Local Definability of Holomorphic Functions

Jonathan Kirby

University of East Anglia

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Abstract

Local Definability of Holomorphic Functions

Given a collection $\mathcal{F}$ of complex or real analytic functions, one can ask what other functions are obtainable from them by finitary algebraic operations. If we just mean polynomial operations we get some field of functions.

If we include as algebraic operations such things as taking implicit functions, maybe in several variables, we get a much more interesting framework, which is closely related to the theory of local definability in an o-minimal setting, starting with suitable restrictions of the functions in $\mathcal{F}$.

O-minimality is a setting for tame topology of real- or complex-analytic functions which does not allow for "bad" singularities. However some more tame singularities can occur. In this talk I will explain work showing what singularities we have to consider to get a characterisation of the locally definable functions in terms of complex analytic operations.

Ax's theorem on the differential algebra version of Schanuel's conjecture is important to give one counterexample, and also for some applications to exponential and elliptic functions.

This is joint work with Gareth Jones, Olivier Le Gal, and Tamara Servi.

Outline

Outline

Motivation 1

Motivating question

Given some complex analytic functions, what other analytic functions can be obtained from them by "local algebraic" operations?
We need a robust notion of "algebraic".

Example

Starting with exp, define $f(z) = \exp(\exp(z))$.
f is transcendental over $\mathbb{C}(z, \exp(z))$, so f is not algebraic in the sense of a field of functions.
Composition of functions is a different sort of algebraic operation we want to include. Likewise inverse and implicit functions.
From exp we get the elementary functions.

Idea

Use some sort of model-theoretic definability as our notion of "algebraic".
Definable means there is a (finite) formula which defines the graph of the function.

Motivation 2

Complex sine is definable from exp: $\sin(z) = \frac{\exp(iz) - \exp(-iz)}{2i}$.
However the real case is different.

Theorem (Bianconi, 1997)

In the model-theoretic structure $\mathbb{R}_{\exp} = \langle \mathbb{R}; +, -, \cdot, 0, 1, \exp \rangle$*, no restriction of the real sine function to an interval is definable.*

Example (Some converse is false)

In $\langle \mathbb{R}; +, -, \cdot, 0, 1, \sin \rangle$, $2\pi\mathbb{Z}$ is setwise definable, as is $\mathbb{Z}$, and this allows us to code sums of infinite series definably. Much of analysis of real and complex functions becomes definable. In particular, real and complex exp are definable.

Theorem (Better converse - Bianconi 1997)

In $\langle \mathbb{R}; +, -, \cdot, 0, 1, \{\sin\restriction_{(a,b)}\}_{a<b} \rangle$*, no restriction of real* exp *to an interval is definable.*

Bianconi's proofs use Ax's differential algebra version of Schanuel's conjecture.

Outline

Definability

Conventions

In this talk, a model-theoretic structure is always of the form $\mathbb{R}_{\mathcal{F}} = \langle \mathbb{R}; +, -, \cdot, 0, 1, \mathcal{F} \rangle$ where $\mathcal{F}$ is a collection of analytic functions $f : U \to \mathbb{R}$ for open sets $U \subseteq \mathbb{R}^n$. (Both U and n can depend on f.)

Complex case

We deal with both the real and complex cases. For the complex case, we identify $\mathbb{C}$ with $\mathbb{R}^2$. Then take the real and imaginary parts of holomorphic functions.

Definition (Definable sets and functions)

A subset of $\mathbb{R}^n$ is then definable if it can be obtained from the graphs of the functions in $\mathcal{F}$, with $+, \cdot$, by the logical operations: intersection, union, complement, and projection.

They correspond to conjunction, disjunction, negation, and existential quantification over the field.

A function is definable if its graph is definable.

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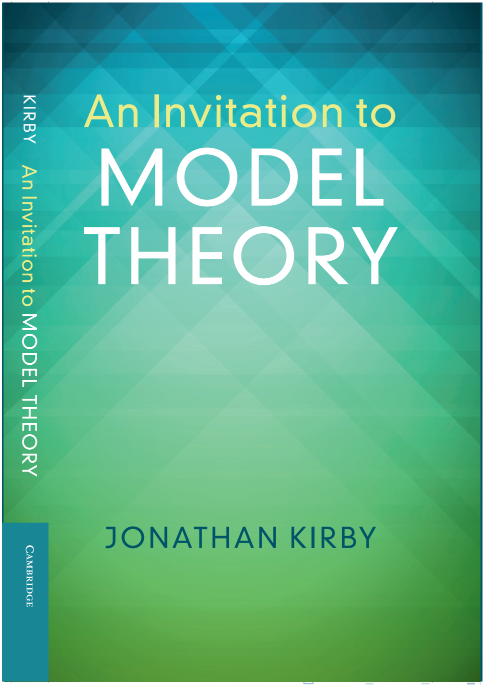

Tame topology: o-minimality

Towards an appropriate notion

If $\mathcal{F} \ni \sin$ then too many functions are definable for the notion to be interesting. However, if we restrict to the case where $\mathcal{F}$ contains only analytic functions on bounded domains, without approaching singularities, then we are in the o-minimal setting where definability is very much tamer.

