Probabilistic Guarantees for Autonomous Systems

Sam Coogan
Assistant professor
Electrical and Computer Engineering
Civil and Environmental Engineering

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Georgia Tech

Formal methods and Autonomous Control of Transportation Systems (FACTS) Lab

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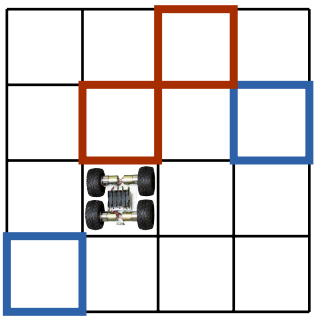
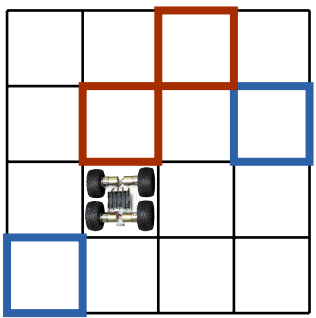
Modeling a stochastic system: Classic grid world

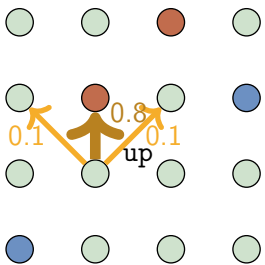


- Robot actions: $\{\texttt{up}, \texttt{down}, \texttt{left}, \texttt{right}\}$
- Actions have probabilistic consequences, *e.g.*:

Example control objective: Visit each blue square infinitely often, never visit a red square.

Modeling the system as a Markov Decision Process (MDP)

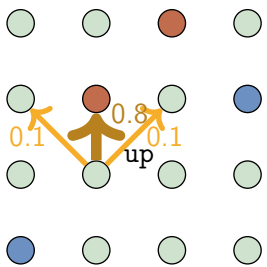




Modeling the system as a Markov Decision Process (MDP)

MDP $\mathcal{M} = (Q, Act, T, \Sigma, L)$:

- Q is set of states, *e.g.*,
  $Q = \{(1,1), (1,2), \ldots, (4,4)\}$
- Act is set of actions, *e.g.*,
  $Act = \{\texttt{up}, \texttt{down}, \texttt{left}, \texttt{right}\}$,
- $T : Q \times Act \times Q \to [0,1]$ is transition matrix, *e.g.*, $T\big((2,2), \text{up}, (3,3)\big) = 0.1$
- Σ is set of labels, *e.g.*,
  $\Sigma = \{\text{red}, \text{blue}, \text{none}\}$
- $L : Q \to \Sigma$ is labeling function, *e.g.*,
  $L\big((1,1)\big) = \text{blue}$



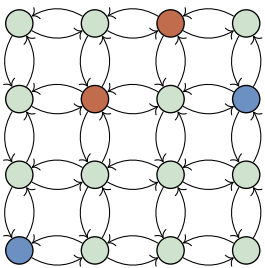
Synthesizing controllers for a canonical reach problem

Consider the goal:

Synthesize a control strategy that maximizes the probability of:
(or, given a control strategy, **verify** the probability of:)

- reaching a good (blue) state infinitely often, while
- encountering a bad (red) state finitely many times (eventually avoid)

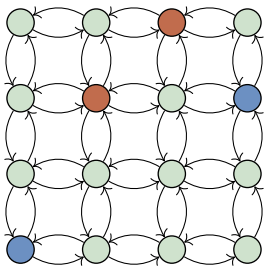
  - Fact 1: optimal controller u is memoryless, *i.e.*, $u : Q \to Act$
  - Fact 2: Maximizing probability solvable via graph-based search for "favorable" strongly connected components, then linear program (i.e., it's easy)



Complex specifications in LTL

Linear temporal logic (LTL) allows for complex specifications such as:

