Algebraic Transformation of Differential Characteristic Decompositions from One Ranking to Another [1]

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Abstract

We propose an algorithm for transforming a characteristic decomposition of a radical differential ideal from one ranking into another. The algorithm is based on a new bound: we show that, in the ordinary case, for *any ranking*, the order of each element of the *canonical characteristic set* of a characterizable differential ideal is bounded by the order of the ideal. Applying this bound, the algorithm determines the number of times one needs to differentiate the given differential polynomials, so that a characteristic decomposition w.r.t. the target ranking could be computed by a purely algebraic algorithm (that is, without further differentiations). We also propose a factorization-free algorithm for computing the canonical characteristic set of a characterizable differential ideal represented as a radical ideal by a set of generators. This algorithm is not restricted to the ordinary case and is applicable for an arbitrary ranking.

Key words: differential algebra, canonical characteristic sets, radical differential ideals, bounds for orders
1991 MSC: 12H05, 13N10, 13P10

1. Introduction

The main result of this paper is an algorithm, which inputs a characteristic decomposition of a radical differential ideal I w.r.t. one ranking and computes a characteristic decomposition of I w.r.t. another ranking. Previously, the problem of efficient transformation of differential characteristic sets from one ranking to another has been addressed in (Boulier, 1999; Boulier et al., 2001; Golubitsky, 2004) in case of prime differential ideals. Our algorithm is different from these approaches in that its most computationally expensive part is performed by a purely algebraic algorithm. Another difference is that the proposed algorithm does not assume that the ideals in the given characteristic decomposition are characterizable w.r.t. the target ranking.

More precisely, the algorithm first applies a bound (described below), in order to determine the number of times one needs to differentiate the given polynomials, so that the target characteristic decomposition could be computed using only algebraic operations. In other words, at the first step, the algorithm reduces the given differential-algebraic problem to a purely algebraic one. The latter problem can be solved using efficient modular methods, e.g. (Dahan et al., 2006), which are not directly generalizable to the differential case due to the difficulties of working over differential fields of positive characteristics. Moreover, in the algebraic case, the complexity of computing a characteristic decomposition (or transforming it to a different ordering on variables) is known to be polynomial in the maximal degree of input polynomials and exponential in the number of variables (Szántó, 1999, Theorem 4.1.7), while for the differential case no complexity bounds are known. Our reduction "almost" allows to obtain a complexity bound for the ordinary differential case. It remains to estimate the complexity of the following algebraic problem: given a characterizable algebraic ideal w.r.t. one ranking, and another ranking, decompose it into ideals that are characterizable w.r.t. *both* rankings. We propose an algorithm for computing such *bi-characteristic decomposition* but do not estimate its complexity.

The bound, on which the above reduction is based, is the following: in the ordinary case, for any ranking on derivatives, the orders of the elements of the canonical characteristic set of a characterizable differential ideal do not exceed the order of the ideal. Here the order of a characterizable differential ideal is defined as the maximum of orders of its minimal prime components. In turn, the order of a prime differential ideal is defined as the sum of orders of the elements of any characteristic set of this ideal w.r.t. an orderly ranking. The order of a prime differential ideal is independent of the choice of the orderly ranking and the characteristic set w.r.t. this ranking.

This bound is the main technical tool of the paper. We prove it in three steps. First, we prove that, for any prime differential ideal and an arbitrary ranking, *there exists*

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a characteristic set, such that the orders of its elements are bounded by the order of the ideal (this is the main step, see Theorem 27). Second, we generalize this existence statement to the case of characterizable differential ideals (see Theorem 29). Finally, in Theorem 31 we show that the bound actually holds for *canonical* characteristic sets of characterizable differential ideals.

The problem of bounding the orders of elements of a differential characteristic set has been previous addressed in (Sadik, 2000, 2006). Our result generalizes (Sadik, 2000, Theorem 24), which gives the same bound for elimination rankings. The bound for arbitrary rankings has been stated in (Sadik, 2006, Theorem 1) without proof, as a consequence of the results of (Sadik, 2000). It would indeed easily follow from (Sadik, 2000, Theorem 25), yet the latter theorem turned out to be incorrect, as we show on a counter-example (see Example 28). It appears that the case of general rankings does not reduce immediately to the case of elimination rankings and requires a detailed proof (see Theorem 27).

The paper is organized as follows. In Section 2, the necessary differential-algebraic notation is introduced. In Sections 3 and 4, the algebraic algorithm for converting characteristic decompositions from one ranking to another is presented. In Section 5, we prove some basic properties of canonical characteristic sets, preparing for the proof of the bound in Section 6. Finally, in the appendix (Section 7) we show how to compute the canonical characteristic set from any other known representation of a characterizable differential ideal.

2. Preliminaries

Differential algebra studies systems of polynomial differential equations from the algebraic point of view. The approach is based on the concept of differential ring introduced by J.F. Ritt. Recent tutorials on the constructive theory of differential ideals are presented in (Hubert, 2003b; Sit, 2002). The classical references for the basic notions we are using are (Kolchin, 1973; Ritt, 1950).

A differential ring is a commutative ring with unity endowed with a set of derivations $\Delta = \{\delta_1, \ldots, \delta_m\}$. The case of $m = 1$, that is, $\Delta = \{\delta\}$, is called *ordinary*. If R is an ordinary differential ring and $y \in R$, we denote $\delta^k y$ by $y^{(k)}$. Construct the multiplicative monoid $\Theta = \left\{\partial_1^{k_1}\partial_2^{k_2}\cdots\partial_m^{k_m} \mid k_i \geqslant 0\right\}$ of *derivative operators*. Let $Y = \{y_1, \ldots, y_n\}$ be a set whose elements are called *differential indeterminates*. The elements of the set $\Theta Y = \{\theta y \mid \theta \in \Theta,\ y \in Y\}$ are called *derivatives*. Derivative operators from Θ act on derivatives as $\theta_1(\theta_2 y_i) = (\theta_1\theta_2)y_i$ for all $\theta_1, \theta_2 \in \Theta$ and $1 \leqslant i \leqslant n$.

The ring of *differential polynomials* in differential indeterminates Y over a differential field $\mathbf{k}$ is a ring of commutative polynomials with coefficients in $\mathbf{k}$ in the infinite set of variables ΘY. This ring is denoted by $\mathbf{k}\{y_1, \ldots, y_n\}$. We consider the case of $\operatorname{char}\mathbf{k} = 0$ only. Let u be a derivative in $\mathbf{k}\{y_1, \ldots, y_n\}$ and $u = \theta y_i$ for a derivative operator $\theta = \delta_1^{k_1}\delta_2^{k_2}\cdots\delta_n^{k_m} \in \Theta$ and a differential indeterminate $y_i \in \{y_1, \ldots, y_n\}$. The *order* of u is defined as $\operatorname{ord} u = \operatorname{ord}\theta = k_1 + \ldots + k_m$. If f is a differential polynomial then $\operatorname{ord} f$ denotes the maximal order of derivatives appearing effectively in f.

A ranking is a well-order $\leq$ on the set of derivatives compatible with differentiation, that is, for any derivatives u, v and derivation $\delta \in \Delta$, $u \leq v$ implies $\delta u \leq \delta v$ and $u < \delta u$ (Kolchin, 1973). A ranking $\leq$ is said to be *orderly* iff $\operatorname{ord} u < \operatorname{ord} v$ implies $u < v$ for all derivatives u and v. A ranking $\leq$ is called an *elimination* ranking iff $y_i < y_j$ implies $\theta_1 y_i < \theta_2 y_j$ for all $\theta_1, \theta_2 \in \Theta$.

For a fixed ranking $\leq$ and a differential polynomial f, denote its leader, rank, initial, and separant by $\mathbf{u}_f = \operatorname{ld} f$, $\operatorname{rk} f$, $\mathbf{i}_f$, and $\mathbf{s}_f$, respectively. For a set F of differential polynomials, the sets of leaders, ranks, initials, and separants of the elements of F are denoted $\operatorname{ld} F$, $\operatorname{rk} F$, I_F, S_F, respectively. Let also $H_F = I_F \cup S_F$. For the differential and radical differential ideals generated by F in $\mathbf{k}\{y_1, \ldots, y_n\}$, we use notations $[F]$ and $\{F\}$, respectively.

In this paper, we often treat a differential polynomial f as an algebraic polynomial over the field $\mathbf{k}$, whose variables are derivatives effectively present in f. We say that a differential polynomial f is algebraically reduced w.r.t. a differential polynomial g, if $\deg_{\mathbf{u}_g} f < \deg_{\mathbf{u}_g} g$; polynomial f is called differentially reduced w.r.t. g, if f is algebraically reduced w.r.t. g and does not contain proper derivatives of $\mathbf{u}_g$. Algebraically autoreduced and differentially autoreduced sets of differential polynomials are defined accordingly. The differential analogue of an algebraically triangular set (which is a set of differential polynomials with distinct leaders) is a weak d-triangular set (Hubert, 2003b, Definition 3.7): a set $\mathcal{C}$ of differential polynomials is called weakly d-triangular, if $\mathcal{C}$ is algebraically triangular and $\operatorname{ld} \mathcal{C}$ is differentially autoreduced.

For an algebraically triangular set $\mathcal{A}$, the algebraic pseudo-remainder of f w.r.t. $\mathcal{A}$ is denoted $\mathsf{algrem}(f, \mathcal{A})$; for a weak d-triangular set $\mathcal{C}$, the differential pseudo-remainder of f w.r.t. $\mathcal{C}$, defined via (Hubert, 2003b, Algorithm 3.13), is denoted $\mathsf{d\text{-}rem}(f, \mathcal{C})$. Since, in this paper, differential versions of the above definitions occur more often than the algebraic ones, we will sometimes omit the descriptor "differential" for brevity.

A ranking on derivatives induces well-orders on the set of ranks and on the set of all finite sets of ranks (Kolchin, 1973). Given that every autoreduced set is finite (Kolchin, 1973), this implies that every family of autoreduced sets has one of the least rank. For a differential ideal I, its autoreduced subset of the least rank is called a characteristic set of I (Kolchin, 1973, page 82).

An algebraically autoreduced set in $\mathbf{k}\{y_1, \ldots, y_n\}$ may be infinite. A ranking induces a total order on the set of all sets of ranks (including the infinite ones), which is not necessarily a well-order. Consequently, not every family of algebraically autoreduced sets has one of the least rank. However, every set of differential polynomials does have an algebraically autoreduced subset of the least rank. For an algebraic ideal J in $\mathbf{k}\{y_1, \ldots, y_n\}$, an algebraically autoreduced subset of J of the least rank is called an algebraic characteristic set of J. An algebraic characteristic set of a finitely generated algebraic ideal is finite.

Let I be an ideal in a commutative ring R and S be a multiplicative subset of $R \setminus \{0\}$ and containing 1. Then $I : S^\infty$ is defined as $\{a \in R | \exists s \in S^\infty : sa \in I\}$. If I is a differential ideal then $I : S^\infty$ is also a differential ideal (see Kolchin (1973)). For a finite set S of differential polynomials denote by S^∞ the multiplicative set containing 1 and generated by S. A differential ideal I is called *characterizable* (Hubert, 2000, Definition 2.6), if there exists a characteristic set $\mathcal{A}$ of I such that $I = [\mathcal{A}] : H_{\mathcal{A}}^\infty$. Any such characteristic set $\mathcal{A}$ is called a *characterizing* set of I. Algebraic characterizable ideals and their algebraic characterizing sets are defined accordingly. Characterizable ideals are radical (Hubert, 2000, Theorem 4.4).

A characteristic set of a characterizable differential ideal may not be unique. Summarizing (Boulier and Lemaire, 2000, Section 2.2.6), we define the *canonical characteristic set* of a characterizable differential ideal. This construction also follows from (Hubert, 2003a, Section 5.4) and (Hubert, 2003b, Theorem 5.5).

Let $\mathcal{A}$ be an autoreduced set in $\mathbf{k}\{y_1, \ldots, y_n\} = \mathbf{k}\{Y\}$, and let $\mathbf{k}[N][L]$ be the polynomial ring associated with $\mathcal{A}$, where L is the set of leaders of polynomials in $\mathcal{A}$ and N is the set of non-leaders, that is, $N = \Theta Y \setminus \Theta L$. Note that the set N may be infinite when $\Delta \neq \varnothing$.

Definition 1 *A characteristic set $\mathcal{C} = C_1, \ldots, C_p$ of a differential ideal I is called canonical if the following conditions are satisfied for every $i = 1, \ldots, p$:*

(1) the initial $\mathbf{i}_{C_i}$ depends only on non-leaders N of $\mathcal{C}$;

(2) the polynomial C_i does not have factors in $\mathbf{k}[N, L]$ belonging to I, other than C_i itself;

(3) the leading coefficient of C_i w.r.t. the induced lexicographic ordering $<_{\text{lex}}$ on monomials over $N \cup L$ is equal to 1.

The above definition is slightly different from that of (Boulier and Lemaire, 2000). In Section 5, we will prove correctness of the above definition and some properties of canonical characteristic sets. An interested reader can also find in Section 7 an algorithm for computing the canonical characteristic set from any other known representation of a characterizable differential ideal.

3. Transformation of characteristic sets of prime differential ideals

As above, let $\mathbf{k}\{Y\}$ be a ring of *ordinary* differential polynomials in n indeterminates with the derivation δ. Let $\mathcal{C}$ be a characteristic set of a prime differential ideal I in $\mathbf{k}\{Y\}$ w.r.t. a ranking $\leq$. We propose an algorithm that computes a characteristic set of I w.r.t. any other ranking $\leq'$ *algebraically.* More precisely, using a bound on the orders of derivatives occurring in the canonical characteristic set $\mathcal{D}$ of I w.r.t. the target ranking, we find a sufficient *differential prolongation* of $\mathcal{C}$ (described below), which defines a prime algebraic sub-ideal $\bar{I}$ in I containing $\mathcal{D}$. After that, it remains to compute an algebraic characteristic set of $\bar{I}$ w.r.t. the target ranking and extract from it a differential characteristic set of I.

