Integrability and limit cycles in polynomial systems of ODEs

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Outline

- Introduction to the center problem

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- Limit cycles: Cyclicity and 16th Hilbert problem

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- Limit cycles: Cyclicity and 16th Hilbert problem
- Algorithmic approach to the problems

A planar system with singular point at the origin:

$$\begin{aligned} \dot{x} &= ax + by + \sum_{p+q=2}^{\infty} \alpha_{pq} x^p y^q, \\ \dot{y} &= cx + dy + \sum_{p+q=2}^{\infty} \beta_{pq} x^p y^q. \end{aligned} \tag{1}$$

The linear approximation:

$$\begin{aligned} \dot{x} &= ax + by, \\ \dot{y} &= cx + dy \end{aligned} \tag{2}$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ \tau = a + d, \ \Delta = ad - bc$$

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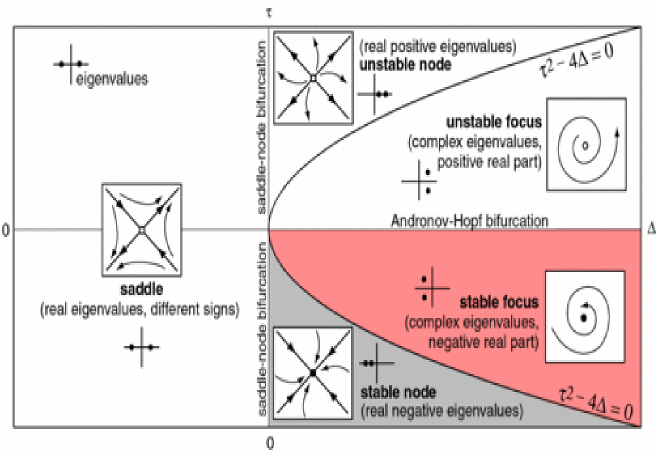
- Topological picture of trajectories near the origin of (1) and (2) is equivalent,
- except of the case $\tau = 0$, $\Delta > 0$ ($\Leftrightarrow$ the eigenvalues of A are $\pm i\omega$).



$\tau = 0$, $\Delta > 0$ ($\Leftrightarrow$ the eigenvalues of A are $\pm i\omega$) - a center for linear system;

in the case of nonlinear system: **either a center or a focus**.

- A center $\Longleftrightarrow$ all solutions near the origin are periodic.
- A focus $\Longleftrightarrow$ all solutions near the origin are spirals.

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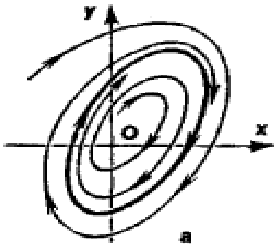
Poincaré center problem

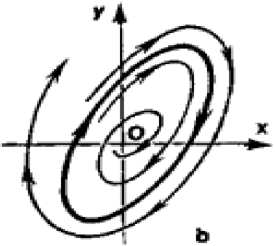
How to distinguish if the system

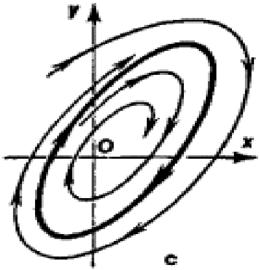
$$\dot{x} = \omega y + \sum_{p+q=2}^{\infty} \alpha_{pq} x^p y^q,$$
$$\dot{y} = -\omega x + \sum_{p+q=2}^{\infty} \beta_{pq} x^p y^q$$

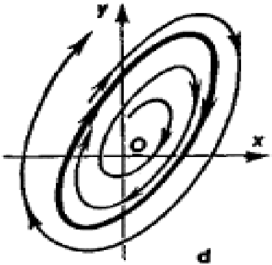
has a center or a focus at the origin?

Limit cycles









Hilbert's 16th problem

$$\dot{x} = P_n(x, y), \quad \dot{y} = Q_n(x, y), \qquad (A)$$

$P_n(x, y)$, $Q_n(x, y)$, are polynomials of degree n.
Let $h(P_n, Q_n)$ be the number of limit cycles of system (A) and let $H(n) = \sup h(P_n, Q_n)$.
The question of the second part of the 16th Hilbert's problem:

- find a bound for $H(n)$ as a function of n.

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The problem is still unresolved even for $n = 2$.

$n = 2$

- I. Petrovskii, E. Landis, On the number of limit cycles of the equation dy/dx = P(x,y)/Q(x,y), where P and Q are polynomials of 2nd degree (Russian), Mat. Sb. N.S. 37(79) (1955), 209-250
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Song Ling Shi, A concrete example of the existence of four limit cycles for plane quadratic systems, Sci. Sinica 23 (1980), 153-158

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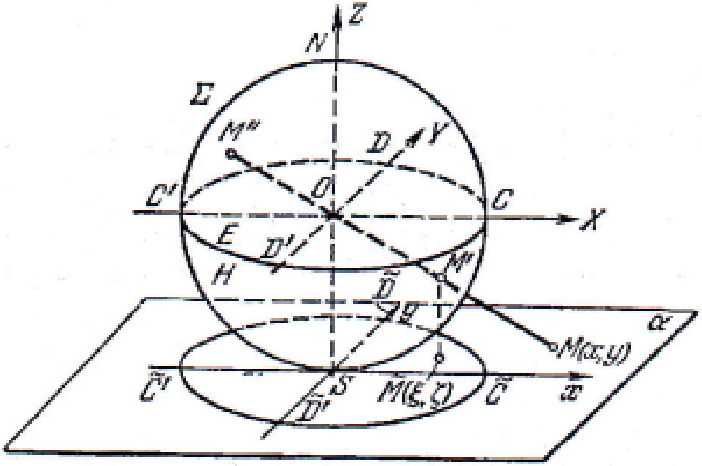
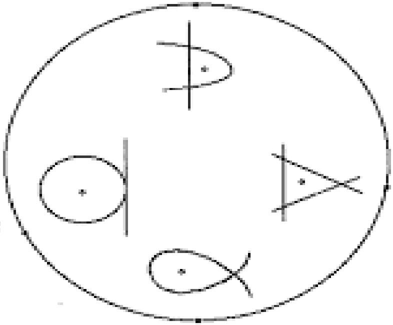
Around 1980 Yu. Ilyashenko found a mistake in Dulac's proof.