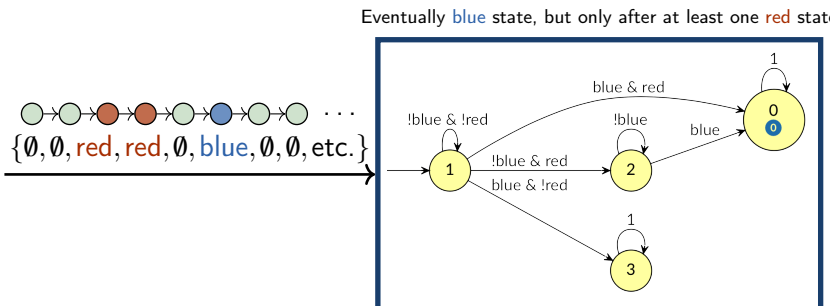
- Eventually visit a blue state, but only after visiting at least one red
- If a red state is visited, then visit a blue state within 3 time steps
- Never visit two blue states in a row



LTL synthesis/verification reduces to canonical reach problem on larger product MDP

LTL Example

- LTL specs are written in a formal syntax (e.g., $\Box$ for "always", $\Diamond$ for "eventually")
- Can be converted into "memory-like" automaton with reach-avoid requirement

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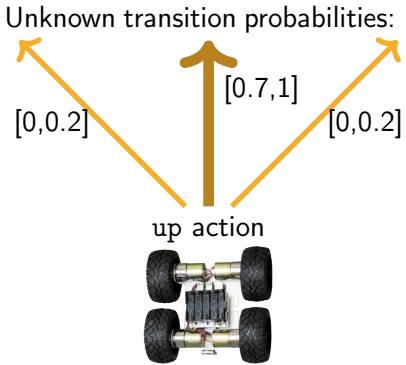
Nondeterministic uncertainty and Bounded-parameter MDPs



Bounded-parameter MDP (BMDP)

$\mathcal{B} = (Q, Act, \check{T}, \hat{T}, \Sigma, L)$ same as MDP but with:

- $\check{T} : Q \times Act \times Q \to [0,1]$, lower transition bounds
- $\hat{T} : Q \times Act \times Q \to [0,1]$, upper transition bounds
- Must have $\check{T}(q,a,q') \leq \hat{T}(q,a,q')$ for all q, a, q' and

$$\sum_{q' \in Q} \check{T}(q,a,q') \leq 1 \leq \sum_{q' \in Q} \hat{T}(q,a,q')$$

In this talk

1. Dynamic systems with uncountable state-space, e.g., $\mathbb{R}^n$?
   Partition the space, approximate as BMDP abstraction
2. Solving simple reach-avoid problems in BMDPs?
   Much more complicated because intervals that include 0 affect qualitative graph structure
3. Solving against complex specifications in, *e.g.*, a temporal logic?
   Reduce to simple reachability problem, then refine state-space partition

Stochastic dynamical system

$$x[k+1] = F(x[k], w[k])$$

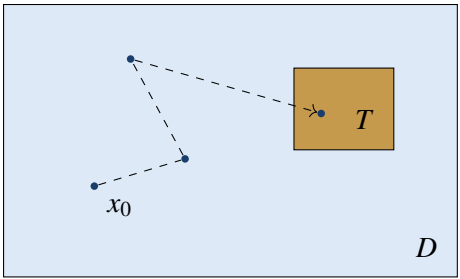
- $x[k] \in D \subset \mathbb{R}^n$ is state
- $w[k] \in W \subset \mathbb{R}^p$ is random disturbance with probability density function f_w

Note:

- Rest of talk: no control input (i.e., we are studying verification), but
- Statespace is $D \subset \mathbb{R}^n$ (Markov Chain with uncountably infinite states)

A reachability problem

A trajectory of $x[k+1] = F(x[k], w[k])$ initialized at x_0 is a random walk on D



Given set $T \subset D$, and target probability threshold p_{thresh}, color:

- Green those states x_0 such that Prob($x[k]$ reaches $T|x_0) \geq p_{\text{thres}}$
- Red all other states

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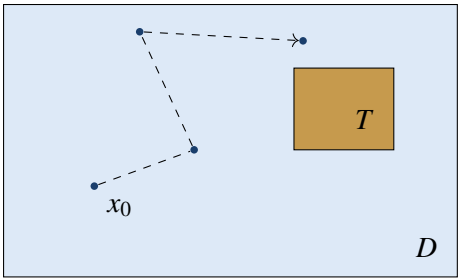