3.1. A bound for characteristic sets of prime differential ideals

First, given a characteristic set $\mathcal{C}$ of a prime differential ideal I w.r.t. an arbitrary ranking $\leq$, we would like to obtain a bound on the orders of derivatives occurring in a characteristic set of I w.r.t. another given ranking $\leq'$. For $\leq$ orderly and $\leq'$ arbitrary, such a bound is given in Section 6. If $\leq$ is not orderly, we first obtain a bound for the orders of the elements of an orderly characteristic set $\mathcal{D}$ of I, and then apply the bound from Section 6.

Indeed, $\mathcal{D}$ can be computed from $\mathcal{C}$ with the help of the Rosenfeld-Gröbner algorithm applied to the system $F_0 = \mathcal{C}$, $H_0 = H_{\mathcal{C}}$ (where the initials and separants of $\mathcal{C}$ in $H_{\mathcal{C}}$ are taken w.r.t. $\leq$). Since I is prime, one of the regular components (A, H) computed by the Rosenfeld-Gröbner algorithm will coincide with I, and the characteristic set of the corresponding regular ideal $[A] : H^\infty$ w.r.t. $\leq'$ can be extracted from the lexicographic Gröbner basis of the algebraic ideal $(A) : H^\infty$ via the algorithm given in (Boulier et al., 1995, Theorem 6). A more efficient algorithm, which uses the fact that the given ideal is prime and thus avoids the computation of redundant regular components, is presented in (Boulier et al., 2001).

Let M be the maximal order of derivatives occurring in $\mathcal{C}$. The only place where the Rosenfeld-Gröbner algorithm differentiates polynomials is the computation of differential pseudo-remainders. However, for an orderly ranking, the order of a polynomial cannot increase as a result of pseudo-reduction. Thus, the orders of derivatives occurring in the characteristic set $\mathcal{D}$ do not exceed M. In fact, the same applies to any other characteristic set of I w.r.t. the same orderly ranking: the leading derivatives of all characteristic sets of I w.r.t. the same ranking coincide, and the orders of non-leading derivatives occurring in a polynomial f cannot exceed the order of the leader of f w.r.t. an orderly ranking.

Now we will use the following

Lemma 2 *The number of elements in a characteristic set $\mathcal{C}$ of a prime differential ideal I in the ring of ordinary differential polynomials $\mathbf{k}\{y_1, \ldots, y_n\}$ does not depend on the ranking.*

Proof. If d is the differential dimension of P then the number of elements of $\mathcal{C}$ is equal to $n - d$ by (Cluzeau and Hubert, 2003, Theorem 4.11) which does not depend on the choice of a differential ranking. □

Remark 3 *The above lemma does not hold in the partial differential case. For example (borrowed from Boulier et al. (2001)), a characteristic set of the prime differential ideal*

$$[u_x^2 - 4u,\ u_{xy}v_y - u + 1,\ v_{xx} - u_x]$$

in $\mathbf{k}\{Y\}$ with derivations $\Delta = \{\partial/\partial x, \partial/\partial y\}$ may have 3 or 4 elements, depending on the ranking.

For the above example, it takes a while to compute the characteristic set of the ideal w.r.t. the elimination ranking $u > v$ using the Rosenfeld-Gröbner algorithm in Maple (Golubitsky, 2006). Consider another example that requires less computational effort.

Example 4 *Consider the following prime differential ideal:*

$$P = [u_{yy},\ v_{xx} + y \cdot u_x + u]\,.$$

This set of generators forms a characteristic set of P w.r.t. the elimination ranking with $v > u$. However, if we change the ranking to $u > v$, then the following set containing 3 *elements will be a characteristic set of P* :

$$\begin{aligned}
&v_{xxyyy},\\
&y^2 \cdot v_{xxxxyy} - 2y \cdot v_{xxxxy} + 2y \cdot v_{xxxyy} + 2v_{xxxx} - 2v_{xxxy} + v_{xxyy},\\
&2u - y^3 \cdot v_{xxxyy} + 2y^2 \cdot v_{xxxy} - 2y \cdot v_{xxx} + 2v_{xx}.
\end{aligned}$$

Applying Lemma 2, we obtain the following bound on the *order* of I (see Section 6):

$$\operatorname{ord} I := \sum_{D \in \mathcal{D}} \operatorname{ord} D \leqslant |\mathcal{C}| \cdot \max_{C \in \mathcal{C}} \operatorname{ord} C. \tag{1}$$

This bound is likely to be non-optimal. As in (Golubitsky et al., 2008, Section 4), for a differential indeterminate $y_i \in Y$ and a set of differential polynomials F, $m_i(F) = m_{y_i}(F)$ denotes the highest order of a derivative of y occurring in F, or zero, if y_i does not occur in F. It is possible that the results of (Ritt, 1950, Chapter VII), together with Lemma 2, imply the following bound, which is better: let $m_1 \geqslant m_2 \geqslant \ldots \geqslant m_n$ be the numbers $m_y(\mathcal{C})$, $y \in Y$, arranged in non-increasing order, then

$$\operatorname{ord} I \leqslant \sum_{i=1}^{|\mathcal{C}|} m_i.$$

For this bound, which so far is a conjecture, one needs to verify that Ritt's proof holds for non-elimination rankings and also adapt it for ideals specified by characteristic sets, rather than sets of generators.

According to Theorem 31 (see Section 6.4), the orders of derivatives occurring in the *canonical* characteristic set of I w.r.t. any ranking do not exceed the order of I. Thus, the number

$$M_1 = |\mathcal{C}| \cdot \max_{C \in \mathcal{C}} \operatorname{ord} C$$

bounds the orders of derivatives occurring in the canonical characteristic set of I w.r.t. any (not necessarily orderly) target ranking $\leq'$. Let

$$M(F) = \sum_{y \in Y} m_y(F).$$

Note that the bound $(n-1)! \cdot M(\mathcal{C})$ obtained in (Golubitsky et al., 2008, Section 4) is also a bound for the orders of derivatives occurring in the characteristic set of I w.r.t. $\leq'$ computed by the Rosenfeld-Gröbner algorithm. In fact, invariant I5 in the proof of (Golubitsky et al., 2008, Proposition 13), together with Lemma 2, yields a better bound

$$M_2 = \frac{(n-1)!}{(n-|\mathcal{C}|-1)!} \cdot M(\mathcal{C}).$$

In most cases, $M_2 > M_1$, but in some, especially for small values of n, it may happen that $M_2 < M_1$. This again suggests that none of the two bounds is optimal. Leaving the important problem of obtaining an optimal bound for future research, we summarize the bounds obtained so far in the following

Lemma 5 *Let $\mathcal{C}$ be a characteristic set of an ordinary prime differential ideal I w.r.t. a ranking $\leq$. Then* $\operatorname{ord} I$ *and the orders of derivatives occurring in the canonical characteristic set of I w.r.t. another ranking $\leq'$ do not exceed*

$$M_{\mathcal{C}} := \min(M_1, M_2) = \min\left(|\mathcal{C}| \cdot \max_{C \in \mathcal{C}} \operatorname{ord} C, \frac{(n-1)!}{(n-|\mathcal{C}|-1)!} \cdot M(\mathcal{C})\right).$$

3.2. Differential prolongation: the prime case

Assume that $\operatorname{ld}_{\leq} \mathcal{C} = \left\{y_1^{(d_1)}, \ldots, y_k^{(d_k)}\right\}$. Let $m_i = M_{\mathcal{C}}$, $1 \leqslant i \leqslant k$. Compute the set

$$\mathcal{A} = \mathsf{Differentiate\&Autoreduce}\left(\mathcal{C}, \{m_i\}_{i=1}^k\right)$$

(for the algorithm Differentiate&Autoreduce, see (Golubitsky et al., 2008, Algorithm 2, Section 4.1)). Informally speaking, the set $\mathcal{A}$ can be thought of as a result of an autoreduction of a *differential prolongation* of the input set $\mathcal{C} = \{C_1, \ldots, C_k\}$, i.e., of the set

$$\tilde{\mathcal{C}} = \left\{\delta^j C_i \mid 1 \leqslant i \leqslant k,\ 0 \leqslant j \leqslant m_i - d_i\right\}.$$

In particular, we have $\operatorname{rk} \mathcal{A} = \operatorname{rk} \tilde{\mathcal{C}}$. See Algorithm 1 for the formal specification of Differentiate&Autoreduce.

Let $\mathcal{D}$ be the *canonical* characteristic set of I w.r.t. $\leq'$. Every polynomial in $\mathcal{D}$, as an element of I, reduces w.r.t. $\mathcal{C}$ and $\leq$ to zero. Since the orders of derivatives occurring in $\mathcal{D}$ do not exceed $M_{\mathcal{C}}$, every polynomial in $\mathcal{D}$ algebraically reduces to zero w.r.t. $\mathcal{A}$. That is, $\mathcal{D} \subset (\mathcal{A}) : H_{\mathcal{A}}^{\infty}$. The algebraic ideal $\bar{I} = (\mathcal{A}) : H_{\mathcal{A}}^{\infty}$ is equal to the intersection of I with the ring

$$R = \mathbf{k}\left[\Theta Y \setminus \Theta \operatorname{ld}_{\leq} \mathcal{C} \cup \operatorname{ld}_{\leq} \mathcal{A}\right].$$

Algorithm 1 Differentiate&Autoreduce($\mathcal{C}, \{m_i\}$)

INPUT: *a weak d-triangular set* $\mathcal{C} = C_1, \ldots, C_k$ *with* $\operatorname{ld} \mathcal{C} = y_1^{(d_1)}, \ldots, y_k^{(d_k)}$,

and a set of non-negative integers $\{m_i\}_{i=1}^k$, $m_i \geqslant m_i(\mathcal{C})$

OUTPUT: *set* $\mathcal{A} = \left\{ A_i^j \;\middle|\; 1 \leqslant i \leqslant k,\ 0 \leqslant j \leqslant m_i - d_i \right\}$ *satisfying*

- $\operatorname{rk} A_i^j = \operatorname{rk} C_i^{(j)}$
- A_i^j *are reduced w.r.t.* $\mathcal{C} \setminus \{A_i\}$
- $m_i(\mathcal{A}) \leqslant m_i$, $i = 1, \ldots, k$
- $m_i(\mathcal{A}) \leqslant m_i(\mathcal{C}) + \sum_{j=1}^k (m_j - d_j)$, $i = k+1, \ldots, n$
- $\mathcal{A} \subset [\mathcal{C}] \subset [\mathcal{A}] : H_{\mathcal{A}}^\infty$
- $H_{\mathcal{A}} \subset H_{\mathcal{C}}^\infty + [\mathcal{C}]$, $\quad H_{\mathcal{C}} \subset (H_{\mathcal{A}}^\infty + [\mathcal{A}]) : H_{\mathcal{A}}^\infty$

or $\{1\}$, *if it is detected that* $[\mathcal{C}] : H_{\mathcal{C}}^\infty = (1)$

Indeed, $\mathcal{A} \subset R$. Vice versa, every element of $I \cap R$ algebraically reduces w.r.t. $\mathcal{A}$ to zero and therefore belongs to $(\mathcal{A}) : H_{\mathcal{A}}^\infty$.

Since I is prime, so is $\bar{I}$. Applying one of the existing efficient algorithms (for instance, see (Boulier et al., 2001) or (Dahan et al., 2006)) to the set $\mathcal{A}$, we compute the canonical algebraic characteristic set $\mathcal{B}$ of $\bar{I}$ w.r.t. the target ranking $\leq'$. We know that the algebraic ideal $\bar{I}$ contains the canonical characteristic set $\mathcal{D}$ of the differential ideal I w.r.t. $\leq'$. In the following section, we will show that, in fact, $\mathcal{D} \subseteq \mathcal{B}$.

3.3. Extracting a differential characteristic set

The following two lemmas hold in the partial differential case. We assume that a ranking is fixed.

Lemma 6 *Let* $\mathbf{k}\{Y\}$ *be a ring of partial differential polynomials, and let* K *be an arbitrary subset of* $\mathbf{k}\{Y\} \setminus \mathbf{k}$. *Let* $\mathcal{C}$ *be a differential characteristic set of* K *and* $\mathcal{A}$ *an algebraic characteristic set of* K. *Let* $\mathcal{T}$ *be a weak d-triangular subset of* $\mathcal{A}$ *of the least rank. Then* $\operatorname{rk} \mathcal{T} \leq \operatorname{rk} \mathcal{C}$.

Proof. Suppose that a polynomial $0 \neq f \in \mathcal{C}$ is differentially reduced w.r.t. $\mathcal{T}$. Then, since $\mathcal{T}$ is a weak d-triangular subset of $\mathcal{A}$ of the least rank, f is algebraically reduced w.r.t. $\mathcal{A}$. Due to the fact that $\mathcal{A}$ is an algebraic characteristic set of K, we have $f = 0$, contradiction. Thus, no element of $\mathcal{C}$ is differentially reduced w.r.t. $\mathcal{T}$, which implies that $\operatorname{rk} \mathcal{T} \leq \operatorname{rk} \mathcal{C}$. □

Lemma 7 *Let* I *be a prime differential ideal, let* $\mathcal{C}$ *be the canonical characteristic set of* I, *and let* $J = I \cap \mathbf{k}[V]$, *where* $V \subset \Theta Y$, *be an algebraic ideal containing* $\mathcal{C}$. *Then the canonical algebraic characteristic set (as in Definition 1)* $\mathcal{D}$ *of* J *contains* $\mathcal{C}$*; more precisely,* $\mathcal{C}$ *is the weak d-triangular subset of* $\mathcal{D}$ *of the least rank.*

Proof. Since $\mathcal{D}$ is triangular, its weak d-triangular subset of the least rank is unique. Let $\mathcal{T}$ be the weak d-triangular subset of $\mathcal{D}$ of the least rank. Since $\mathcal{D}$ is an algebraic characteristic set of the prime ideal J, we have $H_{\mathcal{D}} \cap J = \varnothing$. Moreover, $H_{\mathcal{D}} \subset \mathbf{k}[V]$, therefore $H_{\mathcal{D}} \cap I = \varnothing$ and, hence, $H_{\mathcal{T}} \cap I = \varnothing$. Since $\mathcal{T} \subset I$ and I is prime, this implies

$$[\mathcal{T}] : H_{\mathcal{T}}^\infty \subset I. \tag{2}$$

Algorithm 2 Convert_Prime $(\mathcal{C}, \leq, \leq')$

INPUT: *a prime differential ideal* $P = [\mathcal{C}] : H_{\mathcal{C}}^{\infty} \subset \mathbf{k}\{y_1, \ldots, y_n\}$

with a characteristic set $\mathcal{C}$ *w.r.t. the input ranking* $\leq$

with leading variables $y_1, \ldots, y_k$ *and*

a target ranking $\leq'$.

