Apparent Singularites of D-finite Systems

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Singularities (univariate case)

Let $\partial = \frac{d}{dx}$.

Consider

$$L = p_r \partial^r + p_{r-1} \partial^{r-1} + \cdots + p_0 \in \mathbb{C}[x][\partial],$$

where $p_i \in \mathbb{C}[x]$ with $p_r \neq 0$ and $\gcd(p_r, p_{r-1}, \ldots, p_0) = 1$.

Call r the order of L, denoted by $\operatorname{ord}(L)$.

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Call r the order of L, denoted by $\operatorname{ord}(L)$.

Definition. $c \in \mathbb{C}$ is an ordinary point of L if $p_r(c) \neq 0$.
Otherwise, c is a singularity of L.

Formal power series (univariate case)

Definition. Let $f \in \mathbb{C}[[x]]$ be of the form

$$f = c_m x^m + c_{m+1} x^{m+1} + \cdots ,$$

where $c_m \neq 0$. Call m the initial exponent of f.

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Theorem (Fuchs, 1866). Let $L \in \mathbb{C}[x][\partial] \setminus \{0\}$. Then

the origin is an ordinary point of L

$$\Updownarrow$$

L has $\operatorname{ord}(L)$ sols in $\mathbb{C}[[x]]$ with initial exponents $0, 1, \ldots, \operatorname{ord}(L) - 1$.

Apparent singularities

Assume the origin is a singularity of L.

Definition. The origin is apparent if L has $\text{ord}(L)$ $\mathbb{C}$-linearly independent sols in $\mathbb{C}[[x]]$.

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Example. x^5 is a sol of $xf'(x) - 5f(x) = 0$.

Motivation

Assume the origin is an apparent singularity of L.

Goal. Find $M \in \mathbb{C}[x][\partial] \setminus \{0\}$ s.t.

- $\mathrm{sol}(L) \subset \mathrm{sol}(M)$;
- the origin is an ordinary point of M.

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Remark. If so, then $\mathrm{sol}(L)$ is spanned by formal power series.

Apparent singularites

L has sols of the form:

$$
\begin{aligned}
&x^{\alpha_1} \quad + \cdots, \\
&x^{\alpha_2} \quad + \cdots, \\
&\qquad \vdots \\
&x^{\alpha_r} \quad + \cdots.
\end{aligned}
$$

where $\alpha_1 < \alpha_2 < \cdots < \alpha_r \in \mathbb{N}$, $r = \text{ord}(L)$.

Remark. Some exponents are missing!

Apparent singularites

L has sols of the form:

$$
\begin{array}{lll}
x^{e_1} & + \cdots, & e_1 = 0, \ldots, \alpha_1 - 1, \\
x^{\alpha_1} & + \cdots, & \\
x^{e_2} & + \cdots, & e_2 = \alpha_1 + 1, \ldots, \alpha_2 - 1, \\
x^{\alpha_2} & + \cdots, & \\
 & \vdots & \\
x^{e_r} & + \cdots, & e_r = \alpha_{r-1} + 1, \ldots, \alpha_r - 1, \\
x^{\alpha_r} & + \cdots. &
\end{array}
$$

where $\alpha_1 < \alpha_2 < \cdots < \alpha_r \in \mathbb{N}$, $r = \mathrm{ord}(L)$.

Remark. Some exponents are missing!

Desingularization

Given $L \in \mathbb{C}[x][\partial]$, the origin being apparent, find $M \in \mathbb{C}[x][\partial]$ s.t.

- $M = PL$ for some $P \in \mathbb{C}(x)[\partial]$;
- the origin is an ordinary point of M.

Call M a desingluaried operator of L.

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A first idea (Fuchs). Assume missing exponents are $k_1, \ldots k_\ell$. Compute the least common left multiple of

$$L,\ x\partial - k_1,\ \ldots,\ x\partial - k_\ell$$

in $\mathbb{C}(x)[\partial]$.

Advanced method

Chen, Jaroschek, Kauers and Singer (2013, 2016), construct a desingularized operator M of L s.t.

- all apparent singularities of L are ordinary points of M;
- all singularities of M are non-apparent ones of L;
- the degree of leading coeff of M is minimal.

Contraction of Ore ideals (Z, 2016)

Theorem. A desingularized operator yields generators of

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Contraction of Ore ideals (Z, 2016)

Theorem. A desingularized operator yields generators of

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- Determine the contraction ideals of shift operators
- The ring of constants can replaced by a PID

D-finite systems

Notation.

$$A_n = \mathbb{C}[x_1, \ldots, x_n][\partial_1, \ldots, \partial_n] \quad \subset \quad R_n = \mathbb{C}(x_1, \ldots, x_n)[\partial_1, \ldots, \partial_n]$$

$\Uparrow$ Weyl algebra — $\Uparrow$ Rational algebra

where $\partial_i = \partial/\partial x_i$.

Definition. A left ideal $I \subset R_n$ is D-finite if R_n/I is a finite-dimensional vector space over $\mathbb{C}(x_1, \ldots, x_n)$.

Assume that $G_1, \ldots, G_m$ are generators of I. The system

$$G_i(f) = 0, \quad i = 1, \ldots, m.$$

is called a D-finite system.

D-finite Gröbner bases

Let $\prec_\partial$ be a graded term order on $\partial_1^{k_1} \cdots \partial_n^{k_n}$, a finite set $G \subset A_n$ is a Gröbner basis w.r.t. $\prec_\partial$.

Definition. G is D-finite if $R_n \cdot G$ is D-finite. The set

$$\mathrm{PE}(G) = \left\{ (i_1, \ldots, i_n) \mid \partial_1^{i_1} \cdots \partial_n^{i_n} \text{ is not reducible w.r.t. } G \right\}.$$

is called the set of parametric exponents of G.