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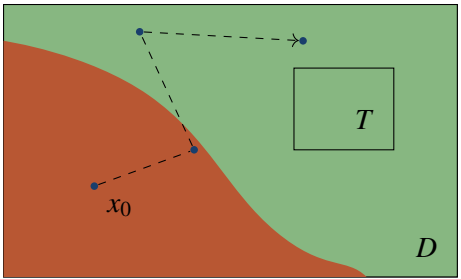

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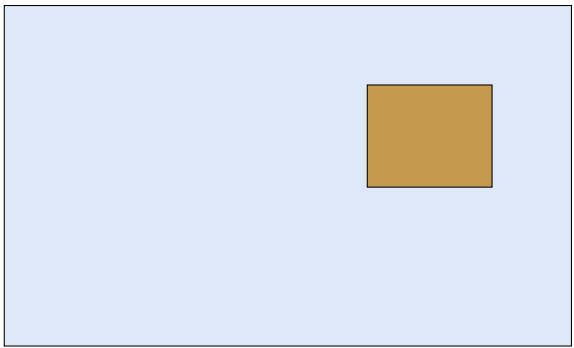
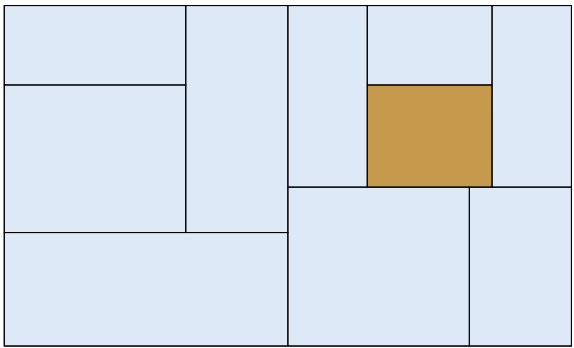
Approximating a solution with a finite abstraction



- Partition state space
- Color each partition:
  - Green,
  - Red, or
  - Yellow if inconclusive

Approximating a solution with a finite abstraction



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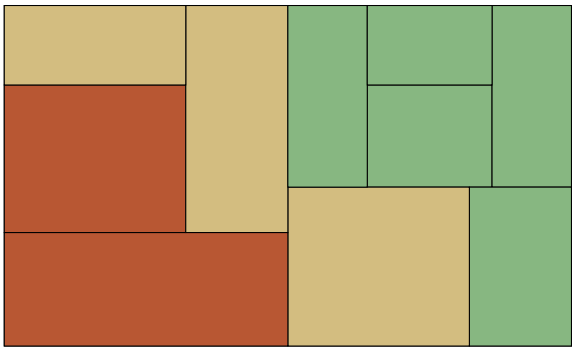

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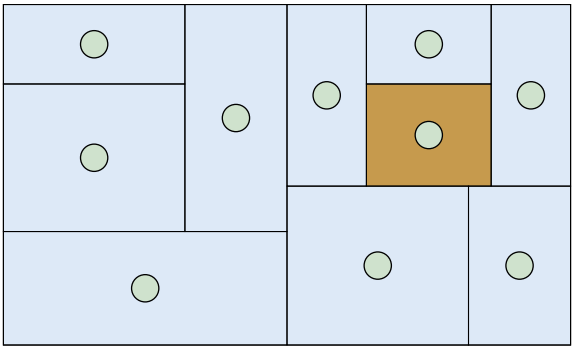
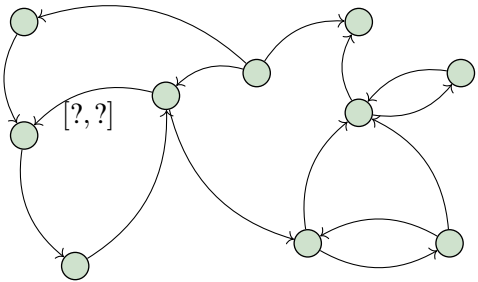
A BMDP abstraction



- Lower transition bound: $\min_{x_0 \in q} \text{Prob}(F(x_0, w) \in q' \mid x_0)$
- Upper transition bound: $\max_{x_0 \in q} \text{Prob}(F(x_0, w) \in q' \mid x_0)$

