Differential Tannakian Categories [1]

Alexey Ovchinnikov

University of Illinois at Chicago
Department of Mathematics, Statistics, and Computer Science
Chicago, IL 60607-7045, USA

Abstract

We define a differential Tannakian category and show that under a natural assumption it has a fibre functor. If in addition this category is neutral, that is, the target category for the fibre functor are finite dimensional vector spaces over the base field, then it is equivalent to the category of representations of a (pro-)linear differential algebraic group. Our treatment of the problem is via differential Hopf algebras and Deligne's fibre functor construction [1].

Key words: differential algebra, Tannakian category, fibre functor
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1 Introduction

A differential Tannakian category is a rigid abelian tensor category with an addition differential structure. In [1] it is shown that if the dimension of each object in a rigid abelian tensor category $\mathcal{C}$ over a field $\mathbf{k}$ of zero characteristic is a non-negative integer then there exists a fibre functor ω. This category is called neutral if the target category are finite dimensional vector spaces over the field $\mathbf{k}$. In this case, the pair $(\mathcal{C}, \omega)$ is equivalent to the category of finite dimensional representations of a (pro-)linear algebraic group [1–3].

We find conditions for a rigid abelian tensor category $\mathcal{C}$ with an additional differential structure over a field $\mathbf{k}$ of characteristic zero so that $\mathcal{C}$ has a fibre functor ω compatible with the differential structure. If, in addition, this category is neutral then we show that $(\mathcal{C}, \omega)$ is equivalent to the category of finite

Email address: `aiovchin@math.uic.edu` (Alexey Ovchinnikov).
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dimensional differential representations of a (pro-)linear differential algebraic group.

Differential algebraic groups were introduced in [4,5] and further used to build the differential Galois theory of parametrised linear differential equations [6]. This work was further generalised to include systems of linear difference equations with differential parameters [7]. Another approach to the Galois theory of systems of linear differential equations with parameters is given in [8], where the authors study Galois groups for generic values of the parameters. Also, it is shown in [9] that over the field $\mathbb{C}(x)$ the differential Galois group of a parametrised system of differential equations will be the same for all values of the parameter outside a countable union of proper algebraic sets. In the usual differential Galois theory the Tannakian approach is a base for algorithms computing differential Galois groups which are linear algebraic groups [10–14]. In the parametrised theory the Galois groups are linear *differential* algebraic groups and the Tannakian theory that we develop should contribute to computation of these groups.

Tannakian approach for linear differential algebraic groups was first introduced in [15], where a differential analogue of Tannaka's theorem was proved (see [16,17]). This was further developed in [18], where neutral differential Tannakian categories were introduced. This approach used a given fibre functor and the axioms for such a category were given heavily employing this fibre functor. In [18] it is shown that such a category is equivalent to the category of finite dimensional differential representations of a (pro-)linear differential algebraic group and this is further applied to the Galois theory of parametrised linear differential equations (see also [7,19,20]). Generalisations of differential Galois theory to non-linear equations via Galois groupoids can be found in [21–29].

In the present paper we extend the results of [18]. In particular, we no longer need the fibre functor to formulate our axioms and, moreover, we show that under additional natural assumptions such a functor exists. Note, that we do *not* require the base field to be differentially closed. The *main difficulty* was to give correct axioms relating tensor product, duals, and the differential structure together without using the fibre functor and also to show that the fibre functor that we construct "respects" the differential structure.

A model theoretic treatment of general differential tensor categories and of some results of [18] is given in [30], in particular, it is shown that a differential tensor category [30, Definition 2.1] together with a fibre functor compatible with the differential structure is equivalent to the category of differential representations of the differential group scheme of differential tensor automorphisms of the fibre functor. In [30] the differential structure is given by a functor to the category of short exact sequences and the tensor structure compatible with this derivation is introduced via Baer sums.

Our axioms are similar to the ones given in [30] but our version is more constructive. It is assumed in [30] that the given fibre functor "commutes" with the differential structure on the category, that is, for each object there is a functorial "derivation" satisfying the Leibniz rule with respect to the tensor structure. We do *not* make this assumption on the fibre functor and show how this functorial commutation for each object comes naturally from the category itself.