Poincaré compactification



Separatrix cycles:



Dulac's mistake

A germ of a map $f : (R^+, 0) \to (R^+, 0)$ is a semi-regular, if it is smooth in a neighborhood of 0 and admits an asymptotic expansion of the form

$$\hat{f}(x) = cx^{\nu_0} + \sum_j P_j(lnx)x^{\nu j},$$

where $c > 0$, $0 < \nu_j \to \infty$, $j > 0$, and P_j are real polynomials.
$\hat{f}$ is an asymptotic expansion of f, if $\forall \nu > 0 \; \exists$ a partial sum of $\hat{f}$, which approximates f f with accuracy better than x^ν, when $x \to 0$.

Dulac's theorem

For any polycycle of an analytic vector field, a cross-section with the vertex zero on the polycycle may be so chosen that the corresponding Poincaré map will be flat, inverse to flat, or semiregular.

Dulac's lemma

Let a semiregular map have an infinite number of fixed points. Then $f(x) \equiv x$.

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Counterexample

$$f(x) = x + (sin\frac{1}{x})e^{-\frac{1}{x}}$$

- Chicone and Shafer (1983) proved that for $n = 2$ a fixed system (A) has only finite number of limit cycles in any bounded region of the phase plane.

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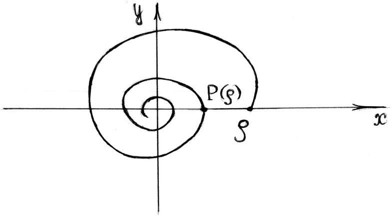
- Bamòn (1986) and V. R (1986) proved that $h(P_2, Q_2, a^*, b^*)$ is finite.
- Il'yashenko (1991) and Ecalle (1992): $h(P_n, Q_n, a^*, b^*)$ is finite for any n.

Poincaré return map:

$$\dot{u} = -v + \sum_{i+j=2}^{n} \alpha_{ij} u^i v^j, \qquad \dot{v} = u + \sum_{i+j=2}^{n} \beta_{ij} u^i v^j . \qquad (3)$$



$$\mathcal{P}(\rho) = \rho + \eta_3(\alpha_{ij}, \beta_{ij})\rho^3 + \eta_4(\alpha_{ij}, \beta_{ij})\rho^4 + \dots .$$

Center: $\eta_3 = \eta_4 = \eta_5 = \cdots = 0.$

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Poincaré center problem

Find all systems in the family

$$\dot{u} = -v + \sum_{i+j=2}^{n} \alpha_{ij} u^i v^j, \qquad \dot{v} = u + \sum_{i+j=2}^{n} \beta_{ij} u^i v^j,$$

which have a center at the origin.

Bautin ideal: $\mathcal{B} = \langle \eta_3, \eta_4, \dots \rangle \subset \mathbb{R}[\alpha_{ij}, \beta_{ij}]$.

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Bautin ideal: $\mathcal{B} = \langle \eta_3, \eta_4, \dots \rangle \subset \mathbb{R}[\alpha_{ij}, \beta_{ij}]$.

Algebraic counterpart

Find the variety $\mathbf{V}(\mathcal{B})$ of the Bautin ideal $\mathcal{B}$ $\mathcal{B} = \langle \eta_3, \eta_4, \eta_5 \dots \rangle$.

$$\mathbf{V}(\mathcal{B}) = \{(\alpha_{ij}, \beta_{ij}) \in \mathcal{E} \mid \eta_3(\alpha_{ij}, \beta_{ij}) = \eta_4(\alpha_{ij}, \beta_{ij}) = \cdots = 0\}$$

- $\mathbf{V}(\mathcal{B})$ is called the center variety.

Cyclicity and Bautin's theorem

$$\dot{u} = -v + \sum_{j+l=2}^{n} \alpha_{jl} u^j v^l, \quad \dot{v} = u + \sum_{j+l=2}^{n} \beta_{jl} u^j v^l \tag{4}$$

Poincare map:

$$\mathcal{P}(\rho) = \rho + \eta_2(\alpha_{ij}, \beta_{ij})\rho^2 + \eta_3(\alpha_{ij}, \beta_{ij})\rho^3 + \cdots + \eta_k(\alpha_{ij}, \beta_{ij})\rho^k + \ldots.$$

Let $\mathcal{B} = \langle \eta_3, \eta_4, \ldots \rangle \subset \mathbb{R}[\alpha_{ij}, \beta_{ij}]$ be the ideal generated by all focus quantities η_i. There is k such that

$$\mathcal{B} = \langle \eta_{u_1}, \eta_{u_2}, \ldots, \eta_{u_k} \rangle.$$

The Bautin ideal and Bautin's theorem

Then for any s

$$\eta_s = \eta_{u_1}\theta_1^{(s)} + \eta_{u_2}\theta_2^{(s)} + \cdots + \eta_{u_k}\theta_k^{(k)},$$

$$\mathcal{P}(\rho) - \rho = \eta_{u_1}(1 + \mu_1\rho + \dots)\rho^{u_1} + \cdots + \eta_{u_k}(1 + \mu_k\rho + \dots)\rho^{u_k}.$$

Bautin's Theorem

If $\mathcal{B} = \langle \eta_{u_1}, \eta_{u_2}, \dots, \eta_{u_k} \rangle$ then the cyclicity of system (4) (i.e. the maximal number of limit cycles which appear from the origin after small perturbations) is less or equal to k.

Proof. Bautin N.N. Mat. Sb. (1952) v.30, 181-196 (Russian); Trans. Amer. Math. Soc. (1954) v.100
Roussarie R. Bifurcations of planar vector fields and Hilbert's 16th problem (1998), Birkhauser.