A BMDP abstraction



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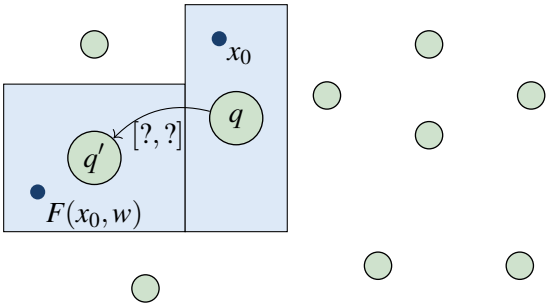

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A BMDP abstraction

A partition Q with transition functions $\check{T} : Q \times Q \to [0,1]$ and $\hat{T} : Q \times Q \to [0,1]$ is a **BMDP abstraction** of $x[k+1] = F(x[k]) + w[k]$ if, for all $q, q' \in Q$,

$$\check{T}(q,q') \leq \min_{x_0 \in q} \text{Prob}(F(x_0,w) \in q' \mid x_0), \text{ and}$$

$$\hat{T}(q,q') \geq \max_{x_0 \in q} \text{Prob}(F(x_0,w) \in q' \mid x_0)$$

- A BMDP abstraction overapproximates the behavior of the true system
- Leads to sound (but possibly conservative) algorithms

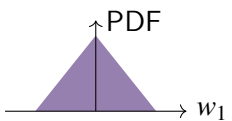
Mixed monotone stochastic systems

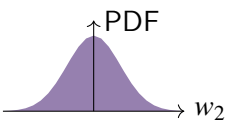
Specialize to

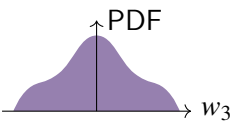
$$x[k+1] = F(x[k]) + w[k]$$

where:

1. $F : D \to D$ is *mixed monotone*
2. Each element of w is drawn from a symmetric, unimodal distribution







Mixed monotonicity [Coogan and Arcak, HSCC '15]

The system

$$x[k+1] = F(x[k])$$

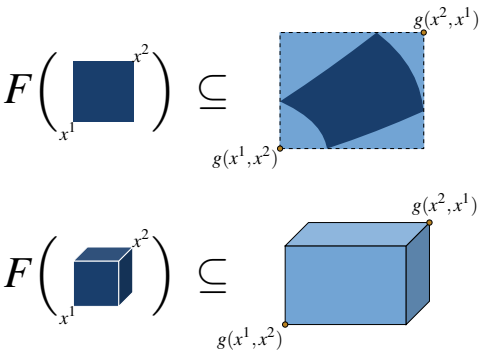
is **mixed monotone** if there exists $g : D \times D \to D$ such that:

1. for all x: $F(x) = g(x,x)$
2. for all $x^1, x^2, y : x^1 \leq x^2$ implies $g(x^1, y) \leq g(x^2, y)$
3. for all $x, y^1, y^2 : y^1 \leq y^2$ implies $g(x, y^2) \leq g(x, y^1)$

- Generalizes monotone systems, for which $g(x,y) = F(x)$

Reachable set approximation for mixed monotone systems: If $x^1 \leq x \leq x^2$, then

$$g(x^1, x^2) \leq F(x) \leq g(x^2, x^1)$$



Mixed monotonicity: A special case

$x^+ = F(x)$ is mixed monotone if

for all i,j, there exists $\delta_{ij} \in \{-1,1\}$ s.t. $\delta_{ij}\frac{\partial F_i}{\partial x_j}(x) \geq 0 \quad \forall x,$

i.e., Jacobian is **sign-stable**

Construct decomposition function:

$$g_i(x,y) = F_i(\ldots, \overset{y_j}{\not{x}_j}, \ldots) \qquad \text{if } \delta_{ij} = -1$$

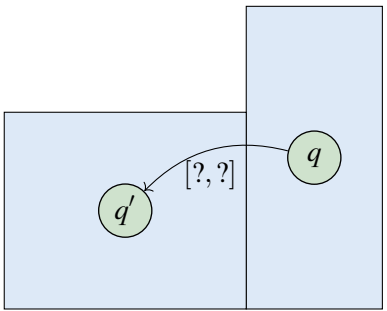
$$g(x,y) = (g_1(x,y), \ldots, g_n(x,y))$$