The paper is organised as follows. We review basic notions of differential algebra in Section 2. In Section 3.1 we introduce differential Tannakian categories. In Remark 6 we show that the usual Tannakian categories can be given a trivial differential structure and, therefore, they are a part of our formalism. We recall what differential comodules and differential algebraic groups are in Section 3.2. We show in Section 4.1, Theorem 14, how starting with a differential Tannakian category one can get a fibre functor based on the ideas of [1, Section 7]. An additional assumption which guarantees that the category is neutral is made in Section 4.2. We recover the differential Hopf algebra structure of the corresponding group in Section 4.3. Lemmas 19 and 22 supply all necessary ingredients to finally recover the (pro-)linear differential algebraic group. In Proposition 23 and Theorem 16 of this section we combine these lemmas with the results from [15,18] to demonstrate the main reconstruction statement of the paper.

2 Basic Definitions

A Δ-ring R, where $\Delta = \{\partial_1, \ldots, \partial_m\}$, is a commutative associative ring with unit 1 and commuting derivations $\partial_i : R \to R$ such that

$$\partial_i(a+b) = \partial_i(a) + \partial_i(b), \quad \partial_i(ab) = \partial_i(a)b + a\partial_i(b)$$

for all $a, b \in R$. If $\mathbf{k}$ is a field and a Δ-ring then $\mathbf{k}$ is called a Δ-field. We restrict ourselves to the case of

$$\operatorname{char} \mathbf{k} = 0.$$

If $\Delta = \{\partial\}$ then we call a Δ-field as ∂-field. For example, $\mathbb{Q}$ is a ∂-field with the unique possible derivation (which is the zero one). The field $\mathbb{C}(t)$ is also a ∂-field with $\partial(t) = f$, and this f can be any rational function in $\mathbb{C}(t)$. Let C be the field of constants of $\mathbf{k}$, that is, $C = \ker \partial$.

Let

$$\Theta = \left\{ \partial^i \mid i \in \mathbb{Z}_{\geqslant 0} \right\}.$$

Since ∂ acts on a ∂-ring R, there is a natural action of Θ on R. A non-commutative ring $R[\partial]$ of linear differential operators is generated as a left R-module by the monoid Θ. A typical element of $R[\partial]$ is a polynomial

$$D = \sum_{i=1}^{n} a_i \partial^i, \; a_i \in R.$$

The right R-module structure follows from the formula

$$\partial \cdot a = a \cdot \partial + \partial(a) \tag{1}$$

for all $a \in R$. Note that (1) defines both left and right R-module structures on $R[\partial]$ that we will use in our constructions. We denote the set of operators in $R[\partial]$ of order less than or equal to p by $R[\partial]_{\leqslant p}$.

Let R be a ∂-ring. If B is an R-algebra, then B is a ∂-R-algebra if the action of ∂ on B extends the action of ∂ on R. If R_1 and R_2 are ∂-rings then a ring homomorphism $\varphi : R_1 \to R_2$ is called a ∂-homomorphism if it commutes with ∂, that is,

$$\varphi \circ \partial = \partial \circ \varphi.$$

We denote these homomorphisms simply by $\mathrm{Hom}(R_1, R_2)$. If A_1 and A_2 are ∂-$\mathbf{k}$-algebras then a ∂-$\mathbf{k}$-homomorphism simply means a $\mathbf{k}[\partial]$-homomorphism. We denote the category of ∂-$\mathbf{k}$-algebras by $\mathbf{Alg}_{\mathbf{k}}(\partial)$. Let $Y = \{y_1, \ldots, y_n\}$ be a set of variables. We differentiate them:

$$\Theta Y := \left\{ \partial^i y_j \;\middle|\; i \in \mathbb{Z}_{\geqslant 0}, \; 1 \leqslant j \leqslant n \right\}.$$

The ring of differential polynomials $R\{Y\}$ in differential indeterminates Y over a ∂-ring R is the ring of commutative polynomials $R[\Theta Y]$ in infinitely many algebraically independent variables ΘY with the derivation ∂, which naturally extends ∂-action on R as follows:

$$\partial\left(\partial^i y_j\right) := \partial^{i+1} y_j$$

for all $i \in \mathbb{Z}_{\geqslant 0}$ and $1 \leqslant j \leqslant n$. A ∂-$\mathbf{k}$-algebra A is called finitely ∂-generated over $\mathbf{k}$ if there exists a finite subset $X = \{x_1, \ldots, x_n\} \subset A$ such that A is a $\mathbf{k}$-algebra generated by ΘX.

