mathcal{C})$. Encode σ using the tuple $(\overline{b}, \sigma(\overline{b}))$. □

Examples

Example

For a linear equation, the binding group is simply the differential Galois group of the associated Picard-Vessiot extension.

Example

For the Ricatti equation, the binding group is $\mathrm{PSL}_2(\mathcal{C})$.

When binding groups become trivial

Let X be an isotrivial irreducible δ-variety, defined over A. Then for any $B \supset A$, the group $\mathrm{Aut}_B(X/\mathcal{C})$ is a subgroup of the binding group $\mathrm{Aut}_A(X/\mathcal{C})$.
If we pick $B = A \cup \{b_1, \cdots, b_n\}$, where $\{b_1, \cdots, b_n\}$ is a fundamental system of solutions, then $\mathrm{Aut}_B(X/\mathcal{C})$ is trivial.

Isotriviality in Families

Definition

A family of isotrivial δ-varieties is a dominant δ-rational map $\pi : Z \to X$ such that each π-fiber is isotrivial and X and Z are irreducible.

Example

Consider the logarithmic derivative: $\delta\log : K \setminus \{0\} \to K, x \to \frac{x'}{x}$, and let $Z = \delta\log^{-1}(\mathcal{C})$. Then $\delta\log : Z \to \mathcal{C}$ is a family of isotrivial δ-varieties.

In fact, if X is any irreducible δ-variety and $Z = \delta\log^{-1}(X)$, then $\delta\log : Z \to X$ is isotrivial. Also true for δ, and in general any map with isotrivial fibers.

Generalizing Binding Groups: Groupoids

If $Z \to X$ is a family of isotrivial δ-varieties, with X defined over A, then each π-fiber has a binding group. We can actually form a groupoid $\mathcal{G}(Z \to X)$ as:

- $\mathrm{Ob}(\mathcal{G})$ is the set of generic points of X
- for any $a, b \in X$, let $\mathrm{Mor}_{\mathcal{G}}(a, b)$ be the set of bijections from $\pi^{-1}(a)$ to $\pi^{-1}(b)$ induced by automorphisms of K fixing $A \cup \mathcal{C}$ pointwise.

Theorem (J.)

The groupoid $\mathcal{G}(Z \to X)$ is isomorphic to the generic set of a δ-variety. That is, it is a groupoid described by differential polynomials, but only generically.

This happens because we have to set $\mathrm{Ob}(\mathcal{G})$ to be the generics of X to get these differential polynomials.
Note that as opposed to the case of a single isotrivial variety, this groupoid is *not* isotrivial.

Examples

Example

In the case of $\delta \log : \delta \log^{-1}(\mathcal{C}) \to \mathcal{C}$, the groupoid is totally disconnected. So it is just a disjoint union of binding groups.

Example

Let X be given by $x' = x$, and let $Z = \delta \log^{-1}(X)$. Then $\delta \log : Z \to X$ is a family of isotrivial δ-varieties. The groupoid $\mathcal{G}$ is connected. Indeed, for any $a, b \in X$ generic, there is an automorphism of K fixing $\mathcal{C}$ and taking a to b. For all $a \in X$, we also have $\mathrm{Mor}(a, a) = G_m(\mathcal{C})$.

A question

When does a family of isotrivial varieties is itself isotrivial?

As isotriviality is preserved by dominant δ-rational maps, it is necessary that X itself is isotrivial.
What else?

One Answer: Preserving Isotriviality

Definition (Moosa)

The family $\pi : Z \to X$ preserves isotriviality if for any isotrivial subvariety $Y \subset X$ passing through a generic point, the variety $\pi^{-1}(Z)$ is isotrivial.

- This implies that this is a family of isotrivial δ-varieties, as we can pick Y to be a generic point of X.
- If X is isotrivial, then Z is isotrivial if and only if $\pi : Z \to X$ preserves isotriviality.

A weird example

Consider the irreducible δ-variety $X = \{x \in K, x' = x^3 - x^2\}$. This variety has the model-theoretic property of being orthogonal to $\mathcal{C}$. It particular, it implies that any irreducible isotrivial subvariety is a point.

Thus, if we consider *any* δ-rational dominant function $\pi : Z \to X$ with isotrivial fibers, it preserves isotriviality.

However, we would like our definition to depend on the behavior of the fibers $\pi : Z \to X$, not just on the base X.

Another Answer: Uniform Isotriviality

Definition

The family $\pi : Z \to X$ is said to be uniformly isotrivial if there is an irreducible algebraic variety V over $\mathcal{C}$, an irreducible δ-variety $\Lambda \subset X \times V(\mathcal{C})$ and a δ-birational map $f : Z \to \Lambda$ (over additional parameters B), making the following diagram commute:

$$\begin{tikzcd} X \times V(\mathcal{C}) \supset \Lambda \arrow[dr] & Z \arrow[l, "f"'] \arrow[d] \\ & X \end{tikzcd}$$

If X is isotrivial, then so is Λ, hence Z is isotrivial if and only if $Z \to X$ is uniformly isotrivial.
So in that case, we do not get anything new compared to preserving isotriviality.