The cyclicity problem

Find an upper bound for the maximal number of limit cycles in a neighborhood of a center or a focus

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Find an upper bound for the maximal number of limit cycles in a neighborhood of a center or a focus

By Bautin's theorem:

Algebraic counterpart

Find a basis for the Bautin ideal $\langle \eta_3, \eta_4, \eta_5, \ldots \rangle$ generated by all coefficients of the Poincaré map

A basis of an ideal and its zero set

Radical of an ideal I is the set of all polynomials f such that some positive integer ℓ $f^\ell \in I$.

Strong Hilbert Nullstellensatz

Let $f \in \mathbb{C}[x_1, \ldots, x_m]$ and let I be an ideal of $\mathbb{C}[x_1, \ldots, x_m]$. Then f vanishes on the variety of I if and only f belongs to the radical of I.

Corollary

If polynomials $f_1, \ldots, f_s$ from an ideal I define the variety of I, $\mathbf{V}(I) = \mathbf{V}(f_1, \ldots, f_s)$, and the ideal I is a radical ideal (that is, $I = \sqrt{I}$), then $I = \langle f_1, \ldots, f_s \rangle$.

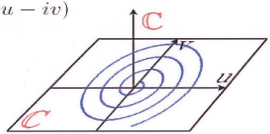
Holds only over $\mathbb{C}$!

Complexification

Complexification: $x = u + iv \quad (\bar{x} = u - iv)$

$$\dot{x} = \; i(x - \sum_{p+q=1}^{n-1} a_{pq} x^{p+1} \bar{x}^q)$$

$$\dot{\bar{x}} = -\, i(\bar{x} - \sum_{p+q=1}^{n-1} \bar{a}_{pq} \bar{x}^{p+1} x^q)$$



$$\dot{x} = i(x - \sum_{p+q=1}^{n-1} a_{pq} x^{p+1} y^q), \quad \dot{y} = -i(y - \sum_{p+q=1}^{n-1} b_{qp} x^q y^{p+1}) \quad (5)$$

The change of time $d\tau = idt$ transforms (5) to the system

$$\dot{x} = (x - \sum_{p+q=1}^{n-1} a_{pq} x^{p+1} y^q), \; \dot{y} = -(y - \sum_{p+q=1}^{n-1} b_{qp} x^q y^{p+1}). \quad (6)$$

Poincaré-Lyapunov Theorem

The system

$$\frac{du}{dt} = -v + \sum_{i+j=2}^{n} \alpha_{ij} u^i v^j, \quad \frac{dv}{dt} = u + \sum_{i+j=2}^{n} \beta_{ij} u^i v^j \qquad (7)$$

has a center at the origin if and only if it admits a first integral of the form

$$\Phi = u^2 + v^2 + \sum_{k+l \geq 2} \phi_{kl} u^k v^l.$$

Definition of a center for complex systems

System

$$\dot{x} = (x - \sum_{p+q=1}^{n-1} a_{pq} x^{p+1} y^q) = P, \; \dot{y} = -(y - \sum_{p+q=1}^{n-1} b_{qp} x^q y^{p+1}) = Q, \tag{8}$$

has a center at the origin if it admits a first integral of the form

$$\Phi(x, y; a_{10}, b_{10}, \ldots) = xy + \sum_{s=3}^{\infty} \sum_{j=0}^{s} v_{j,s-j} x^j y^{s-j}$$

For the complex system

$$\dot{x} = (x - \sum_{p+q=1}^{n-1} a_{pq} x^{p+1} y^q) = P, \ \dot{y} = -(y - \sum_{p+q=1}^{n-1} b_{qp} x^q y^{p+1}) = Q,$$

one looks for a function

$$\Phi(x, y; a_{10}, b_{10}, \ldots) = xy + \sum_{s=3}^{\infty} \sum_{j=0}^{s} v_{j,s-j} x^j y^{s-j}$$

such that

$$\frac{\partial \Phi}{\partial x} P + \frac{\partial \Phi}{\partial y} Q = g_{11} (xy)^2 + g_{22} (xy)^3 + \cdots , \qquad (9)$$

and $g_{11}, g_{22}, \ldots$ are polynomials in a_{pq}, b_{qp}. These polynomials are called *focus quantities*.

The Bautin ideal

The ideal $\mathcal{B} = \langle g_{11}, g_{22}, \ldots \rangle$ generated by the focus quantities is called the *Bautin ideal*.

Center Problem

Find the variety $\mathbf{V}(\mathcal{B})$ of the Bautin ideal $\mathcal{B} = \langle g_{11}, g_{22}, g_{33} \dots \rangle$.

Definition

The variety of the Bautin ideal $\mathbf{V}(\mathcal{B})$ is called the center variety of the system.

By the Hilbert Basis Theorem there is an integer m that $\mathcal{B} = \langle g_{11}, \dots, g_{mm} \rangle$, however it is a difficult problem to find such m. A practical approach is as follows.

- Compute polynomials g_{ss} until the chain of varieties (considering as complex varieties) $V(\mathcal{B}_1) \supseteq V(\mathcal{B}_2) \supseteq V(\mathcal{B}_3) \supseteq \dots$ stabilizes (here $\mathcal{B}_k = \langle g_{11}, \dots, g_{kk} \rangle$), that is, until we find k_0 such that $V(\mathcal{B}_{k_0}) = V(\mathcal{B}_{k_0+1})$.

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  To check that two varieties are equal we use

Radical Membership Test

$I = \langle f_1, \dots, f_s \rangle \in k[x_1, \dots, x_n]$, $f \in k[x_1, \dots, x_n]$.
$f \equiv 0$ on $V(I) \Longleftrightarrow$ Groebner basis of the ideal $I = \langle f_1, \dots, f_s, 1 - f \rangle$ is $\{1\}$.