Definition 1 *The category $\mathcal{V}$ over a ∂-field $\mathbf{k}$ is the category of finite dimensional vector spaces over $\mathbf{k}$:*

(1) objects are finite dimensional $\mathbf{k}$-vector spaces,
(2) morphisms are $\mathbf{k}$-linear maps;

with tensor product $\otimes$, direct sum $\oplus$, dual $*$, and an additional operation:

$$F : V \mapsto V^{(1)} := {}_{\mathbf{k}}\left(\left(\mathbf{k}[\partial]_{\leqslant 1}\right)_{\mathbf{k}} \otimes V \right), \tag{2}$$

which we call a differentiation (or prolongation) functor. If $f \in \mathrm{Hom}(V, W)$ *then we define*

$$F(f) : V^{(1)} \to W^{(1)},\ f(\partial^q \otimes v) = \partial^q \otimes f(v),\ 0 \leqslant q \leqslant 1. \tag{3}$$

Here, $\mathbf{k}[\partial]_{\leqslant 1}$ *is considered as the right* $\mathbf{k}$*-module of differential operators up to order* 1*,* V *is viewed as a left* $\mathbf{k}$*-module, and the whole tensor product is viewed as a left* $\mathbf{k}$*-module as well.*

For each $V \in \mathcal{O}b(\mathcal{V})$ there are: a natural inclusion

$$i_V : V \to V^{(1)}, \quad v \mapsto 1 \otimes v, \tag{4}$$

and a surjection

$$\varphi_V : V^{(1)} \to V, \quad v \mapsto 0,\ \partial \otimes v \mapsto v. \tag{5}$$

We also have a natural splitting of $\mathbf{k}^{(1)} = \mathbf{k}[\partial]_{\leqslant 1} \otimes \mathbf{k}$ as follows

$$\mathbf{k}^{(1)} \ni a \otimes 1 + b \cdot \partial \otimes 1 \mapsto (a, b) \in \mathbf{k} \oplus \mathbf{k} \tag{6}$$

for all $a, b \in \mathbf{k}$. The projection onto the first component gives a left inverse to the map $i_{\mathbf{k}}$. The other projection is just $\varphi_{\mathbf{k}}$.

3 General definition of a differential Tannakian category

Our general definition of differential Tannakian categories is motivated by the following example.

Example 2 *Let* K *be a* $\{\partial_t, \partial_x\}$*-field with* $\mathbf{k}$ *as the* ∂_t*-field of* ∂_x*-constants. Consider a system of ordinary differential equations with respect to the derivation* ∂_x*:*

$$\partial_x Y = AY, \tag{7}$$

where $A \in \mathbf{M}_n(K)$ *(see [10]). Since* ∂_t *and* ∂_x *commute with each other, system* (7) *implies:*

$$\partial_x \partial_t Y = \partial_t \partial_x Y = \partial_t(AY) = (\partial_t A)Y + A(\partial_t Y),$$

which gives a new system "prolonged" with respect to ∂_t*:*

$$\partial_x \begin{pmatrix} \partial_t Y \\ Y \end{pmatrix} = \begin{pmatrix} A & \partial_t A \\ 0 & A \end{pmatrix} \begin{pmatrix} \partial_t Y \\ Y \end{pmatrix}. \tag{8}$$

Equations (7) *and* (8) *define two* $K[\partial_x]$*-modules [10, Section 1.2] that we will denote by* M *and* $M^{(1)}$*, respectively [18, Section 5]. The matrix structure in* (8) *implies a canonical injective map $M \to M^{(1)}$ with a quotient $M^{(1)}/M \cong M$. Thus, with a $K[\partial_x]$-module M we associate the following exact sequence of $K[\partial_x]$-modules:*

$$0 \longrightarrow M \longrightarrow M^{(1)} \longrightarrow M \longrightarrow 0.$$

As we will see, this is the main idea behind **differential** *Tannakian categories.*

3.1 Definition

To define differential Tannakian categories we need to recall a few constructions from tensor categories. Let

$$\psi_{X,Y} : X \otimes Y \to Y \otimes X \tag{9}$$

be the *commutativity morphism* in $\mathcal{C}$ so that $\psi_{X,Y} \circ \psi_{Y,X} = \mathrm{id}_{Y,X}$ [3, page 104]. Let also

$$\Delta_X : X \otimes X^* \to X \otimes X^* \otimes X \otimes X^* \tag{10}$$

be the dual morphism to the morphism that computes the evaluation [3, page 110] of the leftmost X^* with the rightmost X and switching the middle X with X^* that can be more formally written as

$$\begin{aligned}(\mathrm{ev}_X \otimes \psi_{X,X^*}) \circ (\mathrm{id}_{X^*} \otimes \psi_{X,X} \otimes \mathrm{id}_{X^*}) \circ (\mathrm{id}_{X^* \otimes X} \otimes \psi_{X^*,X}) : X^* \otimes X \otimes X^* \otimes X \\ \to X^* \otimes X \otimes X \otimes X^* \to X^* \otimes X \otimes X \otimes X^* \to X^* \otimes X\end{aligned}$$