Alternative Definitions

Recall that if Y is an isotrivial δ-variety, it is witnessed by a δ-birational map $f : Y \to V(\mathcal{C})$, defined over some parameters B. Then we have:

Proposition

The family $\pi : Z \to X$ is uniformly isotrivial if and only if each π-fiber is isotrivial, and there is a set of parameters B such that for each $a \in X$, the δ-birational map $f : Y \to V_a(\mathcal{C})$ witnessing isotriviality is defined over B.

Each fiber also has a binding group $\mathrm{Aut}_a(\pi^{-1}(a)/\mathcal{C})$. Thus, we can reformulate the previous proposition as :

Proposition

The family $\pi : Z \to X$ is uniformly isotrivial if and only if there is a set of parameters B such that for any $a \in X$, we have $\mathrm{Aut}_{B,a}(\pi^{-1}(a)/\mathcal{C}) = \{\mathrm{id}\}$.

Preserving Isotriviality $\neq$ Uniform Isotriviality

Let's go back to the weird example: consider $Y = \{x \in K, x' = x^3 - x^2\}$, $Z = \delta^{-1}(Y)$, and $\delta : Z \to Y$. As we've seen, this family preserves isotriviality.

Proposition

The family $\delta : Z \to Y$ is not uniformly isotrivial.

The proof again uses that X is orthogonal to the constants.

Idea of proof

Assume it is, consider $a \in X$ and $\alpha \in Z$ such that $\delta(\alpha) = a$. Using the alternative definition of isotriviality, we obtain a a differential field $F \supset \mathcal{C}$ and polynomials $P, Q \in F[X]$ such that $\alpha = \frac{P(a)}{Q(a)}$.
Consider the differential field $F(X)$, equipped with the derivation $\delta(X) = X^3 - X^2$, By our choice of X (i.e. X is orthogonal to the constants), we can assume that in $F(X)$ we have :

$$\begin{cases} (\frac{P}{Q})' = X \\ (\frac{P}{Q})'' = ((\frac{P}{Q})')^3 - ((\frac{P}{Q})')^2 \end{cases}$$

With a bit of computational effort, we can show this is impossible.

More counterexamples?

In [1], Chatzidakis, Harrison-Trainor and Moosa prove:

Theorem

If X is a finite dimensional irreducible δ-variety, then the differential tangent bundle $T_\delta(X) \to X$ preserves isotriviality.

Question: Is it uniformly isotrivial?

For a finite-dimensional δ-variety X, there are m, d and polynomials Q_i such that X is given by the equations $Q_i(\partial^k(x_j), 1 \leq j \leq m, 0 \leq k \leq d) = 0$. The defining equations of the differential tangent bundle then are:

$$\begin{cases} Q_i(\partial^k(x_j), 1 \leq j \leq m, 1 \leq k \leq d) \\ S_i = \sum_{k,j} \frac{\partial Q_i}{\partial z_{j,k}}(\partial^k(x_j), 1 \leq j \leq m, 1 \leq k \leq d)\partial^k(y_j), \text{ for } 1 \leq i \leq n \end{cases}$$

Two cases when it is uniformly isotrivial

Properties of the differential tangent bundle:

1. if $a \in X$, then $\delta(a) \in T_\delta(X)_a$
2. T_δ is a covariant functor on the category of δ-varieties.

From these we obtain:

1. if $T_\delta(X)$ is one dimensional, then $T_\delta(X) \to X$ is uniformly isotrivial.
2. if X is a δ-group, then $T_\delta(X) \to X$ is uniformly isotrivial.

No example were it is not uniformly isotrivial so far, but my guess is that it is the norm.

Some Questions

- is the fact that $\delta : \delta^{-1}(X) \to X$, for $X = \{x' = x^3 - x^3\}$, is not uniformly isotrivial a manifestation of a more general phenomenon? One could hope for a general statement involving δ-varieties orthogonal to the constants.
- if X is isotrivial, so is its differential tangent bundle. What if $\pi : Z \to X$ has isotrivial fibers, and X is isotrivial? Is the differential tangent bundle of Z uniformly isotrivial?
- can we give an example were the tangent bundle is not uniformly isotrivial?

All of these questions are likely to require some conceptual work. The direct computational techniques become very tedious as soon as second derivatives are involved.

Splitting

Definition

A δ variety X is said to be split over $\mathcal{C}$ if there are isotrivial δ-varieties X_1 and X_2 and a δ-rational finite-to-one dominant map $f : X \to X_1 \times X_2$.

If X is split over $\mathcal{C}$, then it is almost isotrivial.