OUTPUT: *canonical characteristic set of* P *w.r.t.* $\leq'$.

$M_{\mathcal{C}} := \min\left(|\mathcal{C}| \cdot \max_{C \in \mathcal{C}} \operatorname{ord} C,\ \frac{(n-1)!}{(n-|\mathcal{C}|-1)!} \cdot M(\mathcal{C})\right)$

$m_i := M_{\mathcal{C}},\ 1 \leqslant i \leqslant k$

$\mathcal{A} :=$ Differentiate&Autoreduce $\left(\mathcal{C}, \{m_i\}_{i=1}^{k}\right)$

$\mathcal{D} :=$ Canonical_Algebraic_CharSet $((\mathcal{A}) : H_{\mathcal{A}}^{\infty}, \leq')$

return minimal d-triangular subset $(\mathcal{D}, \leq')$

Let

$$\mathcal{A} = \{\mathsf{d\text{-}rem}(f, \mathcal{T} \setminus \{f\}) \mid f \in \mathcal{T}\}.$$

We have $\mathcal{A} \subset [\mathcal{T}] \subset I$; we will show that set $\mathcal{A}$ is differentially autoreduced and $\operatorname{rk} \mathcal{A} = \operatorname{rk} \mathcal{T}$.

First, show that $\operatorname{rk} \mathcal{A} = \operatorname{rk} \mathcal{T}$. Indeed, suppose that for some $f \in \mathcal{T}$ and $g = \mathsf{d\text{-}rem}(f, \mathcal{T} \setminus \{f\})$, we have $\operatorname{rk} g < \operatorname{rk} f$. Since $\mathcal{T}$ is a weak d-triangular set, $\operatorname{ld} f \notin \Theta \operatorname{ld}(\mathcal{T} \setminus \{f\})$. Thus, (Golubitsky et al., 2008, Lemma 4) applies and tells us that $\mathbf{i}_f \in [\mathcal{T}] : H_{\mathcal{T}}^{\infty}$. Hence, according to (2), $\mathbf{i}_f \in I$. This contradicts with the fact that $H_{\mathcal{T}} \cap I = \varnothing$.

Now, since g is reduced w.r.t. $\mathcal{T} \setminus \{f\}$, $\operatorname{rk} g = \operatorname{rk} f$, and $\operatorname{rk} \mathcal{A} = \operatorname{rk} \mathcal{T}$, g is also reduced w.r.t. $\mathcal{A} \setminus \{g\}$. That is, the set $\mathcal{A}$ is autoreduced. By Lemma 6, $\operatorname{rk} \mathcal{T} \leq \operatorname{rk} \mathcal{C}$. Therefore, $\operatorname{rk} \mathcal{A} \leq \operatorname{rk} \mathcal{C}$. Since $\mathcal{A}$ is an autoreduced subset of I, while $\mathcal{C}$ is an autoreduced subset of I of the least rank, we have $\operatorname{rk} \mathcal{A} \geq \operatorname{rk} \mathcal{C}$. Thus, $\operatorname{rk} \mathcal{A} = \operatorname{rk} \mathcal{T} = \operatorname{rk} \mathcal{C}$.

Let $\bar{\mathcal{D}} = (\mathcal{D} \setminus \mathcal{T}) \cup \mathcal{C}$. Set $\bar{\mathcal{D}}$ is algebraically autoreduced, has the same rank as $\mathcal{D}$, and satisfies the requirements of canonicity: for every $f \in \bar{\mathcal{D}}$, the initial of f does not depend on the leaders of $\bar{\mathcal{D}}$, f is monic and has no factors in $\mathbf{k}[N(\bar{\mathcal{D}})]$, where $N(\bar{\mathcal{D}}) = N(\mathcal{D}) = V \setminus \operatorname{ld} \mathcal{D}$ is the set of non-leaders of $\mathcal{D}$ (or $\bar{\mathcal{D}}$). Since the canonical characteristic set is unique, we have $\bar{\mathcal{D}} = \mathcal{D}$ and $\mathcal{C} = \mathcal{T}$. This concludes the proof. □

Returning to the notation from the previous section and applying the above lemma, we obtain that the canonical characteristic set $\mathcal{D}$ of I is equal to the weak d-triangular subset of $\mathcal{B}$ of the least rank w.r.t. $\leq'$. This concludes the computation of the canonical characteristic set of I w.r.t. the target ranking, which we summarize in Algorithm 2.

4. Transformation of characteristic decompositions of radical differential ideals

We generalize the algebraic method for transforming characteristic sets of a prime differential ideal from one ranking to another to the case of a characterizable differential ideal. Since an ideal characterizable w.r.t. one ranking may not be characterizable w.r.t.

another one, we need to reformulate the problem: given a characterizable differential ideal I with a characteristic set $\mathcal{C}$ w.r.t. a ranking $\leq$, compute a characteristic decomposition of I w.r.t. another ranking $\leq'$ *algebraically*. By analogy with the prime case, an algebraic computation here means finding a sufficient differential prolongation of $\mathcal{C}$, which defines a characterizable algebraic sub-ideal $\bar{I}$ in I, such that a differential characteristic decomposition of I w.r.t. $\leq'$ can be extracted from an algebraic characteristic decomposition of $\bar{I}$ w.r.t. $\leq'$.

We note that, given a characteristic decomposition of a radical differential ideal w.r.t. one ranking, we can obtain its characteristic decomposition w.r.t. another ranking algebraically by solving the above problem for each characterizable component.

All results of this section hold in the partial differential case, except for the bound in Section 4.2, which so far is known only for the ordinary case.

4.1. Differential prolongation

Definition 8 *Let F be a (possibly infinite) subset in a ring $\mathbf{k}\{Y\}$ of partial differential polynomials with a set of derivations Δ. A set $G \subset \Theta F$ is called a differential prolongation of F, if $F \subset G$ and the complement of G, $\Theta F \setminus G$, is invariant w.r.t. differentiation, i.e., for all $f \in \Theta F \setminus G$ and $\delta \in \Delta$, $\delta f \in \Theta F \setminus G$.*

A particular case of a differential prolongation of a weak d-triangular set F is F itself. If $F = \mathcal{C}$ is autoreduced and coherent then, according to (Kolchin, 1973, Lemma 6, page 137) and (Hubert, 2000, Lemma 6.1 and Theorem 6.2), the differential ideal $I = [\mathcal{C}] : H_{\mathcal{C}}^{\infty}$ is prime, respectively characterizable iff the algebraic ideal $J = (\mathcal{C}) : H_{\mathcal{C}}^{\infty}$ is prime, respectively characterizable. The ideal J can be considered either as an algebraic ideal in the ring of differential polynomials $\mathbf{k}\{Y\}$ or as an ideal in the polynomial subring $\mathbf{k}[Z_{\mathcal{C}}]$, where $Z_{\mathcal{C}} = L \cup N$, $L = \operatorname{ld} \mathcal{C}$, $N = \Theta Y \setminus \Theta L$, since the fact that $\mathcal{C}$ is autoreduced implies $\mathcal{C} \subset \mathbf{k}[Z_{\mathcal{C}}]$. The Rosenfeld Lemma states that

$$[\mathcal{C}] : H_{\mathcal{C}}^{\infty} \cap \mathbf{k}[Z_{\mathcal{C}}] = (\mathcal{C}) : H_{\mathcal{C}}^{\infty},$$

where the latter ideal is considered in $\mathbf{k}[Z_{\mathcal{C}}]$. Moreover, a set $\mathcal{D}$ is a differential characteristic set of I iff $\mathcal{D}$ is an algebraic characteristic set of J (if the latter is considered in $\mathbf{k}[Z_{\mathcal{C}}]$, otherwise we need to impose an additional requirement that $\mathcal{D}$ is differentially autoreduced). In particular, the canonical characteristic sets of I and J (differential and algebraic, respectively) coincide (for this statement, it does not matter in which ring to consider J, since the canonical characteristic set of an ideal is the same regardless of the ring in which the ideal is considered).

Now, if we consider a differential prolongation $\mathcal{D}$ of $\mathcal{C}$ and the corresponding polynomial subring $\mathbf{k}[Z_{\mathcal{D}}]$, where $Z_{\mathcal{D}} = \bar{L} \cup N$, $\bar{L} = \operatorname{ld} \mathcal{D}$, $N = \Theta Y \setminus \Theta L = \Theta Y \setminus \Theta \bar{L}$, then $\mathcal{D}$ is not necessarily a subset of $\mathbf{k}[Z_{\mathcal{D}}]$:

Example 9 *Let $\mathcal{C} = y',\ x + y$ with the elimination ranking $y < x$ and a prolongation*

$$\mathcal{D} = y',\ x + y,\ x' + y',\ x'' + y''.$$

Then

$$\bar{L} = y',\ x,\ x',\ x'', \quad N = y.$$

Hence, we have that $x'' + y'' \notin \mathbf{k}[Z_{\mathcal{D}}]$. Also,

$$[\mathcal{C}] : H_{\mathcal{C}}^{\infty} \cap \mathbf{k}[Z_{\mathcal{D}}] = (y',\ x + y,\ x',\ x'')$$

and $x'' \notin (\mathcal{D}) : H_{\mathcal{D}}^{\infty}$.

Therefore, we need to distinguish between two ideals $I_{\mathcal{D}} := (\mathcal{D}) : H_{\mathcal{D}}^{\infty}$ in $\mathbf{k}\{Y\}$ and $\bar{I}_{\mathcal{D}} := I \cap \mathbf{k}[Z_{\mathcal{D}}]$ in $\mathbf{k}[Z_{\mathcal{D}}]$. The algebraic ideal $\bar{I}_{\mathcal{D}}$ depends only on the set of leaders $\bar{L}$ of the differential prolongation of $\mathcal{C}$. In other words, for any characterizing set $\tilde{\mathcal{C}}$ of I and its differential prolongation $\tilde{\mathcal{D}}$ with $\operatorname{ld}\tilde{\mathcal{D}} = \operatorname{ld}\mathcal{D} = \bar{L}$, we have $\bar{I}_{\tilde{\mathcal{D}}} = \bar{I}_{\mathcal{D}}$. We call $\bar{I}_{\bar{L}} := \bar{I}_{\mathcal{D}}$ a *prolongation ideal* of the ideal I.

Next, we study properties of prolongation ideals. The following lemma gives a criterion for a prolongation ideal to be prime or characterizable.

Lemma 10 *Let $\mathcal{C}$ be a coherent autoreduced set, and let $\mathcal{D}$ be a differential prolongation of $\mathcal{C}$. Then the differential ideal $1 \notin I = [\mathcal{C}] : H_{\mathcal{C}}^{\infty}$ is prime, respectively characterizable, iff the corresponding prolongation ideal $\bar{I}_{\mathcal{D}}$ is prime, respectively characterizable.*

Proof. If I is prime then its restriction $I \cap \mathbf{k}[Z_{\mathcal{D}}] = \bar{I}_{\mathcal{D}}$ is also prime. If $\bar{I}_{\mathcal{D}}$ is prime than its restriction $\bar{I}_{\mathcal{D}} \cap \mathbf{k}[Z_{\mathcal{C}}] = (\mathcal{C}) : H_{\mathcal{C}}^{\infty}$ is prime and, thus, I is prime. Let I be a characterizable differential ideal. We will show that the set $\mathcal{A}$ given by formula (3) in Lemma 11 characterizes the prolongation ideal $\bar{I}_{\mathcal{D}}$. We have

$$\bar{I}_{\mathcal{D}} \subset (\mathcal{A}) : H_{\mathcal{A}}^{\infty}.$$

Indeed, by (Golubitsky et al., 2008, Lemma 4), the sets $\mathcal{A}$ and $\mathcal{D}$ have the same ranks, whence they have the same sets of reduced polynomials. In particular, since $\mathcal{D}$ is a differential prolongation of the characteristic set $\mathcal{C}$, the ideal $\bar{I}_{\mathcal{D}}$ has no non-zero polynomials reduced w.r.t. $\mathcal{D}$, and hence w.r.t. $\mathcal{A}$.

Now note that $(\mathcal{A}) : H_{\mathcal{A}}^{\infty} \subset I$ and $\mathcal{A} \subset \mathbf{k}[Z_{\mathcal{D}}]$. Hence, $\bar{I}_{\mathcal{D}} = (\mathcal{A}) : H_{\mathcal{A}}^{\infty}$ and $\mathcal{A}$ is a characteristic set of $\bar{I}_{\mathcal{D}}$. Thus, $\bar{I}_{\mathcal{D}}$ is characterizable. Since $\mathcal{C} \subset \mathbf{k}[Z_{\mathcal{D}}]$ and $(\mathcal{C}) : H_{\mathcal{C}}^{\infty} = I \cap \mathbf{k}[Z_{\mathcal{C}}]$, we have

$$(\mathcal{C}) : H_{\mathcal{C}}^{\infty} = (\mathcal{A}) : H_{\mathcal{A}}^{\infty} \cap \mathbf{k}[Z_{\mathcal{C}}].$$

□

The next lemma establishes a relation between characteristic sets of a characterizable differential ideal I and algebraic characteristic sets of its prolongation ideals.