Definition 3 *A differential Tannakian category $\mathcal{C}$ over a field $\mathbf{k}$ of characteristic zero is a*

(1) rigid
(2) abelian
(3) tensor

category such that $\mathrm{End}(\underline{\mathbf{1}}) = \mathbf{k}$*, where* $\underline{\mathbf{1}}$ *is the unit object with respect to the tensor product; together with:*

(1) a **functor**

$$F : X \mapsto X^{(1)},$$

which is exact on the right and commutes with direct sums.

(2) an inclusion (a morphism with the trivial kernel)

$$i_X : X \to X^{(1)},$$

which is a natural transformation from the id *functor to the functor F;*

(3) a morphism

$$\varphi_X : X^{(1)} \to X$$

in $\mathcal{C}$, which is a natural transformation from the functor F to the functor id *such that the short sequence*

$$0 \longrightarrow X \xrightarrow{i_X} X^{(1)} \xrightarrow{\varphi_X} X \longrightarrow 0 \tag{11}$$

is exact;

(4) a chosen splitting

$$0 \longrightarrow \underline{\mathbf{1}} \xrightarrow{i_{\underline{\mathbf{1}}}} \underline{\mathbf{1}} \oplus \underline{\mathbf{1}} \xrightarrow{\varphi_{\underline{\mathbf{1}}}} \underline{\mathbf{1}} \longrightarrow 0 \tag{12}$$

(5) **functorial morphisms***:*

(a)

$$S_X : (X \otimes X^*)^{(1)} \to X^{(1)} \otimes \left(X^{(1)}\right)^*,$$

(b) an injective (this is the Leibniz rule)

$$T_{X,Y} : (X \otimes Y)^{(1)} \to X^{(1)} \otimes Y^{(1)},$$

(c) and an isomorphism

$$D_X : \left(X^{(1)}\right)^* \to (X^*)^{(1)}$$

making diagrams (13), (14), (15), (16), *and* (17) *given below commutative:*

$$\begin{tikzcd}
(X \otimes X^*)^{(1)} \arrow[r, "S_X"] \arrow[d, "F(\mathrm{ev}_X)"'] & X^{(1)} \otimes \left(X^{(1)}\right)^* \arrow[d, "\mathrm{ev}_{X^{(1)}}"'] \\
\underline{\mathbf{1}}^{(1)} \arrow[r, "i_{\underline{\mathbf{1}}}^{-1}"] & \underline{\mathbf{1}}
\end{tikzcd} \tag{13}$$

where $i_{\underline{\mathbf{1}}}^{-1}$ denotes the left inverse to the map $i_{\underline{\mathbf{1}}}$ in splitting (12)*,*

$$\begin{tikzcd}
X \otimes Y \arrow[r, "i_{X\otimes Y}"] \arrow[d, equal] & (X \otimes Y)^{(1)} \arrow[r, "\varphi_{X\otimes Y}"] \arrow[d, "T_{X,Y}"'] & X \otimes Y \arrow[d, "(i_X \otimes \mathrm{id}_Y, \mathrm{id}_X \otimes i_Y)\circ\Delta"'] \\
X \otimes Y \arrow[r, "i_X \otimes i_Y"] & X^{(1)} \otimes Y^{(1)} \arrow[r, "\left(\mathrm{id}_{X^{(1)}} \otimes \varphi_Y\right)\oplus\left(\varphi_X \otimes \mathrm{id}_{Y^{(1)}}\right)"] & X^{(1)} \otimes Y \oplus X \otimes Y^{(1)}
\end{tikzcd} \tag{14}$$