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- **Show that $V(\mathcal{B}_{k_0}) = V(\mathcal{B})$, that is, that each systems from $V(\mathcal{B}_{k_0})$ admits a first integral of the form (9).**

The center problem is solved for:

• quadratic system: $\dot{x} = x + P_2(x, y), \quad \dot{y} = -y + Q_2(x, y)$ by Dulac (1908) (by Kapteyn (1912) for real systems)

• the linear center perturbed by 3rd degree homogeneous polynomials:
$\dot{x} = x + P_3(x, y), \quad \dot{y} = -y + Q_3(x, y)$
by Sadovski (1974) (by Malkin (1964) for real systems)

• for some particular subfamilies of the cubic system
$\dot{x} = x + P_2(x, y) + P_3(x, y), \quad \dot{y} = -y + Q_2(x, y) + Q_3(x, y)$

• for Lotka-Volterra quartic systems with homogeneous nonlinearities
$\dot{x} = x + xP_3(x, y), \quad \dot{y} = -y + yQ_3(x, y)$
by B. Ferčec, J. Giné, Y. Liu and V. R. (2013)

• for Lotka-Volterra quintic systems with homogeneous nonlinearities
$\dot{x} = x + xP_4(x, y), \quad \dot{y} = -y + yQ_4(x, y)$
by J. Giné and V. R. (2010)

The center variety of the quadratic system

$$\dot{x} = x - a_{10}x^2 - a_{01}xy - a_{-12}y^2, \ \dot{y} = -(y - b_{10}xy - b_{01}y^2 - b_{2,-1}x^2). \quad (10)$$

Theorem (H. Dulac 1908, C. Christopher & C. Rouseeau, 2001)

The variety of the Bautin ideal of system (10) coincides with the variety of the ideal $\mathcal{B}_3 = \langle g_{11}, g_{22}, g_{33} \rangle$ and consists of four irreducible components:

1) $\mathbf{V}(J_1)$, where $J_1 = \langle 2a_{10} - b_{10}, 2b_{01} - a_{01} \rangle$,
2) $\mathbf{V}(J_2)$, where $J_2 = \langle a_{01}, b_{10} \rangle$,
3) $\mathbf{V}(J_3)$, where $J_3 = \langle 2a_{01} + b_{01}, a_{10} + 2b_{10}, \ a_{01}b_{10} - a_{-12}b_{2,-1} \rangle$,
4) $\mathbf{V}(J_4) = \langle f_1, f_2, f_3, f_4, f_5 \rangle$, where
$f_1 = a_{01}^3 b_{2,-1} - a_{-12}b_{10}^3, \ f_2 = a_{10}a_{01} - b_{01}b_{10}$,
$f_3 = a_{10}^3 a_{-12} - b_{2,-1}b_{01}^3$,
$f_4 = a_{10}a_{-12}b_{10}^2 - a_{01}^2 b_{2,-1}b_{01}, \ f_5 = a_{10}^2 a_{-12}b_{10} - a_{01}b_{2,-1}b_{01}^2$.

Proof. Computing the first three focus quantities we have

$g_{11} = a_{10}a_{01} - b_{10}b_{01},$
$g_{22} = a_{10}a_{-12}b_{10}^2 - a_{01}^2 b_{01}b_{2,-1} - \frac{2}{3}(a_{-12}b_{10}^3 - a_{01}^3 b_{2,-1}) -$
$\frac{2}{3}(a_{01}b_{01}^2 b_{2,-1} - a_{10}^2 a_{-12}b_{10}),$
$g_{33} = -\frac{5}{8}(-a_{01}\, a_{-12}b_{10}^4 + 2\, a_{-12}b_{01}b_{10}^4 + a_{01}^4 b_{10}\, b_{2,-1} - 2\, a_{01}^3\, b_{01}\, b_{10}\, b_{2,-1} -$
$2\, a_{10}\, a_{-12}^2\, b_{10}^2\, b_{2,-1} + a_{-12}^2\, b_{10}^3\, b_{2,-1} - a_{01}^3\, a_{-12}\, b_{2,-1}^2 + 2\, a_{01}^2\, a_{-12}\, b_{01}\, b_{2,-1}^2).$

Using the radical membership test we see that

$$g_{22} \notin \sqrt{\langle g_{11} \rangle}, \quad g_{33} \notin \sqrt{\langle g_{11}, g_{22} \rangle}, \; g_{44}, g_{55}, g_{66} \in \sqrt{\langle g_{11}, g_{22}, g_{33} \rangle},$$

i.e., $\mathbf{V}(\mathcal{B}_1) \supset \mathbf{V}(\mathcal{B}_3) \supset \mathbf{V}(\mathcal{B}_3) = \mathbf{V}(\mathcal{B}_4) = \mathbf{V}(\mathcal{B}_5)$. We expect that

$$\mathbf{V}(\mathcal{B}_3) = \mathbf{V}(\mathcal{B}). \tag{11}$$

The inclusion $\mathbf{V}(\mathcal{B}) \subseteq \mathbf{V}(\mathcal{B}_3)$ is obvious, therefore in order to check that (11) indeed holds we only have to prove that

$$\mathbf{V}(\mathcal{B}_3) \subseteq \mathbf{V}(\mathcal{B}). \tag{12}$$

To do so, we first look for a decomposition of the variety $\mathbf{V}(\mathcal{B}_3)$.