Lemma 11 *Let $\mathcal{C}$ be a characteristic set of the differential ideal $1 \notin I = [\mathcal{C}] : H_{\mathcal{C}}^{\infty}$, let $\bar{L}$ be a differential prolongation of $L = \operatorname{ld}\mathcal{C}$, and let $\bar{I}_{\bar{L}}$ be the corresponding prolongation ideal. Then a characterizing set $\mathcal{A}$ of $\bar{I}_{\bar{L}}$ can be obtained from $\mathcal{C}$ as*

$$\mathcal{A} := \{\mathsf{algrem}(f, \mathcal{B} \setminus \{f\}) \mid f \in \mathcal{B},\ \operatorname{ld} f \in \bar{L}\}, \tag{3}$$

where $\mathcal{B}$ is any triangular subset of $\Theta\mathcal{C}$ satisfying $\operatorname{ld}\mathcal{B} = \operatorname{ld}\bar{L}$.

Vice versa, given a characterizing set $\mathcal{A}$ of $\bar{I}_{\bar{L}}$, let $\mathcal{T}$ be a weak d-triangular subset of $\mathcal{A}$ of the least rank. If $\mathcal{T}$ is differentially autoreduced, then it is a characterizing set of I. In particular, if $\mathcal{A}$ is the canonical characteristic set of $\bar{I}_{\bar{L}}$, then $\mathcal{T}$ is the canonical characteristic set of I.

Proof. Since I is characterizable, $\bar{I}_{\bar{L}}$ is also characterizable by Lemma 10 and $\mathcal{A}$ is its characteristic set. The other way follows from Lemma 6. □

In the ordinary case, the triangular set $\mathcal{B}$ considered in the above lemma is unique. Moreover, the set $\mathcal{A}$ can be equivalently obtained as

$$\mathcal{A} := \mathsf{Differentiate\&Autoreduce}(\mathcal{C}, \{m_i\}),$$

where the numbers $\{m_i\}$ are the maximal orders of derivatives of the leading differential indeterminates of $\mathcal{C}$ occurring in the prolongation $\bar{L}$. It is preferable to compute $\mathcal{A}$ in this way, because Differentiate&Autoreduce provides a bound on the orders of non-leading derivatives occurring in $\mathcal{A}$, which can be used for establishing complexity estimates for the entire transformation algorithm.

A generalization of Algorithm Differentiate&Autoreduce to the partial case is an interesting open problem. Moreover, in the partial case, there may be uncountably infinitely many triangular subsets of $\Theta\mathcal{C}$ whose leaders coincide with $\mathrm{ld}\,\Theta\mathcal{C}$. Thus, not every such set can be enumerated by an algorithmic procedure. However, it is easy to write a procedure that would enumerate a particular subset of $\Theta\mathcal{C}$, given $\mathcal{C}$; this procedure makes computation of the set of algebraic pseudo-remainders algorithmic as well. If one would like to choose the subset $\mathcal{B}$ in a systematic way, we suggest to use the ideas from the theory of monomial involutive divisions (Gerdt and Blinkov, 1998).

According to (Hubert, 2003b, Theorem 4.13), there is a one-to-one correspondence between the minimal prime components of a characterizable differential ideal $[\mathcal{C}] : H_{\mathcal{C}}^{\infty}$ and the minimal prime components of the corresponding algebraic ideal $(\mathcal{C}) : H_{\mathcal{C}}^{\infty}$. The following lemma generalizes this result to prolongation ideals.

Lemma 12 *Let $\mathcal{C}$ be a characteristic set of the differential ideal $I = [\mathcal{C}] : H_{\mathcal{C}}^{\infty}$, let $\bar{L}$ be a differential prolongation of $L = \mathrm{ld}\,\mathcal{C}$, and let $\bar{I}_{\bar{L}}$ be the corresponding prolongation ideal. Let*

$$I = P_1 \cap \ldots \cap P_k$$

be the minimal prime decomposition of I, and let $(\bar{P}_i)_{\bar{L}}$ be the prolongation ideals corresponding to P_i, $i = 1, \ldots, k$. Then

$$\bar{I}_{\bar{L}} = \left(\bar{P}_1\right)_{\bar{L}} \cap \ldots \cap \left(\bar{P}_k\right)_{\bar{L}}$$

is the minimal prime decomposition of $\bar{I}_{\bar{L}}$.

Proof. Since $\bar{I}_{\bar{L}} = I \cap \mathbf{k}[Z_{\bar{L}}]$,

$$\bar{I}_{\bar{L}} = (P_1 \cap \mathbf{k}[Z_{\bar{L}}]) \cap \ldots \cap (P_k \cap \mathbf{k}[Z_{\bar{L}}]) = \left(\bar{P}_1\right)_{\bar{L}} \cap \ldots \cap \left(\bar{P}_k\right)_{\bar{L}}$$

is a prime decomposition of the ideal $\bar{I}_{\bar{L}}$. Suppose that it is not minimal. Then, since $(\mathcal{C}) : H_{\mathcal{C}}^{\infty} = \bar{I}_{\bar{L}} \cap \mathbf{k}[Z_{\mathcal{C}}]$,

$$(\mathcal{C}) : H_{\mathcal{C}}^{\infty} = \left(\left(\bar{P}_1\right)_{\bar{L}} \cap \mathbf{k}[Z_{\mathcal{C}}]\right) \cap \ldots \cap \left(\left(\bar{P}_k\right)_{\bar{L}} \cap \mathbf{k}[Z_{\mathcal{C}}]\right)$$

is a prime decomposition of the ideal $(\mathcal{C}) : H_{\mathcal{C}}^{\infty}$, which is also not minimal. But the latter contradicts the fact that $(\bar{P}_i)_{\bar{L}} \cap \mathbf{k}[Z_{\mathcal{C}}] = P_i \cap \mathbf{k}[Z_{\mathcal{C}}]$, $1 \leqslant i \leqslant k$, and

$$(\mathcal{C}) : H_{\mathcal{C}}^{\infty} = (P_1 \cap \mathbf{k}[Z_{\mathcal{C}}]) \cap \ldots \cap (P_k \cap \mathbf{k}[Z_{\mathcal{C}}])$$

is the minimal prime decomposition. □

4.2. A bound for characteristic sets of prime components

Let $I = [\mathcal{C}] : H_{\mathcal{C}}^{\infty}$ be a characterizable differential ideal with a characteristic set $\mathcal{C}$ w.r.t. a ranking $\leq$. Let $L = \mathrm{ld}_{\leq}\,\mathcal{C}$, and let $\bar{L}$ be a differential prolongation of L. From the previous section we know that the prolongation ideal $\bar{I}_{\bar{L}}$ is characterizable (Lemma 10) and its minimal prime components correspond to the minimal prime components of I (Lemma 12). We would like to find a sufficient differential prolongation $\bar{L}$ such that the minimal prime components of $\bar{I}_{\bar{L}}$ contain differential characteristic sets of the corresponding minimal prime components of I w.r.t. any other ranking $\leq'$.

First of all, according to (Hubert, 2003b, Theorem 4.13), a differential characteristic set of a minimal prime component of I coincides with an algebraic characteristic set of the corresponding minimal prime component of the ideal $(\mathcal{C}) : H_{\mathcal{C}}^{\infty}$. This implies that every minimal prime component P of I has a characteristic set $\mathcal{C}_P$ satisfying the bound $m_y(\mathcal{C}_P) \leqslant m_y(\mathcal{C})$ on the orders of derivatives of any differential indeterminate $y \in Y$ occurring in $\mathcal{C}_P$.

For the ordinary case, as was shown in Section 3.1, we thus have a bound $M_{\mathcal{C}}$ on the orders of derivatives occurring in the canonical characteristic sets of the minimal prime components of I w.r.t. any other ranking $\leq'$. For the partial differential case, such a bound is not known, but let us assume that we can compute such a bound $M_{\mathcal{C}}$ also for the partial case [5]. We need to assume that $M_{\mathcal{C}} \geqslant m_y(\mathcal{C})$ for all $y \in Y$.

Let

$$\bar{L} = \{\theta u \mid u \in L,\ \operatorname{ord}\theta u \leqslant M_{\mathcal{C}}\} \tag{4}$$

be the differential prolongation of L up to the order $M_{\mathcal{C}}$. According to Lemma 12, the minimal prime components of $\bar{I}_{\bar{L}}$ contain all polynomials of the corresponding minimal prime components of I of order less than or equal to $M_{\mathcal{C}}$. Thus, they also contain the canonical characteristic sets of the corresponding minimal prime components of I w.r.t. any other ranking $\leq'$. In what follows, we will denote the above differential prolongation $\bar{I}_{\bar{L}}$ simply by $\bar{I}$. Applying Lemma 11, we compute a characteristic set of $\bar{I}$ w.r.t. $\leq$.

4.3. Algebraic bi-characteristic decomposition

So, we have the differential ideal I which is characterizable w.r.t. the ranking $\leq$ and would like to give a characteristic decomposition of I w.r.t. $\leq'$. We have constructed the prolongation algebraic ideal $\bar{I}$ which is characterizable w.r.t. $\leq$ with a characteristic set $\mathcal{A}$ given by formula (3). Let

$$\bar{I} = \bar{J}_1 \cap \ldots \cap \bar{J}_k \tag{5}$$

be a bi-characteristic decomposition of $\bar{I}$ w.r.t. $\leq$ and $\leq'$. That is, each component $\bar{J}_i$, $1 \leqslant i \leqslant k$, is an algebraic ideal characterizable w.r.t. both rankings with the canonical characteristic sets $\mathcal{A}_i$ and $\mathcal{B}_i$ w.r.t. $\leq$ and $\leq'$, respectively.

Let us discuss how one can construct such a decomposition. Algorithm 3 does the following. Given a characterizable algebraic ideal I with the characterizing set $\mathcal{C}$ w.r.t. $\leq_s$, it first computes its (possibly redundant) algebraic characteristic decomposition w.r.t. $\leq_t$ via the procedure

$$\textsf{Algebraic-characteristic-decomposition}(\mathcal{C}, \leq_s, \leq_t).$$

This procedure can be performed, for example, by applying the *Triade* algorithm (Moreno Maza, 1999), which is implemented in the `RegularChains` library in MAPLE (Lemaire et al., 2005). A parallel implementation of this algorithm, on a shared memory machine in `Aldor` is also in progress (Moreno Maza and Xie, 2006).

If one of the characterizable components turns out to be equal to I (note that equality of characterizable algebraic ideals can be checked, e.g., by computing their Gröbner

[5] Of course, $M_{\mathcal{C}}$ can be obtained by computing characteristic sets of the prime components w.r.t. the target ranking, but this would clearly defeat our purpose: we need a bound that can be computed from $\mathcal{C}$ relatively easily.

Algorithm 3 Algebraic-Bicharacteristic-Decomposition $(\mathcal{C}, \leq, \leq')$

INPUT: *characterizing set $\mathcal{C}$ of a characterizable algebraic ideal I w.r.t. an ordering $\leq$ on variables and another ordering $\leq'$*

OUTPUT: *a finite set $T = \{(\mathcal{C}_i, \mathcal{D}_i) \mid i \in \mathfrak{I}\}$, where for every $i \in \mathfrak{I}$, $\mathcal{C}_i$ and $\mathcal{D}_i$ are algebraic characterizing sets of the same ideal I_i w.r.t. $\leq$ and $\leq'$, respectively, and $I = \cap_{i\in\mathfrak{I}} I_i$*

$\leq_s := \leq$, $\leq_t := \leq'$
$\mathfrak{C} := \{\mathcal{C}\}$, $T := \varnothing$
while $\mathfrak{C} \neq \varnothing$ **do**
  $U := \mathfrak{C}$, $\mathfrak{C} := \varnothing$
  for $\mathcal{C} \in U$ **do**
    $J := (\mathcal{C}) : H_{\mathcal{C}}^{\infty}$ *w.r.t.* $\leq_s$
    $\mathfrak{D} :=$Algebraic-characteristic-decomposition$(\mathcal{C}, \leq_s, \leq_t)$
    if $\exists\, \mathcal{D} \in \mathfrak{D}$ *such that* $J = (\mathcal{D}) : H_{\mathcal{D}}^{\infty}$ *w.r.t.* $\leq_t$ **then**
      if $\leq_s = \leq$ **then** $T := T \cup \{(\mathcal{C}, \mathcal{D})\}$ **else** $T := T \cup \{(\mathcal{D}, \mathcal{C})\}$
    else $\mathfrak{C} := \mathfrak{C} \cup \mathfrak{D}$
    end if
  end for
  if $\leq_s = \leq$ **then** $\leq_s := \leq'$, $\leq_t := \leq$ **else** $\leq_s := \leq$, $\leq_t := \leq'$
end while
return T

bases), then I is bi-characterizable; in this case the algorithm terminates and outputs T consisting of a single pair $(\mathcal{C}, \mathcal{D})$ of characterizing sets of I w.r.t. $\leq$ and $\leq'$, respectively. If all characterizable components of I contain it strictly, then, for each characterizable component, we compute its characteristic decomposition w.r.t. $\leq$ and repeat the above strategy.

Correctness of the algorithm follows from the fact that, at each iteration of the **while**-loop, $\mathfrak{C} \cup T$ provides a characteristic decomposition of I w.r.t. $\leq_s$ and T satisfies the requirements of the output. Termination follows from the Noetherian property of the polynomial ring, i.e., that every sequence of strictly nested polynomial ideals is finite.