$$\begin{tikzcd}
(X \otimes X^*)^{(1)} \arrow[r, "T_{X\otimes X^*, X\otimes X^*}\circ F(\Delta_X)"] \arrow[d, "S_X"'] & (X \otimes X^*)^{(1)} \otimes (X \otimes X^*)^{(1)} \arrow[d, "S_X \otimes S_X"'] \\
X^{(1)} \otimes \left(X^{(1)}\right)^* \arrow[r, "\Delta_{X^{(1)}}"] & X^{(1)} \otimes \left(X^{(1)}\right)^* \otimes X^{(1)} \otimes \left(X^{(1)}\right)^*
\end{tikzcd} \tag{15}$$

$$
\begin{array}{ccc}
((X\otimes Y)\otimes(X\otimes Y)^*)^{(1)} & \xrightarrow{\alpha_{X,Y}\circ S_{X\otimes Y}} & X^{(1)}\otimes Y^{(1)}\otimes\left((X\otimes Y)^{(1)}\right)^* \\
\Big\downarrow{\scriptstyle T_{X\otimes X^*,Y\otimes Y^*}\circ F\left(\mathrm{id}_X\otimes\psi_{Y,X^*}\otimes\mathrm{id}_{Y^*}\right)} & & \Big\uparrow{\scriptstyle \mathrm{id}_{X^{(1)}\otimes Y^{(1)}}\otimes T_{X,Y}{}^*} \\
(X\otimes X^*)^{(1)}\otimes(Y\otimes Y^*)^{(1)} & \xrightarrow{\beta_{X,Y}\circ(S_X\otimes S_Y)} & X^{(1)}\otimes Y^{(1)}\otimes\left(X^{(1)}\otimes Y^{(1)}\right)^*
\end{array}
\tag{16}
$$

where

$$\alpha_{X,Y} := T_{X,Y}\otimes\mathrm{id}_{\left((X\otimes Y)^{(1)}\right)^*}$$

and

$$\beta_{X,Y} := \mathrm{id}_{X^{(1)}}\otimes\psi_{\left(X^{(1)}\right)^*,Y^{(1)}}\otimes\mathrm{id}_{\left(Y^{(1)}\right)^*};$$

$$
\begin{array}{ccc}
(X\otimes X^*)^{(1)} & \xrightarrow{\psi_{X^{(1)},\left(X^{(1)}\right)^*}\circ S_X} & \left(X^{(1)}\right)^*\otimes X^{(1)} \\
\Big\downarrow{\scriptstyle S_{X^*}\circ F(\psi_{X,X^*})} & & {\scriptstyle D_X\otimes\mathrm{id}_{X^{(1)}}}\Big\downarrow \\
(X^*)^{(1)}\otimes\left((X^*)^{(1)}\right)^* & \xrightarrow{\mathrm{id}_{(X^*)^{(1)}}\otimes D_X{}^*} & (X^*)^{(1)}\otimes X^{(1)}
\end{array}
\tag{17}
$$

Remark 4 *Diagrams* (13), (14), (15), (16), *and* (17) *are designed in such a way that the multiplication on the algebra of regular functions that we recover satisfies the product rule and the coinverse is a differential homomorphism. We will show this in Lemmas 19 and 22.*

Also, in Example 13 we will explain why the category of differential representations (Definition 11) of a linear differential algebraic group (Definition 12) is a differential Tannakian category. In that example we will also argue why the morphism S_X is essential by showing that, in general,

$$S_X \neq \left(\mathrm{id}_X\otimes(D_X)^{-1}\right)\circ T_{X,X^*}.$$

Remark 5 *If $\mathcal{C}$ is $\mathcal{V}$ then exact sequence* (11) *splits. This is in general not the case. In particular, if $\mathcal{C}$ is the category of differential representations of a linear differential algebraic group G then sequence* (11) *splits (when X is a faithful representation of G) if and only if the group G is conjugate to a group of matrices with constant entries [15, Proposition 3]. The differential algebraic group $\mathbf{G_m}$ is not of that kind, for instance.*

Remark 6 *More generally, the usual Tannakian categories do satisfy the formalism of differential Tannakian categories given in Definition 3. Indeed, they can be supplied with the trivial differential structure:*

$$F : X \mapsto X\oplus X,$$

$$F(\varphi) := \varphi\oplus\varphi,\quad \varphi\in\mathrm{Hom}(X,Y),$$
$$S_X : (X\otimes X^*)\oplus(X\otimes X^*)\to(X\oplus X)\otimes(X^*\oplus X^*),$$
$$T_{X,Y} : (X\otimes Y)\oplus(X\otimes Y)\to(X\oplus X)\otimes(Y\oplus Y),$$

and

$$D_X : (X \oplus X)^* \to (X^* \oplus X^*)$$

that all satisfy (13)*,* (14)*,* (15)*,* (16)*, and* (17)*, which follows from the commutativity of* $\oplus$ *with* $*$ *and the distribution law for* $\oplus$ *and* $\otimes$*.*