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     by: G.-M. Greuel, G. Pfister, H. Schoenemann     \ Aug
FB Mathematik der Universitaet, D-67653 Kaiserslautern \
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```
LIB "primdec.lib";
ring r= 0,(a10,a01,a12,b21,b10,b01),lp;
poly g11=a01*a10 - b01*b10;
poly g22=...
poly g33=...
ideal i=g11,g22,g33;
minAssGTZ(i);
```

```
[1]:
   _[1]=a01^3*b21-a12*b10^3
   _[2]=a10*a12*b10^2-a01^2*b21*b01
   _[3]=a10*a01-b10*b01
   _[4]=a10^2*a12*b10-a01*b21*b01^2
   _[5]=a10^3*a12-b21*b01^3
[2]:
   _[1]=b10
   _[2]=a01
[3]:
   _[1]=a01-2*b01
   _[2]=2*a10-b10
[4]:
   _[1]=2*a12*b21+b10*b01
   _[2]=2*a01+b01
   _[3]=a10+2*b10
```

To verify that (12) holds there remains to show that every system (10) with coefficients from one of the sets $\mathbf{V}(J_1), \mathbf{V}(J_2), \mathbf{V}(J_3), \mathbf{V}(J_4)$ has a center at the origin, that is, there is a first integral $\Psi(x, y) = xy + h.o.t.$

Systems corresponding to the points of $\mathbf{V}(J_1)$ are Hamiltonian with the Hamiltonian

$$H = -\left(xy - \frac{a_{-12}}{3}y^3 - \frac{b_{2,-1}}{3}x^3 - a_{10}x^2y - b_{01}xy^2\right)$$

and, therefore, have centers at the origin (since $D(H) \equiv 0$).
To show that for the systems corresponding to the components $\mathbf{V}(J_2)$ and $\mathbf{V}(J_3)$ the origin is a center we use the Darboux method.

$$\dot{x} = P(x,y), \quad \dot{y} = Q(x,y), \quad x, y \in \mathbb{C} \quad P, Q \text{ are polynomials.} \tag{13}$$

The polynomial $f(x,y) \in \mathbb{C}[x,y]$ defines an *algebraic invariant curve* $f(x,y) = 0$ of system (13) if there exists a polynomial $k(x,y) \in \mathbb{C}[x,y]$ such that

$$D(f) := \frac{\partial f}{\partial x} P + \frac{\partial f}{\partial y} Q = kf. \tag{14}$$

The polynomial $k(x,y)$ is called a *cofactor* of f.

Suppose that the curves defined by

$$f_1 = 0, \dots, f_s = 0$$

are invariant algebraic curves of system (13) with the cofactors $k_1, \dots, k_s$. If

$$\sum_{j=1}^{s} \alpha_j k_j = 0\,, \tag{15}$$

then $H = f_1^{\alpha_1} \cdots f_s^{\alpha_s}$ is a (Darboux) first integral of the system (13).

Systems from $\mathbf{V}(J_2)$ and $\mathbf{V}(J_3)$ admit Darboux integrals.

Consider the variety $\mathbf{V}(J_3)$. In this case the system is

$$\begin{aligned} \dot{x} &= x - a_{10}x^2 + \frac{b_{01}}{2}xy - \frac{a_{10}b_{01}}{4b_{2,-1}}y^2, \\ -\dot{y} &= (y - b_{01}y^2 + \frac{a_{10}}{2}xy - b_{2,-1}x^2). \end{aligned} \tag{16}$$

- $f = \sum_{i+j=0}^{n} c_{ij} x^i y^j, \quad k = \sum_{i+j=0}^{m-1} d_{ij} x^i y^j$. ($m$ is the degree of the system; in our case $m = 1$). To find a bound for n is the Poincaré problem (unresolved).
- Equal the coefficients of the same terms in $D(f) = kf$.
- Solve the obtained system of polynomial equations for unknown variables c_{ij}, d_{ij}.

$$\ell_1 = 1 + 2\, b_{10}\, x - a_{01}\, b_{2,-1}\, x^2 + \ 2\, a_{01}\, y + 2\, a_{01}\, b_{10}\, x\, y - \frac{a_{01}\, b_{10}^2}{b_{2,-1}}\, y^2,$$

$$\ell_2 = (2\, b_{10}\, b_{2,-1}^2 + 6\, b_{10}^2\, b_{2,-1}^2\, x + 3\, {b_{10}}^3\, b_{2,-1}^2\, x^2 - 3\, a_{01}\, b_{10}\, b_{2,-1}^3\, x^2 - a_{01}\, b_{10}^2\, b_{2,-}^3$$
$$6\, a_{01}\, b_{10}\, b_{2,-1}^2\, y - 3\, b_{10}^4\, b_{2,-1}\, x\, y + 6\, a_{01}\, b_{10}^2\, b_{2,-1}^2\, x\, y - 3\, a_{01}^2\, b_{2,-1}^3\, x\, y + 3\, a_{01}\, b_{10}^3\, b_{2,-}^2$$
$$3\, a_{01}^2\, b_{10}\, b_{2,-1}^3 x^2\, y - 3\, a_{01}\, b_{10}^3\, b_{2,-1}\, y^2 + 3\, a_{01}^2\, b_{10}\, b_{2,-1}^2\, y^2 - 3\, a_{01}\, b_{10}^4\, b_{2,-1}\, x\, y^2 + 3\, a_0^2$$
$$a_{01}\, b_{10}^5\, y^3 - a_{01}^2\, b_{10}^3\, b_{2,-1}\, y^3)/(2\, b_{10}\, b_{2,-1}^2)$$

with the cofactors $k_1 = 2\,(b_{10}\, x - a_{01}\, y)$ and $k_2 = 3\,(b_{10}\, x - a_{01}\, y)$.
The equation

$$\alpha_1 k_1 + \alpha_2 k_2 = 0$$

has a solution $\alpha_1 = -3, \alpha_2 = 2, \Longrightarrow$

$$\Psi = \ell_1^{-3} \ell_2^2 \equiv c.$$

Thus, every system from $\mathbf{V}(J_3)$ has a center at the origin.

Systems from $\mathbf{V}(J_4)$ are time-reversible

$$\frac{d\mathbf{z}}{dt} = F(\mathbf{z}) \quad (\mathbf{z} \in \Omega), \tag{17}$$