We note that the components $\bar{J}_i$, for which $\mathrm{ld}_{\leq} \mathcal{A}_i \neq \mathrm{ld}_{\leq} \mathcal{A}$, are redundant, i.e., they can be excluded from the right-hand side of (5) without affecting the intersection. Indeed, if

$$\bar{I} = \bar{P}_1 \cap \ldots \cap \bar{P}_l$$

is the minimal prime decomposition of $\bar{I}$, and

$$\bar{J}_i = \bar{Q}_{i,1} \cap \ldots \cap \bar{Q}_{i,l_i}$$

are the minimal prime decompositions of $\bar{J}_i$, $1 \leqslant i \leqslant k$, then a component $\bar{J}_i$ is redundant, if none of $\bar{P}_j$, $1 \leqslant j \leqslant l$, can be found among $\bar{Q}_{i,t}$, $1 \leqslant t \leqslant l_i$. But this is the case if $\mathrm{ld}_{\leq} \mathcal{A}_i \neq \mathrm{ld}_{\leq} \mathcal{A}$, since by (Hubert, 2003b, Theorem 4.13) the characteristic sets of $\bar{P}_j$ have leaders $\mathrm{ld}_{\leq} \mathcal{A}$, while the characteristic sets of $\bar{Q}_{i,t}$ have leaders $\mathrm{ld}_{\leq} \mathcal{A}_i$. Therefore, we can assume that for all i, $1 \leqslant i \leqslant k$,

$$\mathrm{ld}_{\leq} \mathcal{A}_i = \mathrm{ld}_{\leq} \mathcal{A}.$$

We prove then that *every* minimal prime component of $\bar{J}_i$ is a minimal prime component of $\bar{I}$. Indeed, every $\bar{Q}_{i,t}$ is a prime ideal containing $\bar{I}$. Suppose that $\bar{Q}_{i,t}$ is not minimal, i.e., there is a minimal prime component $\bar{P}_j$ of $\bar{I}$ such that $\bar{P}_j \subsetneq \bar{Q}_{i,t}$. But the latter strict inclusion is impossible according to the following Lemma 13 and Remark 14.

Lemma 13 *Let P and Q be two prime differential ideals whose characteristic sets w.r.t. $\leq$ have the same sets of leaders. Then $P \subseteq Q$ implies $P = Q$.*

Proof. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be these characteristic sets. We have $P = [\mathcal{C}_1] : H_{\mathcal{C}_1}^{\infty}$ and $Q = [\mathcal{C}_2] : H_{\mathcal{C}_2}^{\infty}$. Consider the restricted ideals $\mathfrak{p} = (\mathcal{C}_1) : H_{\mathcal{C}_1}^{\infty}$ and $\mathfrak{q} = (\mathcal{C}_2) : H_{\mathcal{C}_2}^{\infty}$ in the Noetherian ring $\mathbf{k}[L, N(\mathcal{C}_1, \mathcal{C}_2)]$, where $N(\mathcal{C}_1, \mathcal{C}_2)$ is the set of non-leading variables appearing in both $\mathcal{C}_1$ and $\mathcal{C}_2$. From (Hubert, 2000, Theorem 3.2) it follows that both $\mathfrak{p}$ and $\mathfrak{q}$ are of dimension $|N(\mathcal{C}_1, \mathcal{C}_2)|$.

Take any $f \in \mathfrak{p}$. It is partially reduced w.r.t. both $\mathcal{C}_1$ and $\mathcal{C}_2$ (which are coherent and autoreduced) and belongs to $P \subset Q$. By the Rosenfeld lemma $f \in \mathfrak{q}$. Hence, $\mathfrak{p} \subset \mathfrak{q}$. Since the ideals $\mathfrak{p}$ and $\mathfrak{q}$ are prime and their Krull dimensions are equal to the same number $|N(\mathcal{C}_1, \mathcal{C}_2)|$, we obtain $\mathfrak{p} = \mathfrak{q}$.

Thus, $\mathcal{C}_2 \subset \mathfrak{p} \subset P$. Moreover, the elements of $H_{\mathcal{C}_2}$ do not belong to $Q \supseteq \mathfrak{q} = \mathfrak{p}$; since they are partially reduced w.r.t. $\mathcal{C}_2$ (and, therefore w.r.t. $\mathcal{C}_1$, given that $\mathrm{ld}\, \mathcal{C}_1 = \mathrm{ld}\, \mathcal{C}_2$), by the Rosenfeld Lemma, the elements of $H_{\mathcal{C}_2}$ do not belong to P. Thus,

$$Q = [\mathcal{C}_2] : H_{\mathcal{C}_2}^{\infty} \subseteq P : H_{\mathcal{C}_2}^{\infty} = P,$$

which, together with the given inclusion $P \subseteq Q$ implies $P = Q$. $\square$

Remark 14 *In the above lemma, one can assume that the set of derivations is empty, hence the statement also holds for algebraic ideals.*

To summarize, for every bi-characterizable component $\bar{J}_i$, there exists a subset $T_i \subset \{1, \ldots, l\}$ such that

$$\bar{J}_i = \bigcap_{j \in T_i} \bar{P}_j$$

is the minimal prime decomposition of $\bar{J}_i$. Moreover, equality (5) implies that

$$\bigcup_{i=1}^{l} T_i = \{1, \ldots, l\}.$$

4.4. Constructing differential characterizable components from the algebraic ones

Fix any of the above algebraic bi-characterizable components $\bar{J} = \bar{J}_i$, where $1 \leqslant i \leqslant k$; we have a set of indices $T = T_i \subset \{1, \ldots, l\}$ such that

$$\bar{J} = \bigcap_{j \in T} \bar{P}_j.$$

As above, let $\mathcal{A} = \mathcal{A}_i$ and $\mathcal{B} = \mathcal{B}_i$ be the canonical characteristic sets of $\bar{J}$ w.r.t. $\leq$ and $\leq'$, respectively. According to Lemma 12, each minimal prime component $\bar{P}_j$ of $\bar{I}$ is a prolongation ideal of the corresponding minimal prime component P_j of I, that is,

$$\bar{P}_j = P_j \cap \mathbf{k}\left[\bar{L} \cup N\right],$$

where $I = \bigcap_{j=1}^{l} P_j$ is the minimal prime decomposition of I. Since $\mathcal{B}$ is a characterizing set of $\bar{J}$ w.r.t. $\leq'$, the initials and separants of $\mathcal{B}$ w.r.t. $\leq'$ are not zero-divisors modulo $\bar{J}$, that is, they do not belong to the minimal prime components $\bar{P}_j$, $j \in T$. Since $\mathcal{B}$, as well as $H_{\mathcal{B}}$, is a subset of $\mathbf{k}\left[\bar{L} \cup N\right]$, we have, therefore,

$$H_{\mathcal{B}} \cap P_j = \varnothing$$

for all $j \in T$. Let $\mathcal{T} \subset \mathcal{B}$ be the weak d-triangular subset of $\mathcal{B}$ of the least rank w.r.t. $\leq'$. Since $H_{\mathcal{T}} \subset H_{\mathcal{B}}$, we also have

$$H_{\mathcal{T}} \cap P_j = \varnothing$$

for all $j \in T$. Thus, we have

$$[\mathcal{T}] : H_{\mathcal{T}}^{\infty} \subset P_j$$

for all $j \in T$. In particular, this implies that

$$[\mathcal{T}] : H_{\mathcal{T}}^{\infty} \neq (1).$$

Let $\mathcal{D}$ be the result of differential autoreduction of $\mathcal{T}$ w.r.t. $\leq'$, that is,

$$\mathcal{D} = \{\mathsf{d\text{-}rem}(f, \mathcal{T} \setminus \{f\}) \mid f \in \mathcal{T}\}.$$

The set $\mathcal{D}$ is differentially autoreduced. We will show that, in fact, $\mathcal{D} = \mathcal{T}$. By definition of differential remainder, $\mathcal{D} \subset [\mathcal{T}]$. By (Golubitsky et al., 2008, Lemma 4), since $[\mathcal{T}] : H_{\mathcal{T}}^{\infty} \neq (1)$, we have $\operatorname{rk}_{\leq'} \mathcal{D} = \operatorname{rk}_{\leq'} \mathcal{T}$ and, moreover, $H_{\mathcal{D}} \subset H_{\mathcal{T}}^{\infty} + [\mathcal{T}]$. Therefore,

$$[\mathcal{D}] : H_{\mathcal{D}}^{\infty} \subset [\mathcal{T}] : H_{\mathcal{T}}^{\infty} \subset P_j, \;\; j \in T. \tag{6}$$

We will show that $\mathcal{D}$ is a characteristic set of the ideal $[\mathcal{D}] : H_{\mathcal{D}}^{\infty}$ w.r.t. $\leq'$ by proving that every polynomial in the intersection $\bigcap_{j \in T_i} P_j$ reduces w.r.t. $\mathcal{D}$ to zero. Given (6), this will also imply that

$$[\mathcal{D}] : H_{\mathcal{D}}^{\infty} = \bigcap_{j \in T} P_j. \tag{7}$$

Take any polynomial $f \in \bigcap_{j \in T} P_j$, and let $\bar{f} = \mathsf{d\text{-}rem}(f, \mathcal{D})$, where the pseudo-remainder is computed w.r.t. $\leq'$. Since $\mathcal{D} \subset [\mathcal{T}] \subset P_j$, $j \in T$, we have

$$\bar{f} \in \bigcap_{j \in T} P_j.$$

Let $\mathcal{F}_j$ be the canonical characteristic set of P_j w.r.t. $\leq'$, and let $\bar{\mathcal{F}}_j$ be the canonical algebraic characteristic set of the corresponding prolongation ideal $\bar{P}_j$. We have shown in Section 4.2 that $\bar{P}_j$ contains $\mathcal{F}_j$. Thus, from Lemma 7 it follows that $\mathcal{F}_j$ is the weak d-triangular subset of $\bar{\mathcal{F}}_j$ of the least rank w.r.t. $\leq'$. On the other hand, since $\bar{P}_j$ is a minimal prime component of $\bar{J}$, according to (Hubert, 2003b, Theorem 4.13), $\mathrm{ld}_{\leq'} \bar{\mathcal{F}}_j = \mathrm{ld}_{\leq'} \mathcal{B}$. This implies that

$$\mathrm{ld}_{\leq'} \mathcal{F}_j = \mathrm{ld}_{\leq'} \mathcal{T} = \mathrm{ld}_{\leq'} \mathcal{D}.$$

That is, the fact that $\bar{f}$ is reduced w.r.t. $\mathcal{D}$ implies that it is partially reduced w.r.t. $\mathcal{F}_j$. By the Rosenfeld Lemma,

$$\bar{f} \in (\mathcal{F}_j) : H_{\mathcal{F}_j}^{\infty} \subset (\bar{\mathcal{F}}_j) : H_{\bar{\mathcal{F}}_j}^{\infty} = \bar{P}_j, \quad j \in T$$

i.e., $\bar{f} \in \bar{J}$. Now, the fact that $\bar{f}$ is reduced w.r.t. $\mathcal{D}$ implies that it is algebraically reduced w.r.t. $\mathcal{B}$. Since the latter is a characteristic set of $\bar{J}$, we obtain $\bar{f} = 0$ and the required equality (7).

Now we see that the ideal $[\mathcal{D}] : H_{\mathcal{D}}^{\infty}$ is characterizable w.r.t. $\leq'$. The canonical characteristic set of this ideal w.r.t. $\leq'$ is contained in each minimal prime component of the ideal $(\mathcal{D}) : H_{\mathcal{D}}^{\infty}$, therefore it is also contained in every $\bar{P}_j$, $j \in T$, and hence in $\bar{J}$. The ideal $\bar{J}$ is contained in $[\mathcal{D}] : H_{\mathcal{D}}^{\infty}$. Thus, by Lemma 7, the canonical characteristic set of $[\mathcal{D}] : H_{\mathcal{D}}^{\infty}$ is equal to the weak d-triangular subset of $\mathcal{B}$ of the least rank w.r.t. $\leq'$. That is, we have

$$\mathcal{D} = \mathcal{T}$$

which is (w.r.t. the ranking $\leq'$) the canonical characteristic set of the characterizable differential ideal

$$[\mathcal{D}] : H_{\mathcal{D}}^{\infty}.$$

4.5. The final characteristic decomposition

In the previous section, we have shown that for each bi-characterizable component $\bar{J}_i$, $1 \leqslant i \leqslant l$, of $\bar{I}$ with the canonical characteristic set $\mathcal{B}_i$ w.r.t. $\leq'$, if $\mathcal{D}_i$ is the weak d-triangular subset of $\mathcal{B}_i$ of the least rank, then it is the canonical characteristic set of the ideal $[\mathcal{D}_i] : H_{\mathcal{D}_i}^{\infty}$. We have also shown that

$$[\mathcal{D}_i] : H_{\mathcal{D}_i}^{\infty} = \bigcap_{j \in T_i} P_j.$$

Thus, since $\bigcup_{i=1}^{l} T_i = \{1, \ldots, l\}$, the following intersection

$$\bigcap_{i=1}^{l} [\mathcal{D}_i] : H_{\mathcal{D}_i}^{\infty}$$

is a characteristic decomposition of $I = P_1 \cap \ldots \cap P_l$ w.r.t. $\leq'$. This concludes the algebraic computation of a characteristic decomposition of I w.r.t. the target ranking, which we summarize in Algorithm 4.

Now, in order to convert a characteristic decomposition

$$I = \bigcap_{i=1}^{p} [\mathcal{C}_i] : H_{\mathcal{C}_i}^{\infty}$$

of a radical differential ideal I w.r.t. $\leq$ to a ranking $\leq'$, one just applies Algorithm 4 to each characterizable component $[\mathcal{C}_i] : H_{\mathcal{C}_i}^{\infty}$ and then collects all the results together in a single intersection.

Algorithm 4 Convert_Characterizable $(\mathcal{C}, \leq, \leq')$

INPUT: *set $\mathcal{C}$ which characterizes the ideal $[\mathcal{C}] : H_{\mathcal{C}}^{\infty}$ w.r.t. the input ranking $\leq$ and has leading variables $y_1, \ldots, y_k$ and a target ranking $\leq'$.*

OUTPUT: *characteristic decomposition of $[\mathcal{C}] : H_{\mathcal{C}}^{\infty}$ w.r.t. $\leq'$.*

$M_{\mathcal{C}} := \min\left(|\mathcal{C}| \cdot \max_{C \in \mathcal{C}} \operatorname{ord} C,\ \frac{(n-1)!}{(n-|\mathcal{C}|-1)!} \cdot M(\mathcal{C}) \right)$

$m_i := M_{\mathcal{C}},\ 1 \leqslant i \leqslant k$

$\mathcal{A} :=$ Differentiate&Autoreduce $\left(\mathcal{C}, \{m_i\}_{i=1}^{k}\right)$

$\mathfrak{D} :=$ Bi-characterizable_Canonical_Decomposition $((\mathcal{A}) : H_{\mathcal{A}}^{\infty}, \leq, \leq')$

$\mathfrak{C} := \{\text{minimal d} - \text{triangular subset } (\mathcal{D}, \leq') \mid \mathcal{D} \in \mathfrak{D}\}$

return $\mathfrak{C}$

5. Canonical characteristic sets

In this section, we prove correctness of the definition of the canonical characteristic set (see Definition 1) and list some properties of this set, preparing ourselves for the proof of the bound in the next section. Throughout this section we assume that a ranking is fixed.