Similarly to [30, Section 2.2]:

Lemma 7 *The field* $\mathbf{k} = \mathrm{End}(\underline{\mathbf{1}})$ *has a natural derivation induced by the functor* F*.*

PROOF. Let $a \in \mathbf{k}$. The functor F defines a map

$$\mathbf{k} \to \mathrm{End}\left(\underline{\mathbf{1}}^{(1)}\right).$$

Since $\mathrm{End}\left(\underline{\mathbf{1}}^{(1)}\right)$ is a $\mathbf{k}$-algebra, we have another map

$$G : \mathbf{k} \to \mathrm{End}\left(\underline{\mathbf{1}}^{(1)}\right), \quad a \mapsto a \cdot \mathrm{id}_{\underline{\mathbf{1}}^{(1)}}.$$

Since $i_{\underline{\mathbf{1}}}$ is a natural transformation from id to F, the morphism

$$(F - G)(a)$$

is zero on $i_{\underline{\mathbf{1}}}(\underline{\mathbf{1}})$ for all $a \in \mathbf{k}$. Therefore, it induces a morphism

$$H(a) : \underline{\mathbf{1}}^{(1)} / i_{\underline{\mathbf{1}}}(\underline{\mathbf{1}}) \cong \underline{\mathbf{1}} \to \underline{\mathbf{1}}^{(1)}.$$

Moreover, $\varphi_{\underline{\mathbf{1}}} \circ (F - G)(a)$ is the zero morphism $\underline{\mathbf{1}}^{(1)} \to \underline{\mathbf{1}}$. Hence, there exists a morphism

$$D(a) : \underline{\mathbf{1}} \to \underline{\mathbf{1}}$$

such that $i_{\underline{\mathbf{1}}} \circ D(a) = H(a)$. Finally, [30, Claim, Section 2.2] shows that the map $a \mapsto D(a)$ is a *derivation* on $\mathbf{k}$. $\square$

Remark 8 *Note that in the case of the trivial differential structure described in Remark 6 the derivation* D *on* $\mathbf{k}$ *constructed above is the zero map as* $F(a) = G(a)$ *for all* $a \in \mathbf{k}$ *in this case.*

Example 9 *Let the prolongation functor* F *act as in* (3) *on the morphisms. We will show how following Lemma 7 one can recover the differential structure on the field* $\mathbf{k}$ *when, for example,* $\mathcal{C} = \mathbf{Rep}_G$*, where* G *is a linear differential algebraic group (see Section 3.2). So, for all* $a, b \in \mathbf{k}$ *we have*

$$\begin{aligned}
&F - G : \mathbf{k} \to \mathrm{End}(\mathbf{k}[\partial]), \quad a \mapsto (F - G)(a),\\
&(F - G)(a)(1 \otimes b) = 1 \otimes ab - a \otimes b = 0,\\
&(F - G)(a)(\partial \otimes b) = \partial \otimes ab - a\partial \otimes b = (\partial(a)) \otimes b.
\end{aligned}$$

Under the isomorphism $\underline{\mathbf{1}}^{(1)}/i_{\underline{\mathbf{1}}}(\underline{\mathbf{1}}) \cong \underline{\mathbf{1}}$ *we have* $\partial \otimes b \mapsto b$. *This composed with* $i_{\underline{\mathbf{1}}}$ *sends* $\partial \otimes b \mapsto 1 \otimes b$. *Then, for each* $a \in \mathbf{k}$ *the functor* $F - G$ *induces the map*

$$H(a) : \underline{\mathbf{1}} \to \underline{\mathbf{1}}^{(1)}, \quad b \mapsto (\partial(a)) \otimes b,$$

which induces the map

$$D(a) : \underline{\mathbf{1}} \to \underline{\mathbf{1}}, \quad b \mapsto (\partial(a)) \cdot b,$$

that is, as an element of $\mathbf{k}$, $D(a) = \partial(a)$.