Ω is a manifold.

Definition

A (time-)reversible symmetry of (17) is an involution $R : \Omega \mapsto \Omega$, such that

$$R_* \mathcal{X}_F = -\mathcal{X}_F \circ R. \tag{18}$$

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$$\dot{u} = v + vf(u, v^2), \qquad \dot{v} = -u + g(u, v^2), \tag{19}$$

$$u \to u, \qquad v \to -v, \qquad t \to -t$$

leaves the system unchanged $\Rightarrow$ the u–axis is a line of symmetry for the orbits $\Rightarrow$ no trajectory in a neighbourhood of $(0, 0)$ can be a spiral $\Rightarrow$ the origin is a center.

Here $R : u \mapsto u, \; v \mapsto -v$.

$$\begin{aligned} \dot{x} = & \quad x - \sum a_{pq} x^{p+1} y^q = P(x,y), \\ \dot{y} = & \quad -y + \sum b_{qp} x^q y^{p+1} = Q(x,y), \end{aligned} \tag{20}$$

The condition of time-reversibility

$$\gamma Q(\gamma y, x/\gamma) = -P(x,y), \quad \gamma Q(x,y) = -P(\gamma y,^x/\gamma)\,.$$

$\Longrightarrow$ (20) is time–reversible if and only if

$$b_{qp} = \gamma^{p-q} a_{pq}, \qquad a_{pq} = b_{qp} \gamma^{q-p}. \tag{21}$$

$$\Updownarrow$$

$$a_{p_k q_k} = t_k, \quad b_{q_k p_k} = \gamma^{p_k - q_k} t_k \tag{22}$$

for $k = 1, \ldots, \ell$. (22) define a surface in the affine space $\mathbb{C}^{3\ell+1} = (a_{p_1 q_1}, \ldots, a_{p_\ell q_\ell}, b_{q_\ell p_\ell}, \ldots, b_{q_1 p_1}, t_1, \ldots, t_\ell, \gamma)$.

- The set of all time-reversible systems is the projection of this surface onto $\mathbb{C}^{2\ell}$).
- To find this set we have to eliminate t_k and γ from (22).

For an ideal $I \subset k[x_1, \ldots, x_n]$ the *ℓ–elimination ideal* of I is the ideal $I_\ell = I \cap k[x_{\ell+1}, \ldots, x_n]$.

Elimination Theorem

Fix the lexicographic term order on the ring $k[x_1, \ldots, x_n]$ with $x_1 > x_2 > \cdots > x_n$ and let G be a Gröbner basis for an ideal$I$ of $k[x_1, \ldots, x_n]$ with respect to this order. Then for every ℓ, $0 \leq \ell \leq n-1$, the set $G_\ell := G \cap k[x_{\ell+1}, \ldots, x_n]$ is a Gröbner basis for the ℓ–th elimination ideal I_ℓ.

$\mathbf{V}(I_\ell)$ is the smallest affine variety containing $\pi_\ell(V) \subset \mathbb{C}^{n-\ell}$ ($\mathbf{V}(I_\ell)$ is the Zariski closure of $\pi_\ell(V)$).

$$H = \langle a_{p_k q_k} - t_k, b_{q_k p_k} - \gamma^{p_k - q_k} t_k \mid k = 1, \dots, \ell \rangle, \qquad (23)$$

Let $\mathcal{R}$ be the set of all time-reversible systems in the family (20).

Theorem

(V. R., Open Syst. Inf. Dyn., 2008) 1)
$\overline{\mathcal{R}} = \mathbf{V}(\mathcal{I}_R)$ where $\mathcal{I}_R = \mathbb{C}[a, b] \cap H$, that is, the Zariski closure of the set $\mathcal{R}$ of all time-reversible systems is the variety of the ideal $\mathcal{I}_R$.
2) Every system (20) from $\overline{\mathcal{R}}$ admits an analytic first integral of the form $\Psi = xy + \dots$.

For the quadratic system the elimination gives exactly the ideal J_4
$\Longrightarrow$
each system from $\mathbf{V}(J_4)$ also has a center at the origin.

Mechanisms for integrability

- Darboux integrability
- Hamiltonian systems
- Time-reversibility

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- Darboux integrability
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- Hiden symmetries
- Open problem:
  What is the complete list of mechanisms for integrability?

The cyclicity of the quadratic system

Generalized Bautin's theorem (V. R. & D. Shafer, 2009)

If the ideal $\mathcal{B}$ of all focus quantities of system

$$\dot{x} = (x - \sum_{p+q=1}^{n-1} a_{pq} x^{p+1} y^q), \quad \dot{y} = -(y - \sum_{p+q=1}^{n-1} b_{qp} x^q y^{p+1})$$

is generated by the m first focus quantities, $\mathcal{B} = \langle g_{11}, g_{22}, \ldots, g_{mm} \rangle$, then at most m limit cycles bifurcate from the origin of the corresponding real system

$$\dot{u} = \lambda u - v + \sum_{j+l=2}^{n} \alpha_{jl} u^j v^l, \quad \dot{v} = u + \lambda v + \sum_{j+l=2}^{n} \beta_{jl} u^j v^l,$$

that is the cyclicity of the system is less or equal to m.

The problem has been solved for:

- The quadratic system ($\dot{x} = P_n,\ \dot{y} = Q_n,\ n = 2$) - Bautin (1952) (Żoładek (1994), Yakovenko (1995), Françoise and Yomdin (1997), Han, Zhang & Zhang (2007)).