The difference of our definition from that of (Boulier and Lemaire, 2000) is that we did not require the canonical characteristic set to be a characterizing set of the differential ideal. Thus, (Boulier and Lemaire, 2000) implies the existence of the canonical characteristic set (for characterizable differential ideals) in the sense of Definition 1. Its uniqueness is shown in (Boulier and Lemaire, 2000, Theorem 3). We prove this below for arbitrary differential ideals.

We have also replaced the set $N_{\mathcal{C}}$ of non-leaders effectively occurring in $\mathcal{C}$ by the set $N = \Theta Y \setminus \Theta L$ of all non-leaders (where L is the set of leaders of $\mathcal{C}$). This replacement yields an equivalent definition, which is more convenient, because it provides the ring $\mathbf{k}(N)[L]$ independently of the choice of the characteristic set $\mathcal{C}$, while the field of coefficients $\mathbf{k}(N_{\mathcal{C}})$ of the polynomial ring $\mathbf{k}(N_{\mathcal{C}})[L]$ depends on $\mathcal{C}$.[6]

Proposition 15 *Let $\mathcal{C}$ be a characteristic set of a characterizable differential ideal I, whose initials do not depend on the leaders of $\mathcal{C}$. Then $\mathcal{C}$ characterizes the ideal I, that is, $I = [\mathcal{C}] : H_{\mathcal{C}}^{\infty}$.*

Proof. By (Hubert, 2000, Theorems 3.2 and 4.5), for every minimal prime component P of I, the set of leaders of any characteristic set $\mathcal{D}$ of P coincides with $\operatorname{ld} \mathcal{C}$. Since the initials of $\mathcal{C}$ do not depend on the leaders of $\mathcal{C}$, they are reduced w.r.t. $\mathcal{D}$ and, hence, do not belong to P. Thus, the initials of $\mathcal{C}$ are not zero-divisors modulo I.

Hence, the initials of $\mathcal{C}$ are not zero-divisors modulo the algebraic ideal $\bar{I} = I \cap [N \cup L]$, that is, $\bar{I} : I_{\mathcal{C}}^{\infty} = I$. By the Rosenfeld Lemma, $\bar{I} \subseteq (\mathcal{C}) : I_{\mathcal{C}}^{\infty}$. Since $\mathcal{C} \subset \bar{I}$, we obtain therefore

$$\bar{I} \subseteq (\mathcal{C}) : I_{\mathcal{C}}^{\infty} \subseteq \bar{I} : I_{\mathcal{C}}^{\infty} = \bar{I}.$$

[6] The idea of constructing a canonical field of coefficients by considering the infinite set of all non-leading derivatives was communicated to the first author by E. Hubert.

Since I is characterizable, it is radical, whence so is $\bar{I} = I \cap \mathbf{k}[N \cup L]$. Thus, by (Hubert, 2000, Proposition 3.3), we have

$$\bar{I} = (\mathcal{C}) : I_{\mathcal{C}}^{\infty} = (\mathcal{C}) : H_{\mathcal{C}}^{\infty},$$

that is, $\bar{I} = (\mathcal{C}) : H_{\mathcal{C}}^{\infty}$ is a characterizable algebraic ideal characterized by $\mathcal{C}$. According to (Hubert, 2000, Lemma 6.1), the latter implies that the differential ideal $[\mathcal{C}] : H_{\mathcal{C}}^{\infty}$ is also characterizable and characterized by $\mathcal{C}$.

Let $\mathcal{A}$ be a characterizing set for I, that is, $\mathcal{A}$ is a characteristic set of I such that $[\mathcal{A}] : H_{\mathcal{A}}^{\infty} = I$. Since $\mathcal{C}$ is a characteristic set of I, we have

$$I \subseteq [\mathcal{C}] : H_{\mathcal{C}}^{\infty}.$$

In particular,

$$\mathcal{A} \subset I \subseteq [\mathcal{C}] : H_{\mathcal{C}}^{\infty}.$$

Thus, for all $f \in [\mathcal{C}] : H_{\mathcal{C}}^{\infty}$, we have

$$\bar{f} = \mathsf{d\text{-}rem}(f, \mathcal{A}) \in [\mathcal{C}] : H_{\mathcal{C}}^{\infty}.$$

Since $\mathcal{C}$ characterizes $[\mathcal{C}] : H_{\mathcal{C}}^{\infty}$, either $\bar{f} = 0$, or $\bar{f}$ is reducible w.r.t. $\mathcal{C}$. But the latter is impossible, because $\operatorname{rk} \mathcal{A} = \operatorname{rk} \mathcal{C}$ (since both are characteristic sets of I), and $\bar{f}$ is reduced w.r.t. $\mathcal{A}$. Therefore, $\bar{f} = 0$, which means that every $f \in [\mathcal{C}] : H_{\mathcal{C}}^{\infty}$ reduces w.r.t. $\mathcal{A}$ to zero and

$$[\mathcal{C}] : H_{\mathcal{C}}^{\infty} \subseteq [\mathcal{A}] : H_{\mathcal{A}}^{\infty} = I.$$

This concludes the proof. □

The following statement can be obtained by combining Lemmas 3.5 and 3.9 from (Hubert, 2000), yet it appears to be easier to prove it directly.

Proposition 16 *Let $\mathcal{C}$ be a characteristic set of a differential ideal I, whose initials do not depend on the leaders of $\mathcal{C}$. Then $\mathcal{B} = \{f/\mathbf{i}_f \mid f \in \mathcal{C}\}$ is the reduced Gröbner basis of the zero-dimensional algebraic ideal J generated by $I \cap \mathbf{k}[N \cup L]$ in $\mathbf{k}(N)[L]$ w.r.t. the lexicographic ordering on monomials over L induced by the ranking.*

Proof. Every element of the ideal $I \cap \mathbf{k}[N \cup L]$ algebraically pseudo-reduces w.r.t. $\mathcal{C}$ to zero. Since the initials of $\mathcal{C}$ are in $\mathbf{k}(N)$, the ideal J is generated by $\mathcal{B}$ in $\mathbf{k}(N)[L]$. Also, the leading monomials of $\mathcal{B}$ w.r.t. the induced lexicographic ordering are elements of $\operatorname{rk} \mathcal{C}$, whence $\mathcal{B}$ is autoreduced w.r.t. the induced lexicographic ordering, $\mathcal{B}$ is a Gröbner basis (since its leading monomials are pairwise relatively prime), and the ideal J in $\mathbf{k}(N)[L]$ is zero-dimensional by (Adams and Loustanau, 1996, Theorem 2.2.7). □

Corollary 17 *Let $\mathcal{C}$ be a characteristic set of a differential ideal I, whose initials do not depend on the leaders of $\mathcal{C}$. Then any other characteristic set of I, whose initials do not depend on the leaders, can be obtained via multiplying/dividing the elements of $\mathcal{C}$ by some polynomials from $\mathbf{k}[N]$.*

Corollary 18 *If a canonical characteristic set $\mathcal{C}$ exists for a differential ideal I, it is unique, and every other characteristic set of I, whose initials do not depend on the leaders, can be obtained via multiplying the elements of $\mathcal{C}$ by some polynomials from $\mathbf{k}[N]$.*

The following property of canonical characteristic sets will be used further in Lemma 26 and will help us to obtain the bound on the orders of the elements of canonical characteristic sets. The next section will tell about this in detail.

Proposition 19 *Let $\mathcal{C} = C_1, \ldots, C_p$ be the canonical characteristic set of a characterizable differential ideal I. Let v be a derivative appearing in some C_i, $1 \leqslant i \leqslant p$. Then,*

$$\frac{\partial C_i}{\partial v} \notin I.$$

Proof. Suppose that $\dfrac{\partial C_i}{\partial v} \in I$. Then v appears effectively in the initial $\mathbf{i}_{C_i}$. Indeed, suppose that v is not in $\mathbf{i}_{C_i}$, then $\dfrac{\partial C_i}{\partial v}$ is not reducible w.r.t. $\mathcal{C}$. This contradicts the fact that $\mathcal{C}$ is a characteristic set of I and $\dfrac{\partial C_i}{\partial v} \in I$. Now, since v appears effectively in $\mathbf{i}_{C_i}$, the set

$$\mathcal{C}' = \mathcal{C} \setminus \{C_i\} \cup \left\{\frac{\partial C_i}{\partial v}\right\}$$

is autoreduced and has the same rank as $\mathcal{C}$, hence $\mathcal{C}'$ is a characteristic set of I. Moreover, the initial of $\dfrac{\partial C_i}{\partial v}$ is equal to $\dfrac{\partial \mathbf{i}_{C_i}}{\partial v}$, hence it does not depend on the leaders of $\mathcal{C}$. Yet $\dfrac{\partial C_i}{\partial v}$ is not a multiple of C_i, which contradicts Corollary 18. □

6. Main tool: bounds for the orders of characteristic sets

Here are the main steps towards the bound for the orders of elements of the canonical characteristic set of a characterizable differential ideal:

- existence of a bounded characteristic set for prime differential ideals (Section 6.2),
- extension of the existence result to characterizable ideals (Section 6.3),
- reduction to canonical characteristic sets (Section 6.4).

The first step is the most technically difficult one and requires preparation. The last two steps are easier.

6.1. Preparation

Let $R = \mathbf{k}\{y_1, \ldots, y_n\}$ with $\Delta = \{\delta\}$. So, we are in the ordinary case. *Differential dimension* of a differential ideal I is the maximal number q such that $I \cap \mathbf{k}\{y_{i_1}, \ldots, y_{i_q}\} = \{0\}$. Recall that the order of a differential polynomial f is the maximal order of derivatives appearing effectively in f. Fix any differential ranking. Let $\mathcal{A} = A_1, \ldots, A_p$ be an autoreduced set. Define the *order* of $\mathcal{A}$ by the following equality:

$$\operatorname{ord} \mathcal{A} = \operatorname{ord} A_1 + \ldots + \operatorname{ord} A_p.$$

Let an *orderly* differential ranking be fixed. If $\mathcal{C}$ is a characteristic set of a *prime* differential ideal P then, by definition, the *order* of the ideal P equals $\operatorname{ord} \mathcal{C}$ and denoted by $\operatorname{ord} P$.

Denote by $P(s)$ the set of elements of P whose order is less than or equal to s. The set $P(s)$ is a prime algebraic ideal in the corresponding polynomial ring. According to

(Kolchin, 1973, II.12, Theorem 6) or (Kondratieva et al., 1999, Theorems 5.4.1, 5.4.4) the dimension of $P(s)$ is a polynomial in s for $s \geqslant h = \operatorname{ord} P$. More precisely,

$$\dim P(s) = q(s+1) + \operatorname{ord} P,$$

where q is the differential dimension of the ideal P. Moreover, $q = n - p$, where p is the number of elements of a characteristic set of the ideal P w.r.t. any orderly ranking. Thus, the numbers $\operatorname{ord} P$ and p do not depend on the choice of an orderly ranking. We are going to define the order of a characterizable differential ideal, and we should be very careful because of the following example.

Example 20 *Consider the radical differential ideal $\{x(x+y')\} = I$ characterizable w.r.t. the elimination ranking $x >_{el} y$. While $I = [x] \cap [x+y']$ and the leaders of x and $x+y'$ w.r.t. the ranking are the same, the orders of the components are different. This is because the ideal I is not characterizable w.r.t. any orderly ranking.*

Hence, we give the following definition.

Definition 21 *For a characterizable differential ideal $I = \bigcap_{i=1}^{k} P_i$, where P_i are minimal differential prime components of I, define*

$$\operatorname{ord} I = \max_{1 \leqslant i \leqslant k} \operatorname{ord} P_i.$$

Remark 22 *The theory of differential dimensional polynomials is due to Johnson (1969); Kolchin (1973). Carrà Ferro and Sit continued to develop this subject (Carrà Ferro, 1987, 1989; Sit, 1978). Many of the results concerning differential dimension polynomials are summarized in (Kondratieva et al., 1999). The latter book also presents algorithms for computing these polynomials.*

Lemma 23 *(Sadik, 2000, Proposition 17) Consider a prime differential ideal P of differential dimension q and order h. For every subset $\{y_{i_1}, \ldots, y_{i_{q+1}}\}$ of $\{y_1, \ldots, y_n\}$, the ideal P contains a differential polynomial in the indeterminates $\{y_{i_1}, \ldots, y_{i_{q+1}}\}$ of order less than or equal to h.*

It is not possible to bound the orders of elements of an arbitrary characteristic set. For example, consider the ideal $[x] \in \mathbf{k}\{x, y\}$ and the elimination ranking with $x > y$. Then the set $y^{(q)}x$ is a characteristic set of the ideal $[x]$ for any $q \geqslant 0$. In order to avoid this problem, the concept of *irreducible* characteristic set is introduced in (Sadik, 2000) right before (Sadik, 2000, Lemma 19) for prime differential ideals:

Definition 24 *Let $\mathcal{A} = A_1, \ldots, A_p$ be an autoreduced set and V_{i-1} be the set of all derivatives appearing in the polynomials $A_1, \ldots, A_{i-1}$, $I_{i-1} := I_{A_1,\ldots,A_{i-1}}$, and U_i be the set of derivatives from A_i that are not in V_{i-1}. Consider the unique factorization domain*

$$R_i = \operatorname{Quot}(\mathbf{k}[V_{i-1}] \,/\, (A_1, \ldots, A_{i-1}) : I_{i-1}^{\infty})[U_i],$$

where Quot *means the total ring of quotients. The set $\mathcal{A}$ is called* **irreducible** *if A_i is irreducible in R_i for all i, $1 \leqslant i \leqslant p$.*

The key property of irreducible characteristic sets, which we need for the proof of our bound, is formulated in (Sadik, 2000, Lemma 20). In addition, our proof of the bound will require existence of a characteristic set satisfying the statement of the above Proposition 19, which is a property of canonical characteristic sets. Lemma 26 below provides the necessary combination of the two properties. Note that it does not imply that the canonical characteristic set must be irreducible, which, in fact, is not always the case:

Example 25 *Consider the ideal* $I = \{x^2-t, (zx+1)y+1\} \subset \mathbf{k}\{x,t,z,y\}$ *and any ranking such that* $y > z > x > t$. *The set* $x^2 - t$, $(zx+1)y+1$ *is an irreducible characteristic set of* I. *The canonical characteristic set of* I, *which is equal to* $x^2 - t$, $(z^2t-1)y + zx - 1$, *is not irreducible because*

$$\begin{aligned}(z^2t-1)y + zx - 1 &= (z^2x^2-1)y + zx - 1 = \\ &= (zx-1)(zx+1)y + (zx-1) = \\ &= (zx-1)((zx+1)y+1)\end{aligned}$$

in the polynomial ring $\mathrm{Quot}\left(\mathbf{k}[x,t]/(x^2-t)\right)[y,\ z]$.