3.2 Differential comodules

Let A be a ∂-$\mathbf{k}$-algebra. Assume that A is supplied with the following operations:

- differential algebra homomorphism $m : A \otimes A \to A$ is the multiplication map on A,
- differential algebra homomorphism $\Delta : A \to A \otimes A$ which is a comultiplication,
- differential algebra homomorphism $\varepsilon : A \to \mathbf{k}$ which is a counit,
- differential algebra homomorphism $S : A \to A$ which is a coinverse.

We also assume that these maps satisfy commutative diagrams (see [5, page 225]):

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & A \otimes A \arrow[d, "\mathrm{id}_A \otimes \Delta"] \\ A \otimes A \arrow[r, "\Delta \otimes \mathrm{id}_A"] & A \otimes A \otimes A \end{tikzcd} \quad \begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\mathrm{id}_B"'] & A \otimes A \arrow[d, "\mathrm{id}_A \otimes \varepsilon"] \\ A \arrow[r, "\sim"] & A \otimes \mathbf{k} \end{tikzcd} \quad \begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\varepsilon"'] & A \otimes A \arrow[d, "m \circ (S \otimes \mathrm{id}_A)"] \\ \mathbf{k} \arrow[r, hook] & A \end{tikzcd} \tag{18}$$

Definition 10 *Such a commutative associative* ∂-$\mathbf{k}$-*algebra* A *with unit* 1 *and operations* m, Δ, S, *and* ε *satisfying axioms* (18) *is called a differential Hopf algebra (or* ∂-$\mathbf{k}$-*Hopf algebra).*

Definition 11 *A finite dimensional vector space* V *over* $\mathbf{k}$ *is called an* A-*comodule if there is a given* $\mathbf{k}$-*linear morphism*

$$\rho : V \to V \otimes A,$$

satisfying the axioms:

$$\begin{tikzcd} V \arrow[r, "\rho"] \arrow[d, "\rho"] & V \otimes A \arrow[d, "\mathrm{id}_V \otimes \Delta"] \\ V \otimes A \arrow[r, "\rho \otimes \mathrm{id}_A"] & V \otimes A \otimes A \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, "\rho"] \arrow[d, "\mathrm{id}_V"] & V \otimes A \arrow[d, "\mathrm{id}_V \otimes \varepsilon"] \\ V \arrow[r, "\sim"] & V \otimes \mathbf{k} \end{tikzcd}$$

If A is a differential coalgebra with Δ and ε then for a comodule V over A the $\mathbf{k}$-space $\mathbf{k}[\partial]_{\leqslant 1} \otimes V = V^{(1)}$ has a natural A-comodule structure

$$\rho(f \otimes v) := f \otimes \rho(v) \tag{19}$$

for $f \in \mathbf{k}[\partial]_{\leqslant 1}$ and $v \in V$.

We denote the category of comodules over a differential coalgebra A by $\mathbf{CoDiff}_A$ with the induced differentiation (19). For a ∂-$\mathbf{k}$-Hopf algebra A the functor

$$G : \mathbf{Alg}_{\mathbf{k}}(\partial) \to \{\mathrm{Groups}\}, \quad R \mapsto \mathrm{Hom}(A, R)$$

is called an *affine differential algebraic group scheme* generated by A (see [15, Section 3.3]). In this case $V \in \mathcal{O}b(\mathbf{CoDiff}_A)$ is called a *differential representation* of G ([15, Definition 7, Theorem 1]). The category $\mathbf{CoDiff}_A$ is also denoted by $\mathbf{Rep}_G$.

The differential $\mathbf{GL}_n$ by definition is the functor represented by the ∂-$\mathbf{k}$-Hopf algebra

$$\mathbf{k}\{X_{11}, \ldots, X_{nn}, 1/\det(X)\},$$

where X_{ij} are differential indeterminates. The comultiplication Δ and coinverse S are defined on X_{ij} in the usual way. Their prolongation on the derivatives of X_{ij} can be obtained by differentiation.

Definition 12 *An affine differential algebraic group scheme G is called a linear differential algebraic group if there exists an embedding:*

$$G \to \mathbf{GL}_n$$

for some $n \in \mathbb{Z}_{\geqslant 1}$.