Definition

Given $U \subseteq \mathbb{R}^n$, open, $f : U \to \mathbb{R}$ analytic, a proper restriction of f is $f\restriction_\Delta$, where Δ is an open box with rational corners such that $\bar{\Delta} \subseteq U$.
(No singularities on the boundary of Δ.)
$\mathrm{PR}(\mathcal{F})$ *is the set of all proper restrictions of all functions in $\mathcal{F}$.*

Definition

$\mathbb{R}_{\mathrm{an}}$ *is the case of $\mathbb{R}_{\mathrm{PR}(\mathcal{F})}$ where $\mathcal{F}$ is all analytic functions.*

Theorem (Essentially due to Gabrielov)

$\mathbb{R}_{\mathrm{an}}$ *is o-minimal.*

Local definability

Definition

Let $\mathcal{F}$ be a collection of analytic functions.
$\mathrm{PR}(\mathcal{F})$ *is the set of all proper restrictions of functions in* $\mathcal{F}$.
$\mathbb{R}_{\mathrm{PR}(\mathcal{F})}$ *is the structure consisting of the real field equipped with (the real and imaginary parts of) all the proper restrictions of functions in* $\mathcal{F}$.

Definition

A function $f : U \to \mathbb{R}$ (or $\mathbb{C}$) is ***locally definable*** *if and only if all (the real and imaginary parts of) its proper restrictions are definable.*
Equivalently, for every $a \in U$ there is $U_a \ni a$ such that $f\restriction_{U_a}$ is definable.

Definition

A function g is ***locally definable from $\mathcal{F}$*** *if it is locally definable in* $\mathbb{R}_{\mathrm{PR}(\mathcal{F})}$.

Problem – real and complex versions

For arbitrary $\mathcal{F}$, characterise the analytic g which are locally definable from $\mathcal{F}$ using **analytic / algebraic** operations, rather than logical ones.

Outline

Wilkie's conjecture

Notation

$\mathrm{LD}(\mathcal{F}) = \{$analytic functions which are locally definable from $\mathcal{F}\}$.

Observations

$\mathrm{LD}(\mathcal{F})$ is closed under the following operations:

- All polynomials in $\mathbb{Q}[\bar{X}]$ are in $\mathrm{LD}(\mathcal{F})$.
- partial differentiation (using the usual ϵ-δ definition)
- composition of functions
- implicit definition (using the implicit function theorem)
- (in complex case) $i \in \mathrm{LD}(\mathcal{F})$ and Schwarz reflection: $f^{SR}(z) = \overline{f(\bar{z})}$

Define $\tilde{\mathcal{F}}$ to be the closure of $\mathcal{F}$ under these operations.

Conjecture (Wilkie, 2005, published 2008)

These operations are enough to characterise $\mathrm{LD}(\mathcal{F})$. *That is,* $\mathrm{LD}(\mathcal{F}) = \tilde{\mathcal{F}}$.

Wilkie's theorem

Theorem (Wilkie, 2005, published 2008)

At generic points in $\mathbb{R}^n$ or $\mathbb{C}^n$ (most points), the conjecture is true.
That is, if $g : U \to \mathbb{R}$ (or $\mathbb{C}$) is definable in $\mathbb{R}_{\mathrm{PR}(\mathcal{F})}$ and $a \in U$ is generic then there is $U_a \ni a$ such that $g\restriction_{U_a} \in \tilde{\mathcal{F}}$.

Outline

Counterexample 1: Removable singularities

Let $\mathcal{F} = \{\exp\}$ and $g(z) = \begin{cases} \dfrac{e^z - 1}{z} & , \quad z \neq 0 \\ 1 & , \quad z = 0 \end{cases}$. So $g \in \mathrm{LD}(\mathcal{F})$.

Theorem

$g \notin \tilde{\mathcal{F}}$. However it can be obtained if we allow monomial division.