Lemma 26 *Let* $\mathcal{A} = A_1, \ldots, A_p$ *be an irreducible characteristic set of a prime differential ideal* P *in* $\mathbf{k}\{y_1, \ldots, y_n\}$. *Let* $i \in \{1, \ldots, p\}$, *and let* $y_t^{(s)}$ *be a derivative appearing in* A_i *and not appearing in* $A_1, \ldots, A_{i-1}$. *Then*

$$\frac{\partial A_i}{\partial y_t^{(s)}} \notin P.$$

Proof. Suppose that the second condition is failed for an irreducible characteristic set $A_1, \ldots, A_p$, which exists by (Sadik, 2000, Lemma 19). Let z be a derivative that does not appear in $\mathcal{A} = A_1, \ldots, A_{i-1}$ but does appear in A_i and satisfies $\dfrac{\partial A_i}{\partial z} \in P$. Take the canonical characteristic set $C_1, \ldots, C_p$ of the ideal P. Consider the unique factorization domain (see Definition 24):

$$R_i = \mathrm{Quot}(\mathbf{k}[V_{i-1}] \,/\, (A_1, \ldots, A_{i-1}) : I_{i-1}^{\infty})[U_i],$$

The derivative z is an indeterminate in this ring. Since $\mathcal{A}$ is irreducible, the polynomial A_i is irreducible in R_i. The polynomial C_i is reducible to zero w.r.t. $\mathcal{A}$. Hence C_i is reducible to zero w.r.t. A_i in R_i, since $A_1, \ldots, A_{i-1}$ is a characteristic set of the prime ideal

$$(A_1, \ldots, A_{i-1}) : I_{i-1}^{\infty}.$$

Then, there exists a polynomial $D_i \in R_i$ such that

$$\mathbf{i}_{A_i} C_i = D_i A_i,$$

because C_i and A_i have the same rank. Since $D_i A_i$ is divisible by C_i, A_i is irreducible, $\mathbf{i}_{A_i}$ does not depend on the leading variable of A_i, and, again, A_i and C_i have the same rank, we have $C_i = E_i A_i$ for some factor E_i of D_i. Thus, the polynomial C_i must contain the derivative z. Since the polynomial $f = \mathbf{i}_{C_i} A_i - \mathbf{i}_{A_i} C_i \in P$ is reduced w.r.t. A_i, we have

$$f \in J := (A_1, \ldots, A_{i-1}) : I_{i-1}^{\infty}.$$

Since z does not appear in $A_1, \ldots, A_{i-1}$, there exist generators $g_1, \ldots, g_k$ of the ideal J not containing this derivative. Then there exist polynomials $a_1, \ldots, a_k$ such that

$$f = a_1 g_1 + \ldots + a_k g_k.$$

Hence, $\dfrac{\partial f}{\partial z} \in J \subset P$. On the other hand,

$$\frac{\partial f}{\partial z} = \frac{\partial A_i}{\partial z}\mathbf{i}_{C_i} - \frac{\partial C_i}{\partial z}\mathbf{i}_{A_i} + \frac{\partial \mathbf{i}_{C_i}}{\partial z}A_i - \frac{\partial \mathbf{i}_{A_i}}{\partial z}C_i \equiv \frac{\partial A_i}{\partial z}\mathbf{i}_{C_i} - \frac{\partial C_i}{\partial z}\mathbf{i}_{A_i} \pmod{P}.$$

Thus, from $\dfrac{\partial A_i}{\partial z} \in P$ and Proposition 19, we have $\mathbf{i}_{A_i} \in P$. But the initials of a characteristic set of a prime ideal cannot belong to it. Contradiction. □

6.2. Bound for prime differential ideals

The theorem below generalizes (Sadik, 2000, Theorem 24) to arbitrary rankings. The induction carried out in (Sadik, 2000) appears to be applicable only in case of elimination rankings. Instead of proving the statement by induction, we construct the set $\tilde{\mathcal{C}}$ and choose a special element $C_i \in \tilde{\mathcal{C}}$. Both statements are related to the Jacobi bound (Kondratieva et al., 1982), but deducing them from the bound does not seem to be easier than the elementary proof below. After Theorem 27 we give a counter-example (Example 28) to (Sadik, 2000, Theorem 25), from which the bound could easily follow.

Theorem 27 *Let P be a prime differential ideal of order h in $\mathbf{k}\{y_1, \ldots, y_n\}$ and $\le$ be* **any** *differential ranking. Then there exists a characteristic set $\mathcal{C} = C_1, \ldots, C_p$ of the ideal P w.r.t. the ranking $\le$ such that the order in y_t of each C_i does not exceed h for all $1 \leqslant t \leqslant n$.*

Proof. For a characteristic set $\mathcal{C}$ of P denote the set

$$\{y_k \mid \theta y_k \text{ is not a leader of any } C_j,\ 1 \leqslant j \leqslant p, \theta \in \Theta\}$$

by $\mathfrak{N}$. If for some $\theta \in \Theta$ and t, $1 \leqslant t \leqslant n$, the derivative θy_t is *the leader* of some C_j then we will show that $\operatorname{ord}(C_q, y_t) \leqslant h$ for all $1 \leqslant q \leqslant p$ using Lemma 23. Indeed, since $\mathcal{C}$ is autoreduced, we have

$$\operatorname{ord}(C_q, y_t) \leqslant \operatorname{ord}\theta, \tag{8}$$

for all q, $1 \leqslant q \leqslant p$. Since $\dim P = \#\mathfrak{N}$, by Lemma 23 there exists a polynomial

$$0 \neq f \in \mathbf{k}\{y_t, \mathfrak{N}\} \cap P$$

of order not greater than h. This polynomial depends only on non-leading differential indeterminates $\mathfrak{N}$ and the leading differential indeterminate y_t. Moreover, f is reducible to zero w.r.t. $\mathcal{C}$. Hence,

$$\operatorname{ord}\theta = \operatorname{ord}(C_j, y_t) \leqslant \operatorname{ord}(f, y_t) \leqslant h. \tag{9}$$

Inequalities (8) and (9) give us

$$\operatorname{ord}(C_q, y_t) \leqslant h$$

for all q, $1 \leqslant q \leqslant p$.

Now let $y_t \in \mathfrak{N}$ and $\mathcal{C}$ be an irreducible characteristic set (see Lemma 26). Let also y_{C_j} denote the differential indeterminate such that θy_{C_j} is the leader of C_j for some $\theta \in \Theta$, that is, y_{C_j} is the leading differential indeterminate of C_j. The main idea is to reduce the polynomial *of the smallest order with respect to* y_{C_j}

$$f_j \in \mathbf{k}\{y_{C_j}, \mathfrak{N}\} \cap P$$

given by Lemma 23 w.r.t. $\mathcal{C}$. Let $u = y_{C_j}^{(r)}$ be the derivative of y_{C_j} of the highest order in f_j. If we represent f_j as a univariate polynomial in u then denote by I_{f_j} its leading coefficient. Notice that I_{f_j} does not have to be the initial of f_j w.r.t. our ranking, but we still use this notation for convenience. For instance, I_{f_j} would *be* the initial of f_j w.r.t. the elimination ranking $y_{C_j} > \mathfrak{N}$. We emphasize that

$$I_{f_j} \notin P.$$

Suppose that for some j, $1 \leqslant j \leqslant p$, we have

$$\operatorname{ord}(C_j, y_t) > h. \tag{10}$$

Since f_j is reducible to zero w.r.t. $\mathcal{C}$, we must have

$$\operatorname{ord}\left(f_j, y_{C_j}\right) \geqslant \operatorname{ord}\left(C_j, y_{C_j}\right). \tag{11}$$

Denote by "arg max ord" the *set* of all elements which provide the maximum of the order. Consider

$$\tilde{\mathcal{C}} = \arg\max_{C_j \in \mathcal{C}} \operatorname{ord}(C_j, y_t)$$

and then choose $C_i \in \tilde{\mathcal{C}}$ of the *lowest* possible rank. We can have many elements in $\tilde{\mathcal{C}}$. But we take the special one, C_i. Let $\mathbf{u}_i = \theta_i y_i$ for some $\theta_i \in \Theta$ and $\mathbf{u}_i$ be the leader of C_i for simplicity. From (10) and (11) we have

$$s = \operatorname{ord}(C_i, y_t) > h \tag{12}$$

and

$$r_f = \operatorname{ord}(f_i, y_i) \geqslant \operatorname{ord}(C_i, y_i) = r_C,$$

where

$$f_i = f_i(y_i, \mathfrak{N}) = I_{f_i}\left(y_i^{(r_f)}\right)^{n_f} + a_1\left(y_i^{(r_f)}\right)^{n_f-1} + \ldots + a_{n_f}.$$

Let us reduce each term (coefficients a_j, "initial" I_{f_i}, and its "leader" $y_i^{(r_f)}$) of f_i first by C_i. We need to differentiate C_i q times and get the remainder $\tilde{f}$, where $0 \leqslant q \leqslant r_f - r_C$. Remember that f_i depends only on $y_i, \mathfrak{N}$, and their derivatives. By reduction here we mean the following. Any proper derivative θ of C_i is linear in $\theta\mathbf{u}_i$ and its initial is equal to the separant of C_i. We simply multiply f_i by a sufficient power (say, n_f) of the separant and replace $y_i^{(r_f)}$ and the derivatives of y_i of lower order in f_i by the corresponding tails.

Hence, applying further steps of reduction to the terms of $\tilde{f}$ w.r.t. all C_j we need to differentiate them *less* than q times if $C_j \in \tilde{\mathcal{C}}$. Indeed, the fact that $C_i < C_j$, as C_i has the smallest rank in $\tilde{\mathcal{C}}$, implies

$$\operatorname{ord}\left(C_i, y_{C_j}\right) < \operatorname{ord}\left(C_j, y_{C_j}\right).$$

We need to differentiate them *at most* q times if $C_j \notin \tilde{\mathcal{C}}$. Indeed, the set $\mathcal{C}$ is autoreduced, so

$$\operatorname{ord}\left(C_i, y_{C_j}\right) \leqslant \operatorname{ord}\left(C_j, y_{C_j}\right).$$

In addition, the variables to reduce can come just from derivatives of variables from C_i.

In the case of $r_f = r_C$ the polynomial f_i can be algebraically reduced to zero using just C_i and elements $C \in \mathcal{C} \setminus \tilde{\mathcal{C}}$ because of our choice of C_i. Moreover, the elements of $\mathcal{C} \setminus \tilde{\mathcal{C}}$ do not contain $y_t^{(s)}$. Hence, we can apply (Sadik, 2000, Lemma 20) to get the inequality

$$\operatorname{ord}(f_i, y_t) \geqslant \operatorname{ord}(C_i, y_t). \tag{13}$$

Since $\operatorname{ord}(f_i, y_t) \leqslant h$, inequality (13) contradicts to inequality (12).

Consider the other case of $r_f > r_C$. Here, after we reduce all leaders of $\mathcal{C}$ from f_i we get a polynomial depending effectively on $y_t^{(s+q)}$ and $s+q \geqslant s$. Its leading coefficient w.r.t. the derivative $y_t^{(s+q)}$ is equal to

$$\mathbf{i}_{C_1}^{i_1} \cdot \ldots \cdot \mathbf{i}_{C_p}^{i_p} \cdot \mathbf{s}_{C_1}^{j_1} \cdot \ldots \cdot \mathbf{s}_{C_p}^{j_p} \cdot \tilde{I}_{f_i} \cdot \left(\frac{\partial C_i}{\partial y_t^{(s)}}\right)^{n_f}, \tag{14}$$

where $i_1, \ldots, i_p$, $j_1, \ldots, j_p \in \mathbb{Z}_{\geqslant 0}$ and $\tilde{I}_{f_i}$ is the remainder of I_{f_i} w.r.t. $\mathcal{C}$. Remember that P is a prime ideal. Hence,

$$\mathbf{i}_{C_1}^{i_1} \cdot \ldots \cdot \mathbf{i}_{C_p}^{i_p} \cdot \mathbf{s}_{C_1}^{j_1} \cdot \ldots \cdot \mathbf{s}_{C_p}^{j_p} \notin P, \tag{15}$$

because $\mathbf{i}_{C_j}$ and $\mathbf{s}_{C_j} \notin P$ for all j, $1 \leqslant j \leqslant p$. Moreover, $P = [\mathcal{C}] : H_{\mathcal{C}}^{\infty}$ and $\mathcal{C}$ is a characteristic set of $[\mathcal{C}] : H_{\mathcal{C}}^{\infty}$. Also,

$$\tilde{I}_{f_i} \notin P, \tag{16}$$

because $I_{f_i} \notin P$ due to our choice of f_i. By the Rosenfeld lemma, the remainder of f_i we are computing belongs to the prime algebraic ideal $(\mathcal{C}) : H_{\mathcal{C}}^{\infty}$. Thus, according to (Sadik, 2000, Lemma 22), its leading coefficient given by (14) is reducible to zero w.r.t. $\mathcal{C}$. For a prime differential ideal the fact that an element is reducible to zero w.r.t. a characteristic set means that the element belongs to the ideal. Using (15) and (16) we conclude that the polynomial $\frac{\partial C_i}{\partial y_t^{(s)}}$ belongs to P. Finally, this contradicts Lemma 26. □

Example 28 *Consider the prime differential ideal*

$$P = [x + z',\ y + x']$$

in the ring $\mathbf{k}\{x, y, z\}$. *Since the characteristic set of* P *w.r.t. the orderly ranking with* $x > y > z$ *is equal to* $z' + x$, $x' + y$, *the order of* P *is* 2. *On the other hand, the set*

$$x + z',\ x' + z'',\ x''(y - z''),\ y' + x''$$

is an algebraic characteristic set of the prime ideal

$$P(2) := P \cap \mathbf{k}\left[x, x', x'', y, y', y'', z, z', z''\right]$$

with respect to the ranking on these variables induced by the elimination differential ranking $z < x < y$. *We note that according to (Sadik, 2000, Theorem 25) the set*

$$\mathcal{C} = x + z', x''(y - z'')$$

must be a characteristic set of P *with respect to the elimination ranking* $z < x < y$, *but this is* **not** *correct since* $\mathcal{C}$ *is not autoreduced.*

6.3. Characterizable ideals: estimate for the bound

We do *not* need the ordinary case for the following result. Fix a ring of differential polynomials $\mathbf{k}\{y_1, \ldots, y_n\}$.