1. $g(x,x) = F(x)$
2. $\frac{\partial g}{\partial x}(x,y) \geq 0$
3. $\frac{\partial g}{\partial y}(x,y) \leq 0$

- Can generalize to systems with bounded Jacobian
- Mixed monotonicity is quite general, but its usefulness depends on conservatism of decomposition function

Computing transition probabilities

$$x[k+1] = F(x[k]) + w[k]$$



- Maximizing transition probability: Shift disturbance closest to center of q'
- Minimizing transition probability: Shift disturbance furthest from center of q'
- Integrate disturbance PDF over q'

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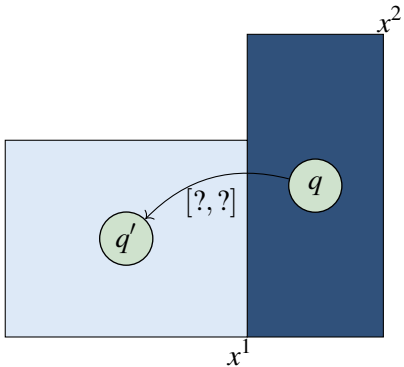

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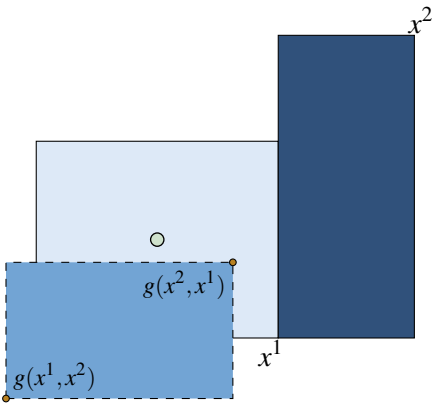

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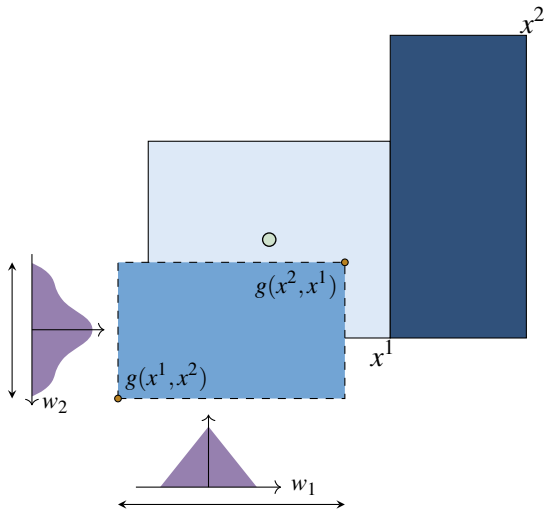

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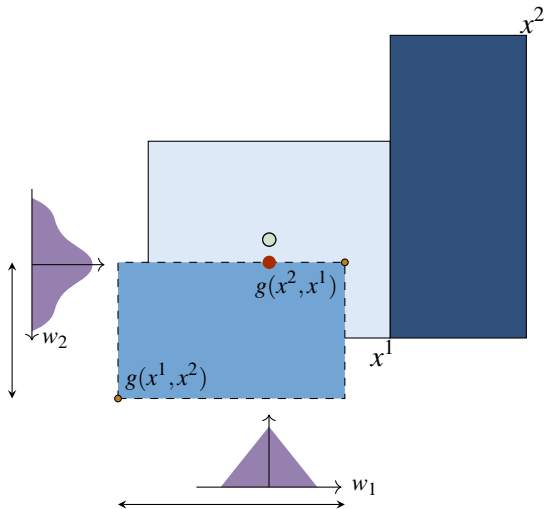