Lemma

Suppose $\mathcal{F}$ contains the polynomials and is closed under partial differentiation and (in the complex case) Schwarz reflection. Then $g \in \tilde{\mathcal{F}}$ iff g is the implicit function of some system of equations using functions from $\mathcal{F}$.

Proof idea for theorem

Replace $\mathcal{F}$ by $\mathcal{F}_1$, the set of all polynomials in variables $\bar{z}$ and their exponentials. Then $\mathcal{F}_1$ is closed under partial differentiation and Schwarz reflection. We use Ax's theorem to show that at 0, f cannot be the implicit function of any system of equations using functions from $\mathcal{F}_1$.

Counterexample 2: Deramification

Let $g(z)$ be holomorphic, $f(z) = g(z^2)$, and $\mathcal{F} = \{f(z)\}$.
g is the square deramification of f. We have $g \in \mathrm{LD}(f)$, defined by

$$g(z) = w \iff \exists x[x^2 = z \quad \& \quad f(x) = w]$$

which is well-defined for this f.

Theorem

If g is a sufficiently generic function, then no restriction of g to a neighbourhood of 0 is obtainable from f by the previous operators, including monomial division.

Proof ideas.

1. The operator of square deramification is not equal to any operator built from the previous operators.
2. If g is strongly transcendental then distinct operators applied to it cannot give rise to the same function.
3. Strongly transcendental holomorphic functions exist.

Counterexample 2: Deramification 2

Sketch proof

- Germs of holomorphic functions at 0 are determined by their Taylor expansions. The operators can be considered as operators on germs, hence on formal power series.
- Suppose $g = \mathcal{L}(f_1, \ldots, f_n)$ with the f_i in $\mathcal{F}$, and $\mathcal{L}$ is some operator built from polynomials, partial differentiation, composition, implicit definition, Schwarz reflection, and monomial division.
- Define $d_{\mathcal{L}} : \mathbb{N} \to \mathbb{N}$ by, for any power series $h_1, \ldots, h_n$, $d_{\mathcal{L}}(n)$ is the greatest number such that to compute the coefficients of degree $\leqslant n$ in the Taylor series of $\mathcal{L}(h_1, \ldots, h_n)$, it is enough to know the coefficients of degree $\leqslant d_{\mathcal{L}}(n)$ of $h_1, \ldots, h_n$.
- If N is the number of times partial differentiation and monomial division are used, we have $d_{\mathcal{L}}(n) \leqslant n + N$.
- However, if $\mathcal{L}$ is the square deramification operator $\mathcal{L}(f)(z) = f(z^{1/2})$, we have $d_{\mathcal{L}}(n) = 2n$.

Strongly transcendental functions (due to Le Gal)

Definition

Suppose $U \subseteq \mathbb{C}$, $f : U \to \mathbb{C}$, $a = (a_1, \ldots, a_n) \in U^n$, $k \in \mathbb{N}$.

The multi-jet $j^k_n f(a)$ is the $n(k+1)$-tuple of all derivatives $\frac{d^i f}{dz^i}(a_l)$ for $i = 0, \ldots, k$, $l = 1, \ldots, n$.

f is strongly transcendental if for every $n, k \in \mathbb{N}$ and every $a \in U^n$ we have

$$\mathrm{td}_{\mathbb{Q}}\, \mathbb{Q}(j^k_n f(a), \overline{j^k_n f(a)}) \geqslant 2nk$$

Note: we could take all $a_l \in \mathbb{Q}$, so this is the strongest transcendence condition we could ask for on the derivatives of f.

Strongly transcendental implies differentially transcendental, but is a much stronger condition, also with number-theoretic transcendence content.

Lemma

One can show that in a suitable topology on a space of holomorphic functions, the strongly transcendental functions are residual. In particular, by the Baire category theorem, they exist.

Counterexample 3: Blowing down

Definition

The blow-up of $0 \in \mathbb{C}^2$ is the map $\pi : V \to \mathbb{C}^2$, where
$V = \{(z, p) \in \mathbb{C}^2 \times \mathbb{P}_1(\mathbb{C}) \mid z \in p\}$ and $\pi(z, p) = z$.
If $U \subseteq \mathbb{C}^2$ and $f : U \to \mathbb{C}$, then $f \circ \pi : \pi^{-1}(U) \to \mathbb{C}$ is the blow-up of f and f is the blow-down of $f \circ \pi$.

Thanks to the compactness of the fibres of π, blow-downs preserve local definability.