- The system with homogeneous cubic nonlinearities - Sibirsky (1965) (Żołądek (1994))

In both cases the analysis is relatively simple because the Bautin ideal is a radical ideal.

Bautin's theorem for the quadratic system

The cyclicity of the origin of system

$$\dot{u} = \lambda u - v + \alpha_{20} u^2 + \alpha_{11} uv + \alpha_{02} v^2, \ \ \dot{v} = u + \lambda v + \beta_{20} u^2 + \beta_{11} uv + \beta_{02} v^2$$

equals three.

Proof. (V. R., 2007) We have for all k

$$g_{kk}\big|_{\mathbf{V}(\mathcal{B}_3)} \equiv 0 \tag{24}$$

where $\mathcal{B}_3 = \langle g_{11}, g_{22}, g_{33} \rangle$.
Hence, if $\mathcal{B}_3$ is a radical ideal then (24) and Hilbert Nullstellensatz yield that $g_{kk} \in \mathcal{B}_3$. Thus, to prove that an upper bound for the cyclicity is equal to three it is sufficient to show that $\mathcal{B}_3$ is a radical ideal.
With help of SINGULAR we check that

$$std(radical(\mathcal{B}_3)) = std(\mathcal{B}_3). \tag{25}$$

Hence, $\mathcal{B}_3 = \mathcal{B}$. This completes the proof.

The cyclicity problem can be "easily" solved if the Bautin ideal is radical.

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An approach which works for some systems with non-radical Bautin ideal:

- - V. Levandovskyy, V. R. , D. S. Shafer, The cyclicity of a cubic system with non-radical Bautin ideal, J. Differential Equations, 246 (2009) 1274-1287.
  - V. Levandovskyy, G. Pfister and V. R. Evaluating cyclicity of cubic systems with algorithms of computational algebra, Communications in Pure and Applied Analysis, **11** (2012) 2023 - 2035.

Another point of view at the center problem

$$\dot{x} = (x - \sum_{i+j=1}^{n-1} a_{ij} x^{i+1} y^j) = P, \;\; \dot{y} = -(y - \sum_{i+j=1}^{n-1} b_{ji} x^j y^{i+1}) = Q, \qquad (26)$$

$$\Phi(x, y; a_{10}, b_{10}, \ldots) = xy + \sum_{s=3}^{\infty} \sum_{k=0}^{s} v_{k,s-k} x^k y^{s-k}$$

$$W = \sum W_{(\nu_1,\nu_2,\ldots,\nu_{2l})} a_{10}^{\nu_1} a_{01}^{\nu_2} \ldots a_{p_\ell,q_\ell}^{\nu_\ell} b_{q_\ell,p_\ell}^{\nu_{\ell+1}} \ldots b_{10}^{\nu_{2\ell l-1}} b_{01}^{\nu_{2\ell}} \qquad (27)$$

be a formal series with $W(\bar{0}) = 1$. Denote $|a| = \sum a_{ij}$, $|b| = \sum b_{ij}$,

$$\mathcal{A}(W) = \sum \frac{\partial W}{\partial a_{ij}} a_{ij}(i-j-i|a|+j|b|) + \sum \frac{\partial W}{\partial b_{ij}} b_{ij}(i-j-i|a|+j|b|). \quad (28)$$

Theorem

System (26) has a center at the origin for all values of the parameters a_{kn}, b_{nk} if and only if there is a formal series (27) satisfying the equation

$$\mathcal{A}(W) = W(|a| - |b|). \qquad (29)$$

There is a ring $\mathcal{P}$ of some functions of a, b such that the following diagram is commutative

$$\begin{array}{ccc} \mathcal{P}[[x,y]] & \xrightarrow{\pi} & \mathcal{P} \\ D\downarrow & & \downarrow\mathcal{A} \\ \mathcal{P}[[x,y]] & \xrightarrow{\pi} & \mathcal{P}, \end{array} \tag{30}$$

where π is an isomorphism defined by

$$\pi : \sum c_{\alpha,\beta}(a,b)x^{\alpha}y^{\beta} \longrightarrow \sum c_{\alpha,\beta}(a,b), \tag{31}$$

and $D(\Phi)$ is the operator

$$D(\Phi) := \frac{\partial \Phi}{\partial x}P + \frac{\partial \Phi}{\partial y}Q.$$

If we consider a system which has a Darboux integral

$$f_1^{\alpha_1} f_2^{\alpha_2} \dots f_s^{\alpha_s}$$

then the exponents α_i are, generally speaking, functions of the coefficients a_{ij}, b_{ji} of our system. Therefore, noting that for w_i of the form $w_i = 1 + h.o.t$ the property $\mathcal{A}(w_i) = k_i w_i$ yields

$$\mathcal{A}(\log w_i) = k_i,$$

we see that an analog of the equations for the cofactors in the Darboux method is the equation

$$\sum_{i=1}^{s} \alpha_i k_i + \sum_{i=1}^{s} \mathcal{A}(\alpha_i) \log(w_i) = 0 \tag{32}$$

(if we look for a first integral of the form $1 + \sum_{i=1}^{\infty} h_i(x, y)$ with $h_i(x, y)$ being homogeneous polynomials of the degree i) or

$$\sum_{i=1}^{s} \alpha_i k_i + \sum_{i=1}^{s} \mathcal{A}(\alpha_i) \log(w_i) = |a| - |b| \tag{33}$$

(if we look for a Lyapunov first integral $\Phi = xy + h.o.t.$).

Quadratic system