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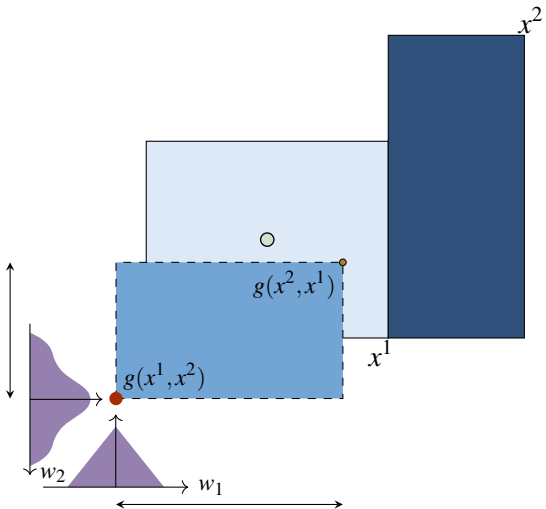

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BMDP abstraction of mixed monotone systems

Theorem: Given partition $\mathcal{Q}$. Let s^q be center of partition $q \in \mathcal{Q}$, $\check{r}^q = g(x^{1,q}, x^{2,q})$ be lower corner of reach set for $q \in \mathcal{Q}$, $\hat{r}^q = g(x^{2,q}, x^{1,q})$ be upper corner of reach set for $q \in \mathcal{Q}$. Define

$$\hat{T}(q,q') = \prod_{i=1}^{n} \int_{x_i^{1,q'}}^{x_i^{2,q'}} \mathsf{PDF}_{w_i}(x_i - s_{i,max}^{q,q'}) dx_i, \qquad \check{T}(q,q') = \prod_{i=1}^{n} \int_{x_i^{1,q'}}^{x_i^{2,q'}} \mathsf{PDF}_{w_i}(x_i - s_{i,min}^{q,q'}) dx_i$$

where

$$s_{i,max}^{q,q'} = \begin{cases} s_i^q, & \text{if } s_i^q \in [\check{r}_i^q, \hat{r}_i^q] \\ \hat{r}_i^q & \text{if } s_i^q > \hat{r}_i^q \\ \check{r}_i^q & \text{if } s_i^q < \check{r}_i^q \end{cases}, \quad s_{i,min}^{q,q'} = \begin{cases} \check{r}_i^q & \text{if } s_i^q < (\hat{r}_i^q + \check{r}_i^q)/2 \\ \hat{r}_i^q & \text{otherwise.} \end{cases}$$

Then $\mathcal{Q}, \hat{T}, \check{T}$ is a BMDP abstraction.

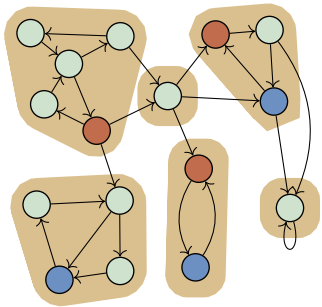
Reach-avoid verification in a Markov chain

Given Markov chain with states Q, bad states $B \subset Q$, good states $G \subset Q$, find the probability of:

- reaching a good (blue) state infinitely often, while
- encountering a bad (red) state finitely many times (eventually avoid)

Standard approach:

- Identify strongly connected components (SCC) via qualitative graph analysis
- Compute probability of reaching **accepting** bottom SCCs
- **Winning component**: states that reach accepting BSCC w.p. 1



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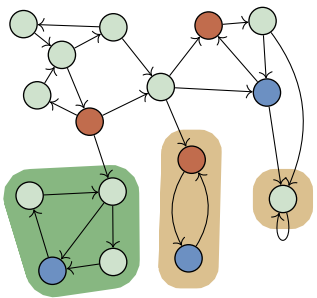
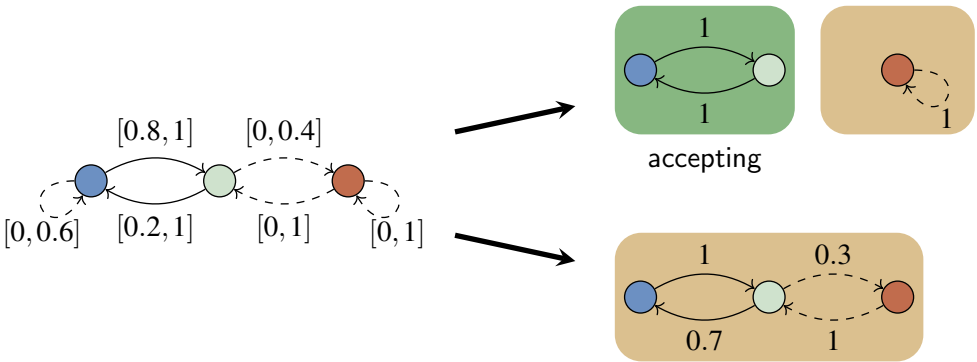