Question: when does isotriviality imply splitting?

Jin and Moosa's answer for $\delta\log$ pullbacks, Setup

Consider a δ-variety X defined by $\{x' = f(x)\}$, where f is a polynomial defined over some parameters A, and assume it is isotrivial. In [2], Jin and Moosa ask the question of when is $Z = \delta\log^{-1}(X)$ itself isotrivial. This gives rise to a family of isotrivial varieties.

The behavior of this family depends on the binding group $\mathrm{Aut}_A(X/\mathcal{C})$. There are four possibilities for this group:

- $\mathrm{Aut}_A(X/\mathcal{C}) = G_a(\mathcal{C})$
- $\mathrm{Aut}_A(X/\mathcal{C}) = G_m(\mathcal{C})$
- $\mathrm{Aut}_A(X/\mathcal{C})$ is isomorphic to $G_a(\mathcal{C}) \rtimes G_m(\mathcal{C})$
- $\mathrm{Aut}_A(X/\mathcal{C})$ is isomorphic to $\mathrm{PSL}_2(\mathcal{C})$

Jin and Moosa's Answer

Theorem

In the first three cases, $Z = \delta \log^{-1}(X)$ is isotrivial if and only if it splits over $\mathcal{C}$.

Theorem

There are isotrivial δ-varieties X with $\mathrm{Aut}_A(X/\mathcal{C})$ isomorphic to $\mathrm{PSL}_2(\mathcal{C})$, such that $Z = \delta \log^{-1}(X)$ is isotrivial but does not split over $\mathcal{C}$.

Alternative Groupoid-inspired Method

Let $\pi : Z \to X$ be a family of isotrivial δ-varieties, defined over A. Suppose that Z is isotrivial. This produces two isotrivial δ-algebraic groups: first $\mathrm{Aut}_A(Z/\mathcal{C})$, and second $\mathrm{Aut}_A(X/\mathcal{C})$. For each fiber, we also obtain an isotrivial group $\mathrm{Aut}_{A,a}(\pi^{-1}(a)/\mathcal{C})$. There is a short exact sequence:

$$1 \to H \to \mathrm{Aut}_A(Z/\mathcal{C}) \to \mathrm{Aut}_A(X/\mathcal{C}) \to 1$$

The group H is also an isotrivial δ-algebraic group, isomorphic to a subgroup of $\prod_{a \in X} \mathrm{Aut}_{A,a}(\pi^{-1}(a)/\mathcal{C})$.

A Criteria for Splitting

Definition

We say that an isotrivial δ variety X is fundamental if the map $f : X \to V(\mathcal{C})$ witnessing isotriviality can be defined over some $a \in X$.

Proposition

Let $\pi : Z \to X$ be a family of δ-variety, with Z isotrivial. If the short exact sequence:

$$1 \to H \to \mathrm{Aut}_A(Z/\mathcal{C}) \to \mathrm{Aut}_A(X/\mathcal{C}) \to 1$$

splits, the δ-variety X is fundamental and $\mathrm{Aut}_A(X/\mathcal{C})$ acts transitively on generic points of X, then Z splits over $\mathcal{C}$.

The proof uses the groupoid defined earlier.

Good News and Bad News

Recall the setup $X = \{x, x' = f(x)\}$ with f a polynomial defined over A. If X is not in $\mathcal{C}$, the transitivity assumption is always satisfied.

If $\mathrm{Aut}_A(X/\mathcal{C}) = G_a(\mathcal{C})$ or $G_m(\mathcal{C})$, then X is fundamental, and the proposition applies.

In the other two cases, the δ-variety X is not fundamental. Rahim and I are still working out how to proceed in that case.

Also note that $1 \to G_m(\mathcal{C}) \to GL_2(\mathcal{C}) \to PSL_2(\mathcal{C}) \to 1$ is not split.

Splitting of Short Exact Sequences

Let's focus on the cases of G_a and G_m. We assume that $Z = \delta \log^{-1}(X)$ is isotrivial, we want to show that it splits. For any $a \in X$, we have $\mathrm{Aut}_{A,a}(\delta \log^{-1}(a)/\mathcal{C}) = G_m(\mathcal{C})$. Thus, the group H of the proposition is an algebraic torus.

Any algebraic group extension of G_a or G_m by some torus is a solvable affine algebraic group. Using that fact, we can show that any such extension is split.

This is enough to recover Jin and Moosa's theorem if $\mathrm{Aut}_A(X/\mathcal{C}) = G_a(\mathcal{C})$ or $G_m(\mathcal{C})$.

Potential Generalizations

In their paper, Jin and Moosa ask about potential generalizations, if one replaces the logarithmic derivative by, for example, the derivative. This new method has the advantage to potentially generalize to this case. This time, we would be concerned with extensions by algebraic subgroups of $G_a(\mathcal{C})^n$, for some n.

Thank you!

References

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