$$\mathcal{A}(W) := a_{01}(|b|-1)\frac{\partial W}{\partial a_{01}} + a_{10}(1-|a|)\frac{\partial W}{\partial a_{10}} + a_{-12}(|a|+2|b|-3)\frac{\partial W}{\partial a_{-12}} +$$

$$b_{01}(|b|-1)\frac{\partial W}{\partial b_{01}} + b_{10}(1-|a|)\frac{\partial W}{\partial b_{10}} + b_{2,-1}(-2|a|-|b|+3)\frac{\partial W}{\partial b_{2,-1}}$$

$$|a| = a_{10} + a_{01} + a_{-12}, \qquad |b| = b_{01} + b_{10} + b_{2,-1}.$$

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$$b_{01}(|b|-1)\frac{\partial W}{\partial b_{01}} + b_{10}(1-|a|)\frac{\partial W}{\partial b_{10}} + b_{2,-1}(-2|a|-|b|+3)\frac{\partial W}{\partial b_{2,-1}}$$

$|a| = a_{10} + a_{01} + a_{-12}, \qquad |b| = b_{01} + b_{10} + b_{2,-1}.$
Hamiltonian system: $\mathbf{V}(J_1)$, where $J_1 = \langle 2a_{10} - b_{10}, 2b_{01} - a_{01}\rangle \Rightarrow$
$a_{01} = 2b_{01}, \; b_{10} = 2a_{10}$

$$\mathcal{A}(W) := a_{10}(1-|a|)\frac{\partial W}{\partial a_{10}} + a_{-12}(|a|+2|b|-3)\frac{\partial W}{\partial a_{-12}} +$$
$$b_{01}(|b|-1)\frac{\partial W}{\partial b_{01}} + b_{2,-1}(-2|a|-|b|+3)\frac{\partial W}{\partial b_{2,-1}}$$

$$H = -\left(xy - \frac{a_{-12}}{3}y^3 - \frac{b_{2,-1}}{3}x^3 - a_{10}x^2y - b_{01}xy^2\right)$$

$W = 1 - a_{-12}/3 - b_{2,-1}/3 - a_{10} - b_{01}$ is a solution to
$\mathcal{A}(W) = W(|a| - |b|)$.

$\mathbf{V}(J_3)$, where $J_3 = \langle 2a_{01} + b_{01}, a_{10} + 2b_{10},\ a_{01}b_{10} - a_{-12}b_{2,-1}\rangle$,

$$\begin{cases} \mathcal{A}(W) := a_{01}(|b|-1)\frac{\partial W}{\partial a_{01}} + \ a_{10}(1-|a|)\frac{\partial W}{\partial a_{10}} \ + a_{-12}(|a|+2|b|-3)\frac{\partial W}{\partial a_{-12}} \\ +b_{01}(|b|-1)\frac{\partial W}{\partial b_{01}} + \ b_{10}(1-|a|)\frac{\partial W}{\partial b_{10}} \ + b_{2,-1}(-2|a|-|b|+3)\frac{\partial W}{\partial b_{2,-1}} \\ = W(|a|+|b|) \\ 2a_{01}+b_{01} = a_{10}+2b_{10} = a_{01}b_{10} - a_{-12}b_{2,-1} = 0 \end{cases}$$

$\Rightarrow$

$$\mathcal{A}(W) := a_{01}(|b|-1)\frac{\partial W}{\partial a_{01}} + b_{10}(1-|a|)\frac{\partial W}{\partial b_{10}} + b_{2,-1}(-|b|-2|a|+3)\frac{\partial W}{\partial b_{2,-1}} = W(|a|+|b|) \quad (34)$$

$$\ell_1 = 1 + 2\, b_{10}\, x - a_{01}\, b_{2,-1}\, x^2 + \; 2\, a_{01}\, y + 2\, a_{01}\, b_{10}\, x\, y - \frac{a_{01}\, b_{10}^2}{b_{2,-1}}\, y^2,$$

$$\ell_2 = (2\, b_{10}\, b_{2,-1}^2 + 6\, b_{10}^2\, b_{2,-1}^2\, x + 3\, {b_{10}}^3\, b_{2,-1}^2\, x^2 - 3\, a_{01}\, b_{10}\, b_{2,-1}^3\, x^2 - a_{01}\, b_{10}^2\, b_{2,-}$$
$$6\, a_{01}\, b_{10}\, b_{2,-1}^2\, y - 3\, b_{10}^4\, b_{2,-1}\, x\, y + 6\, a_{01}\, b_{10}^2\, b_{2,-1}^2\, x\, y - 3\, a_{01}^2\, b_{2,-1}^3\, x\, y + 3\, a_{01}\, b_{10}^3\, b_{2,-}^2$$
$$3\, a_{01}^2\, b_{10}\, b_{2,-1}^3 x^2\, y - 3\, a_{01}\, b_{10}^3\, b_{2,-1}\, y^2 + 3\, a_{01}^2\, b_{10}\, b_{2,-1}^2\, y^2 - 3\, a_{01}\, b_{10}^4\, b_{2,-1}\, x\, y^2 + 3\, a_0^2$$
$$a_{01}\, b_{10}^5\, y^3 - a_{01}^2\, b_{10}^3\, b_{2,-1}\, y^3)/(2\, b_{10}\, b_{2,-1}^2)$$

with the cofactors $k_1 = 2\,(b_{10}\, x - a_{01}\, y)$ and $k_2 = 3\,(b_{10}\, x - a_{01}\, y)\,.$
First integral

$$\Psi = \ell_1^{-3} \ell_2^2 \equiv c.$$

$L_1 = 1 + 2a_{01} + 2b_{10} + 2a_{01}b_{10} - (a_{01}b_{10}^2)/b_{2,-1} - a_{01}b_{2,-1}$
$L_2 = (a_{01}b_{10}^5 - 3a_{01}b_{10}^3 b_{2,-1} - a_{01}^2 b_{10}^3 b_{2,-1} - 3b_{10}^4 b_{2,-1} -$
$3a_{01}b_{10}^4 b_{2,-1} + 2b_{10}b_{2,-1}^2 + 6a_{01}b_{10}b_{2,-1}^2 + 3a_{01}^2 b_{10}b_{2,-1}^2 + 6b_{10}^2 b_{2,-1}^2 +$
$6a_{01}b_{10}^2 b_{2,-1}^2 + 3a_{01}^2 b_{10}^2 b_{2,-1}^2 + 3b_{10}^3 b_{2,-1}^2 + 3a_{01}b_{10}^3 b_{2,-1}^2 - 3a_{01}^2 b_{2,-1}^3 -$
$3a_{01}b_{10}b_{2,-1}^3 - 3a_{01}^2 b_{10}b_{2,-1}^3 - a_{01}b_{10}^2 b_{2,-1}^3 + a_{01}^2 b_{2,-1}^4)/(2b_{10}b_{2,-1}^2))$

$$\mathcal{A}(L_1) = K_1 L_1, \quad \mathcal{A}(L_2) = K_2 L_2,$$

$K_1 = -2(a_{01} - b_{10}), \quad K_1 = -3(a_{01} - b_{10}),$
$U = L_1^{-3} L_2^2$ is a solution to $\mathcal{A}(U) = 0$

$$W = \frac{U - 1}{-6a_{01}b_{10} - (3b_{10}^3)/b_{2,-1} - (3a_{01}^2 b_{2,-1})/b_{10})}$$

is a solution to

$$\mathcal{A}(W) = W(|a| + |b|).$$

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Thank you for your attention!