Reach-avoid verification in bounded-parameter Markov chains

- Edges with lower bound=zero affect qualitative graph structure

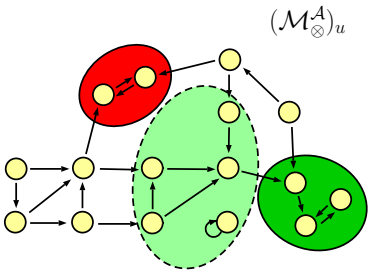


Solving BDMP verification with largest winning and losing components

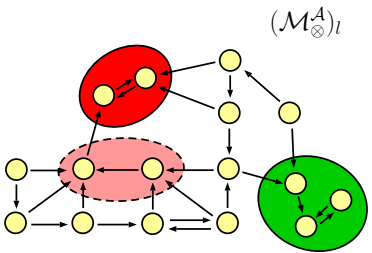
Theorem: Lower and upper bounds of reach-avoid are computed by:

- Upper bound = probability of reaching largest winning component
- Lower bound = 1−(probability of reaching largest losing component)

Best case

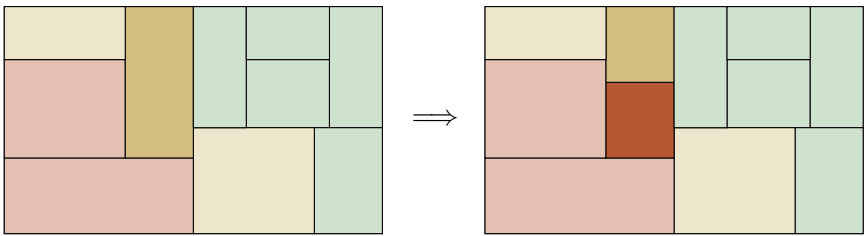


Worst case

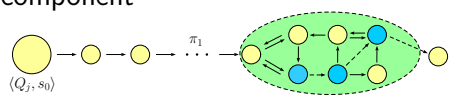


State-space refinement

Refining partition to reduce uncertain states



- **Observation:** it is important to refine intelligently
- **Heuristic algorithm** that assigns scores based on potential to make/break winning or losing component



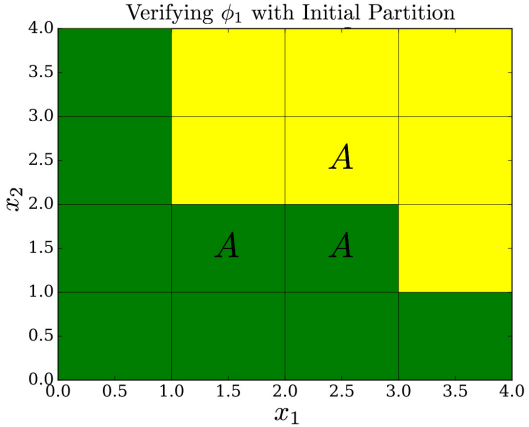
Example

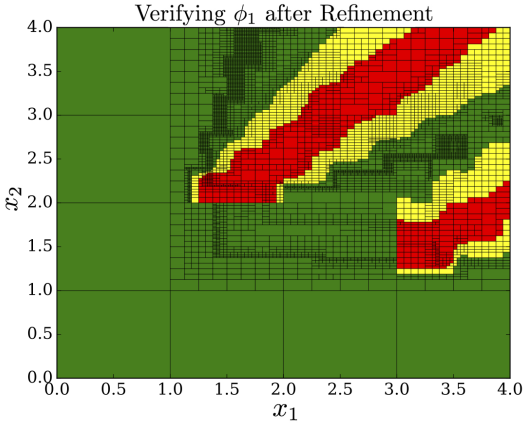
$$x_1[k+1] = x_1[k] + (-a x_1[k] + x_2[k]) \cdot \Delta T + w_1$$
$$x_2[k+1] = x_2[k] + \left(\frac{(x_1[k])^2}{(x_1[k])^2 + 1} - b x_2[k] \right) \cdot \Delta T + w_2,$$