Theorem 29 *Suppose a function* h *from the set of prime differential ideals to the set* $\mathbb{Z}_{\geqslant 0}$ *is such that for any prime differential ideal* P *there exists its characteristic set* $C_1, \ldots, C_p$ *with the property* $\operatorname{ord} C_i \leqslant h(P)$ *for all* i, $1 \leqslant i \leqslant p$. *Then for any characterizable* *differential ideal I there exists its characteristic set $\mathcal{B} = B_1, \dots, B_k$ characterizing this ideal ($I = [\mathcal{B}] : H_{\mathcal{B}}^{\infty}$) such that*

$$\operatorname{ord} B_i \leqslant \max_{1 \leqslant j \leqslant n} h(P_j) =: h(I)$$

for all i, $1 \leqslant i \leqslant k$, where the set of ideals $\{P_j \mid 1 \leqslant j \leqslant n\}$ is the minimal prime decomposition of I.

Proof. Take the minimal prime decomposition $I = \bigcap_{j=1}^{t} P_j$ and choose a characteristic set $\mathcal{C}_j = C_{j,1}, \dots, C_{j,p_j} \subset P_j$ with $\operatorname{ord} C_{j,i} \leqslant h(P_j) \leqslant h(I)$ for all i, $1 \leqslant i \leqslant p_j$, and j, $1 \leqslant j \leqslant n$. We have

$$I = \bigcap_{j=1}^{t} [\mathcal{C}_j] : H_{\mathcal{C}_j}^{\infty}.$$

Let $\mathcal{B}$ be any characteristic set of I characterizing this radical differential ideal, that is, $I = [\mathcal{B}] : H_{\mathcal{B}}^{\infty}$, and L be the set of its leaders which is uniquely determined by I and does not depend on the choice of $\mathcal{B}$. Let N be the (infinite) set of *all* other variables from $\mathbf{k}\{y_1, \dots, y_n\}$. From (Hubert, 2000, Theorem 4.5) we know that

$$J = (\mathcal{B}) : H_{\mathcal{B}}^{\infty} = \bigcap_{j=1}^{t} (\mathcal{C}_j) : H_{\mathcal{C}_j}^{\infty}.$$

in the ring $\mathbf{k}[N, L]$ and $\mathcal{B}$ is an algebraic characteristic set of J which can be computed, e.g., from the reduced Gröbner basis G of the ideal J. We just need to notice that G can be computed from all $\mathcal{C}_j$ without involving extra variables from the set N. To conclude that $I = [\mathcal{B}] : H_{\mathcal{B}}^{\infty}$ we use (Hubert, 2000, Lemmas 3.5, 3.9, and 6.1). □

Let us switch to the *ordinary case* and see what Theorem 29 gives us.

Corollary 30 *In the ordinary case for a characterizable differential ideal I there exists a characteristic set $\mathcal{C} = C_1, \dots, C_p$ with the following properties:*

- $I = [\mathcal{C}] : H_{\mathcal{C}}^{\infty}$.
- $\operatorname{ord} C_i \leqslant \operatorname{ord} I$ *(see Definition 21) for all i, $1 \leqslant i \leqslant p$.*

Proof. Follows from Theorem 27 and Theorem 29 setting $h(P) = \operatorname{ord} P$. □

6.4. Bounding orders in canonical characteristic sets

We need the ordinary case for the following assertions about bounds.

Theorem 31 *Let $\mathcal{C} = C_1, \dots, C_p$ be the canonical characteristic set of a characterizable differential ideal I. Then*

$$\operatorname{ord} C_i \leqslant \operatorname{ord} I$$

for all $i \leqslant i \leqslant p$.

Proof. Let $\mathcal{B} = B_1, \dots, B_p$ be a characteristic set of I given by Corollary 30. We have $\operatorname{ord} B_i \leqslant \operatorname{ord} P$ for all $i, 1 \leqslant i \leqslant p$. Take the canonical characteristic set $\mathcal{C}$ of the algebraic ideal $(\mathcal{B}) : H_{\mathcal{B}}^{\infty}$ in the ring $\mathbf{k}[U]$, where U is the set of derivatives effectively present in

$\mathcal{B}$. Then $\operatorname{rk}\mathcal{C} = \operatorname{rk}\mathcal{B}$, and $\mathcal{C}$ is a differentially autoreduced subset of I. That is, $\mathcal{C}$ is a characteristic set of I. Moreover, $\mathcal{C}$ satisfies the requirements of Definition 1. Thus, $\mathcal{C}$ is the canonical characteristic set of I. The fact that $\mathcal{C} \subset \mathbf{k}[U]$ implies the statement. □

7. Computation of the canonical characteristic set

We do *not* assume the ordinary case now. Fix a differential ranking. Given any characteristic set $\mathcal{A}$ of a characterizable differential ideal I, it is easy to compute the canonical characteristic set. In (Boulier and Lemaire, 2000, Section 5), the canonical characteristic set is computed by inverting the initials. Alternatively, by the remark after (Hubert, 2000, Lemma 3.9), the reduced Gröbner basis $\mathcal{B}$ of $(A) : H_{\mathcal{A}}^{\infty}$ in $\mathbf{k}(N)[L]$ w.r.t. the lexicographic monomial ordering induced by the ranking has the same rank as $\mathcal{A}$. By clearing out the denominators of $\mathcal{B}$, we thus obtain a characteristic set $\mathcal{C}$ of $(A) : H_{\mathcal{A}}^{\infty}$, whose initials do not depend on the leaders. By Corollary 18, $\mathcal{C}$ satisfies the properties required for the canonical characteristic set by Definition 1. Thus, due to the uniqueness of the canonical characteristic set, $\mathcal{C}$ must be this set. Moreover, elements of $\mathcal{C}$ do not have factors in $\mathbf{k}[N]$.

Note that an ideal which has a canonical characteristic set may not be characterizable. For example, such is the algebraic ideal generated by the polynomial xy, where $x < y$. However, the polynomial xy, which constitutes the canonical characteristic set of this ideal, has a factor $x \in \mathbf{k}[N]$. It is not known whether a non-characterizable radical differential ideal may have a canonical characteristic set whose elements do not have factors in $\mathbf{k}[N]$.

Algorithm 5 computes the canonical characteristic set, given a set of generators of a characterizable differential ideal. Alternatively, one can assume that the characterizable differential ideal is given as an intersection of other characterizable differential ideals—in that case, start the algorithm from the second line.

It may seem that Algorithm 5 allows one to check whether a radical differential ideal is characterizable (by computing the canonical characteristic set). But this is not the case. As we have seen above, there exist non-characterizable radical differential ideals, which have canonical characteristic sets.

Remark 32 *Note that in the second line of Algorithm 5 it would not be sufficient to consider only the characterizable components having characteristic sets of the highest rank in $\mathfrak{C}$. Indeed, let $x > y > z$, and consider the following algebraic characterizable ideal and its decomposition into characterizable components:*

$$I = (y^2 + z, x^3 + x^2 y + xy - z) = (y^2 + z, x + y) \cap (y^2 + z, x^2 + y).$$

Characteristic sets of both components have the same set of leaders, $\{x, y\}$. The component of the highest rank is $(y^2 + z, x^2 + y)$ and, clearly, $I \neq (y^2 + z, x^2 + y)$.

Proposition 33 *Algorithm 5 computes the canonical characteristic set of a given characterizable differential ideal $\{F\}$.*

Proof. Let $\mathcal{C}$ be the canonical characteristic set of the characterizable ideal $I = \{F\}$. First, let us prove an auxiliary

Lemma 34 *Let P be a prime differential ideal with a characteristic set $\mathcal{A}$ whose set of leaders coincides with that of $\mathcal{C}$, where $\mathcal{C}$ is a characteristic set of $[\mathcal{C}] : H_{\mathcal{C}}^{\infty} = I$. Assume also that $I \subseteq P$. Then $(\mathcal{C}) : H_{\mathcal{C}}^{\infty} \subset (\mathcal{A}) : H_{\mathcal{A}}^{\infty}$.*

Algorithm 5 Characteristic Set of a Characterizable Differential Ideal

INPUT: *a finite set F of differential polynomials such that*

the radical differential ideal $\{F\}$ is characterizable.

OUTPUT: *the canonical characteristic set of $\{F\}$.*

let $\mathfrak{C} = \mathsf{Rosenfeld\_Gröbner}(F)$ *and* $\mathfrak{C} = \mathcal{C}_1, \ldots, \mathcal{C}_q$

let $[\mathcal{C}_{i_j}] : H^{\infty}_{\mathcal{C}_{i_j}}$ *be the components whose characteristic sets have sets of leaders of the highest possible rank in* $\mathfrak{C}$ *and* $1 \leqslant j \leqslant k$

let $I' = \bigcap_{j=1}^{k} (\mathcal{C}_{i_j}) : H^{\infty}_{\mathcal{C}_{i_j}}$

$L := \mathsf{Leaders}(\mathcal{C}_{i_1})$

$N := \Theta Y \setminus \Theta L$

$GB := \mathsf{Reduced\_Gröbner\_Basis}(I')$ *in* $\mathbf{k}(N)[L]$

$N' := \{x \in N \mid x \text{ appears in } GB\}$.

$\mathcal{D} := \mathsf{Clear\_out\_denominators}(GB)$ *in* $\mathbf{k}(N)[L]$

divide each element of $\mathcal{D}$ by its leading coefficient from $\mathbf{k}$

return $\mathcal{D}$

Proof. Let $f \in (\mathcal{C}) : H^{\infty}_{\mathcal{C}}$. Then f is partially reduced w.r.t. $\mathcal{C}$. Since the leaders of $\mathcal{A}$ and $\mathcal{C}$ coincide, f is partially reduced w.r.t. $\mathcal{A}$. Since $f \in I$ and $I \subseteq P$, we have $f \in P$. Hence, by the Rosenfeld lemma, $f \in (\mathcal{A}) : H^{\infty}_{\mathcal{A}}$. □

Consider the prime decomposition $I = \bigcap P_i$, where P_i's are the minimal prime components of I. Let $\mathcal{A}_i$ be a characteristic set of P_i, then, according to (Hubert, 2000, Theorem 4.5), the ideal $P'_i = (\mathcal{A}_i) : H^{\infty}_{\mathcal{A}_i}$ is a minimal prime component of the algebraic ideal $(\mathcal{C}) : H^{\infty}_{\mathcal{C}}$. Consider also the minimal prime decompositions $J_l = \bigcap Q_{lj}$ of the characteristic components $J_l = [\mathcal{C}_l] : H^{\infty}_{\mathcal{C}_l}$ of I. The intersection of these decompositions is a finite prime decomposition of I. According to (Ritt, 1950, Section I.16), every minimal prime component appears in every finite prime decomposition of the radical ideal I, which implies that every P_i can be found among Q_{lj}. Moreover, according to (Hubert, 2000, Theorem 3.2), the leaders of $\mathcal{A}_i$ coincide with the leaders of $\mathcal{C}$, hence P_i can be found among those Q_{lj} whose characteristic sets have leaders coinciding with the leaders of $\mathcal{C}$.

Applying (Hubert, 2000, Theorem 3.2) again, we obtain that P_i can be found among the minimal prime components of those J_l whose characteristic sets $\mathcal{C}_l$ have leaders coinciding with the leaders of $\mathcal{C}$. Now, since for each l, $I \subseteq J_l$, the rank of the set of leaders of $\mathcal{C}_l$ is lower than or equal to the rank of the set of leaders of $\mathcal{C}$. Hence, P_i can be found among the minimal prime components of those J_l, for which the set of leaders of $\mathcal{C}_l$ has the highest rank, that is, among the minimal prime components of $J_{i_1}, \ldots, J_{i_k}$. Thus, by (Hubert, 2000, Theorem 4.5), every minimal prime P'_i of the algebraic ideal $(\mathcal{C}) : H^{\infty}_{\mathcal{C}}$ can be found among the minimal primes of the algebraic ideals

$$(\mathcal{C}_{i_1}) : H^{\infty}_{\mathcal{C}_{i_1}}, \ldots, (\mathcal{C}_{i_k}) : H^{\infty}_{\mathcal{C}_{i_k}},$$

and we obtain

$$(\mathcal{C}) : H_{\mathcal{C}}^{\infty} \supseteq \bigcap_{j=1}^{k} \left(\mathcal{C}_{i_j}\right) : H_{\mathcal{C}_{i_j}}^{\infty} = I'.$$

The inverse inclusion follows from the above Lemma 34. Hence, $I' = (\mathcal{C}) : H_{\mathcal{C}}^{\infty}$, and the canonical characteristic set $\mathcal{D}$ of I' computed by the above algorithm coincides with that of $(\mathcal{C}) : H_{\mathcal{C}}^{\infty}$ and of I. □

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