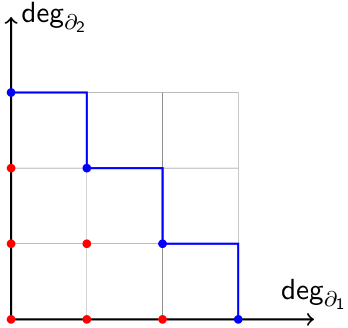
$|\mathrm{PE}(G)|$ is called the rank of G.

Example 1

Consider

$$M = \{\partial_1^3, \partial_1^2\partial_2, \partial_1\partial_2^2, \partial_2^3\}.$$

Then $\mathrm{PE}(M) = \{(0,0),(1,0),(0,1),(2,0),(1,1),(0,2)\}$.



PE(M)

Ordinary points and singularities

Assume that $G \subset A_n$ is a Gröbner basis and its elements are all primitive.

Definition. $c \in \mathbb{C}^n$ is an ordinary point of G if c is not a zero of

$$\prod_{g \in G} \mathrm{lc}(g).$$

Otherwise, c is a singularity of G.

Ordinary points and singularities

Example 1 (cont.) Consider

$$M = \{\partial_1^3, \partial_1^2\partial_2, \partial_1\partial_2^2, \partial_2^3\}.$$

where $\prod_{g \in M} \mathrm{lc}(g) = 1$. The origin is an ordinary point of M.

Example 2. Consider

$$G = \{x_2^2\partial_2 - x_1^2\partial_1 + x_1 - x_2, \partial_1^2\},$$

where $\prod_{g \in G} \mathrm{lc}(g) = x_2^2$. The origin is a singularity of G.

Formal power series

Let $\prec_x$ be the order induced by $\prec_\partial$ on $x_1^{k_1} \cdots x_n^{k_n}$.

Let $f \in \mathbb{C}[[x_1, \ldots, x_n]]$ be of form

$$f = c_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n} + \text{higher terms w.r.t. } \prec_x,$$

where $c_{i_1,\ldots i_n} \in \mathbb{C}$ is nonzero.

Definition. Call $(i_1, \ldots, i_n)$ the initial exponent of f.

Main result

Let G be a D-finite Gröbner basis and its elements are all primitive.

Theorem 1. The origin of $\mathbb{C}^n$ is an ordinary point of G

$$\Updownarrow$$

$\forall\ (i_1, \ldots, i_n) \in \mathrm{PE}(G)$, $\exists\ f \in \mathbb{C}[[x_1, \ldots, x_n]]$ with initial exponent $(i_1, \ldots, i_n)$ s.t. f is a solution of G.

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Remark. an algorithm for computing formal power series sols of D-finite systems at ordinary points.

Apparent singularities

Assume the origin is a singularity of G.

Definition. The origin is apparent if G has $|\mathrm{PE}(G)|$ $\mathbb{C}$-linearly independent sols in $\mathbb{C}[[x_1, \ldots, x_n]]$.

Example 2 (cont.) Consider

$$G = \{x_2^2\partial_2 - x_1^2\partial_1 + x_1 - x_2, \partial_1^2\},$$

$\{x_1 + x_2, x_1x_2\}$ are sols of G. The origin is apparent.

We can decide whether a given point is apparent or not and remove it using "a first idea".

Detecting and removing apparent singularities

Example 2 (cont.) Consider

$$G = \{x_2^2\partial_2 - x_1^2\partial_1 + x_1 - x_2, \partial_1^2\},$$

Set

$$S = \{(0,0),(0,1),(2,0),(0,2)\}.$$

Let $M \subset A_2$ be a Gröbner basis with

$$R_2 M = R_2 G \cap \left(\bigcap_{(s,t)\in S} R_2\{x_1\partial_1 - s, x_2\partial_2 - t\} \right)$$

We find

$$M = \{\partial_1^3, \partial_1^2\partial_2, \partial_1\partial_2^2, \partial_2^3\}.$$

The origin is an ordinary point of M.

Formal power series solutions at apparent singularities

Example 3 Consider the D-finite Gröbner basis of rank 2:

$$H = \{x_2\partial_2 + \partial_1 - x_2 - 1, \partial_1^2 - \partial_1\}.$$

- Let $M \subset A_2$ be a Gröbner basis of the left ideal

  $$R_2 H \cap R_2\{x_1\partial_1 - 1, \partial_2\}.$$

  Then the origin is an ordinary point of M, which is of rank 3.
- By Theorem 1, $\mathrm{sol}(M)$ at the origin is spanned by

  $$f_1 = \exp(x_1 + x_2) - x_1 - x_2 \exp(x_2), \quad f_2 = x_1,$$
  $$f_3 = x_2 \exp(x_2).$$

Formal power series solutions at apparent singularities

- Make an ansatz $f = \sum_{i=1}^{3} c_i f_i$, where c_i is unknown. Then one can show that f is a solution of

$$H_1(f) = 0, \quad H_2(f) = 0,$$

if and only if $(c_1, c_2, c_3)^t$ is a solution of $A\mathbf{x} = \mathbf{0}$, where

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

A basis of its right kernel is $\{(1,1,0)^t, (0,0,1)^t\}$. It give rise to a basis of $\mathrm{sol}(H)$ at the origin:

$$\{\exp(x_1 + x_2) - x_2 \exp(x_2),\ x_2 \exp(x_2)\}.$$

Conclusion

- Characterization of ordinary points of D-finite systems
- Detect and remove apparent singularities of D-finite systems
- An algorithm for computing formal power series sols of D-finite systems at apparent singularities.

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Thanks!