- w_1, $w_2 \sim$ truncated Gaussians
- Two specifications:
  - At least 80% chance of: remaining in an A state for 2 time steps after entering an A state
    $\mathcal{P}_{\geq 0.80}[\Box((\neg A \wedge \bigcirc A) \rightarrow (\bigcirc\bigcirc \wedge \bigcirc\bigcirc\bigcirc A))]$
  - Less than 90% chance of: reaching a B state if it eventually always remains in A state, and always stays outside of B state if it reaches a C state
    $\mathcal{P}_{\leq 0.90}[(\Diamond\Box A \rightarrow \Diamond B) \wedge (\Diamond C \rightarrow \Box \neg B)]$

Example 1: $\mathcal{P}_{\geq 0.80}[\Box((\neg A \wedge \bigcirc A) \rightarrow (\bigcirc\bigcirc \wedge \bigcirc\bigcirc\bigcirc A))]$

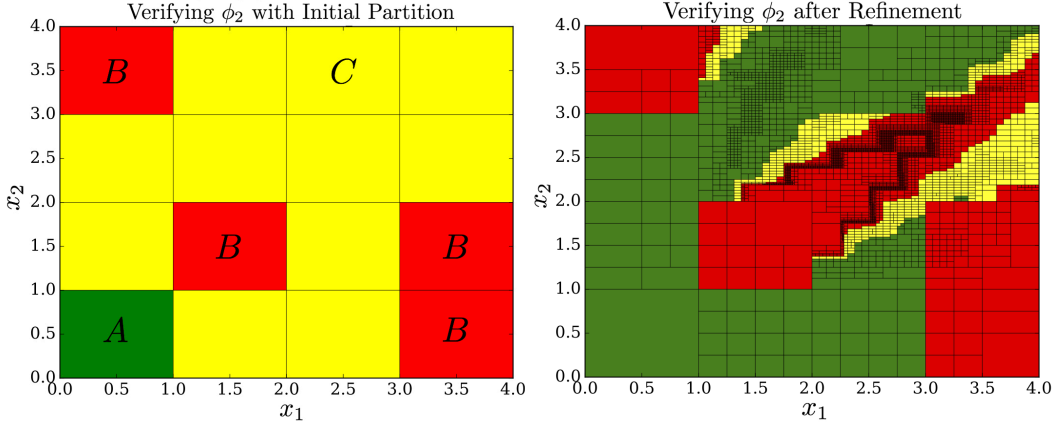
At least 80% chance of: remaining in an A state for 2 time steps after entering an A state





Example 2: $\mathcal{P}_{\leq 0.90}[(\Diamond\Box A \rightarrow \Diamond B) \wedge (\Diamond C \rightarrow \Box\neg B)]$

Less than 90% chance of: reaching a B state if it eventually always remains in A state, and always stays outside of B state if it reaches a C state



Conclusions

1. Bounded Markov Decision Processes (BMDPs) are a good model for formal methods applied to dynamical systems
2. Introduces several challenges:
   - How to compute BMDP abstractions?
   - How to reason about BMDPs?
   - How to intelligently refine BMDP abstractions?
3. Synthesis with continuous inputs (*i.e.*, $u \in \mathbb{R}^m$) introduces new and unique challenges (forthcoming paper)

- Max Dutreix, S. Coogan, "Specification-guided verification and abstraction refinement of mixed-monotone stochastic systems" arXiv, in submission.
- Max Dutreix, S. Coogan, "Efficient verification for stochastic mixed monotone systems," International Conference on Cyber-Physical Systems (ICCPS), 2018.
- S. Coogan, Murat Arcak, "Efficient finite abstraction of mixed monotone systems," International Conference on Hybrid Systems: Computation and Control (HSCC), 2015.