Key idea

Blow-downs are not local operators, but all previous operators are (including monomial division and deramification).
That is, if $\mathcal{L}$ is one of the previous operators, and $g = \mathcal{L}(f)$, then to obtain the germ of g at 0 it is enough to have the germ of f at finitely many points $a_1, \ldots, a_k$.
However, if g is strongly transcendental and $f = g \circ \pi$ then to obtain the germ of g at 0, one needs the germ of f at all points of $\pi^{-1}(0)$.

Here the strongly transcendental function is a function of 2 variables, but the definition is easily modified.

Outline

Recall: for $\mathcal{F}$ be a collection of analytic functions

$\tilde{\mathcal{F}}$ is the closure of $\mathcal{F}$ under the following operations:

- All polynomials in $\mathbb{Q}[\bar{X}]$ are in $\mathrm{LD}(\mathcal{F})$.
- partial differentiation
- implicit definition
- (in complex case) $i \in \tilde{\mathcal{F}}$ and Schwarz reflection: $f^{SR}(z) = \overline{f(\overline{z})}$

New operations

Let $\tilde{\tilde{\mathcal{F}}}$ be the closure of $\mathcal{F}$ under the above operations and:

- Monomial division: $f(z) \mapsto f(z)/z_1$
- Deramification: $f \mapsto f(z^{1/r})$
- Blow downs

Apply each operation only when the result is a well-defined analytic function.

Conjecture (Revised Conjecture)

For any set $\mathcal{F}$ of analytic functions, $\tilde{\tilde{\mathcal{F}}} = \mathrm{LD}(\mathcal{F})$.

A revised conjecture

New operations

Let $\tilde{\tilde{\tilde{\mathcal{F}}}}$ be the closure of $\mathcal{F}$ under the above operations and:

- Monomial division: $f(z) \mapsto f(z)/z_1$
- Deramification: $f \mapsto f(z^{1/r})$
- Blow downs

Apply each operation only when the result is a well-defined analytic function.
These operations are those needed for resolution of singularities in algebraic / tame analytic settings.

Conjecture (Revised Conjecture)

For any set $\mathcal{F}$ of analytic functions, $\tilde{\tilde{\tilde{\mathcal{F}}}} = \mathrm{LD}(\mathcal{F})$.

Work in progress by Le Gal, Servi, Vieillard Baron to prove the conjecture in the real case.
The complex case is open.

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Thank you for your attention!

Outline

6 Application to exponential and elliptic functions

Elliptic curves and $\wp$-functions

Weierstrass $\wp$-functions

Take $\Lambda = \mathbb{Z} + \tau\mathbb{Z} \subseteq \mathbb{G}_a(\mathbb{C})$ with $\tau \in \mathbb{C} \setminus \mathbb{R}$.
The quotient $\mathbb{C}/\Lambda$ is a complex Lie group homeomorphic to a torus.
As a complex manifold it is isomorphic to the projective algebraic variety (elliptic curve)

$$E(\mathbb{C}) = \left\{ [X : Y : Z] \in \mathbb{P}_2(\mathbb{C}) \;\middle|\; Y^2 Z = 4X^3 - g_2 X Z^2 - g_3 Z^3 \right\}$$

for suitable $g_2, g_3 \in \mathbb{C}$.The quotient map $\exp_E : \mathbb{C} \to E(\mathbb{C})$ is given as $z \mapsto [\wp(z) : \wp'(z) : 1]$ where

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right)$$

is a meromorphic function.

diff eq

Interdefinability theorems

We work in the complex setting.

Theorem (Jones, K., Servi 2016)

- *No Weierstrass $\wp$ is in $\mathrm{LD}(\exp)$.*
- $\exp$ *is not in* $\mathrm{LD}($*all* $\wp$*-functions*$)$.
- *More generally: let $\mathcal{F} = \{\exp, \wp_1, \ldots, \wp_n\}$, with $\wp_i$ corresponding to a lattice Λ_i. Then some $\wp_{n+1}$ is in $\mathrm{LD}(\mathcal{F})$ if and only if the lattice Λ_{n+1} is isogenous to one of $\Lambda_1, \ldots, \Lambda_n$ or their complex conjugates.*
- *Also stronger results of similar type.*

The proof uses the Ax-Schanuel differential algebra theorem explaining linear dependences between the differential forms associated with these functions in terms of isogenies / algebraic subgroups of algebraic groups.

Current work with Jones and Schmidt extends these results to other elliptic functions: Weierstrass-ζ functions and φ-functions. The Weierstrass σ is more difficult but once the relevant Ax-Schanuel theorem is understood it should be doable.