Example 13 *Let G be a linear differential algebraic group. The category $\mathbf{Rep}_G$ is a rigid abelian tensor category with $\mathrm{End}(\underline{\mathbf{1}}) = \mathbf{k}$ and $\mathbf{k}$ is a differential field. The differential structure on $\mathbf{Rep}_G$ is given by* (2), (3), *and* (19). *Therefore, the $\mathbf{k}$-linear maps* (4) *and* (5) *are G-morphisms. Splitting* (12) *comes from splitting* (6). *It remains to show that there exist G-equivariant maps*

$$S_X, T_{X,Y}, \text{and } D_X$$

for all $X, Y \in \mathcal{O}b\,(\mathbf{Rep}_G)$. The morphism S_X is defined in [15, Lemma 4, part (3)]. For $T_{X,Y}$ and D_X this is shown in [15, Lemmas 7 and 11], respectively. We will now show that, in general,

$$S_X \neq \left(\mathrm{id}_X \otimes (D_X)^{-1}\right) \circ T_{X,X^*}.$$

Indeed, in the notation of Section 4.3 and [15] for $X = \mathrm{span}\{v_1, \ldots, v_n\}$ and

$X^* = \operatorname{span}\{v^1, \dots, v^n\}$ *with* $v^j(v_i) = \delta_{i,j}$, $1 \leqslant i, j \leqslant n$, *we have*

$$\operatorname{ev}_X \left(S_X \left(\partial \otimes \left(v_i \otimes v^j\right)\right)\right) = \operatorname{ev}_X \left((\partial \otimes v_i) \otimes F\left(v^j\right)\right) = F\left(v^j\right)(\partial \otimes v_i) = \\ = \partial\left(v^j(v_i)\right) = \partial(\delta_{i,j}) = 0.$$

where

$$F : X^* \to \left(X^{(1)}\right)^*,\ F(u)(v) = u(v),\ F(u)(\partial \otimes v) = \partial(u(v)),\ v \in X,\ u \in X^*.$$

On the other hand (see [15, proof of Lemma 11] for $(D_X)^{-1}$*),*

$$\begin{aligned} &\operatorname{ev}_X \circ \left(\operatorname{id}_X \otimes (D_X)^{-1}\right) \circ T_{X,X^*}\left(\partial \otimes \left(v_i \otimes v^j\right)\right) = \\ &= \operatorname{ev}_X \circ \left(\operatorname{id}_X \otimes (D_X)^{-1}\right)\left(\partial \otimes v_i \otimes 1 \otimes v^j + 1 \otimes v_i \otimes \partial \otimes v^j\right) = \\ &= \operatorname{ev}_X \left(\partial \otimes v_i \otimes (\partial \otimes v_j)^* + 1 \otimes v_i \otimes F\left(v^j\right)\right) = \delta_{i,j} + \delta_{i,j} \neq 0. \end{aligned}$$

4 Main result

4.1 Constructing a differential fibre functor

Let $\mathcal{C}$ be a differential Tannakian category over a ∂-field $\mathbf{k}$ of characteristic zero. As in [1, Theorem 7.1], **suppose** that for each $X \in \mathcal{O}b(\mathcal{C})$ we have

$$\dim X \in \mathbb{Z}_{\geqslant 0},$$

where $\dim X = \operatorname{Tr}(\operatorname{id}_X)$ and $\operatorname{Tr} : \operatorname{End}(X) = \operatorname{Hom}(\underline{\mathbf{1}}, X^* \otimes X) \to \operatorname{End}(\underline{\mathbf{1}})$ given by the evaluation morphism (see [3, formula (1.7.3)]). As it is shown in [1, Section 7.18], one can construct a ring object A of the category $\mathcal{I}\mathrm{nd}\,\mathcal{C}$ (see [1, page 167]) with $\mu : A \otimes A \to A$ and $e : \underline{\mathbf{1}} \to A$ such that for all $X \in \mathcal{O}b(\mathcal{C})$ we have

$$A \otimes X \cong A^{\dim X} \tag{20}$$

and

$$\omega(X) := \operatorname{Hom}(\underline{\mathbf{1}}, A \otimes X) \tag{21}$$

is a fibre functor. However, this functor does not necessarily preserve the differential structure, that is, one does not necessarily have a functorial isomorphism

$$\omega\left(X^{(1)}\right) \to \omega(X)^{(1)}.$$

Therefore, in order to achieve our goal we have to modify the functor. Based on A, we shall construct another ring object B of $\mathcal{I}\mathrm{nd}\,\mathcal{C}$ giving the required functorial isomorphism.

Theorem 14 *There exists a ring object B of $\mathcal{I}\mathrm{nd}\,\mathcal{C}$ such that if we define the functor ω by*

$$\omega(X) = \mathrm{Hom}(\underline{\mathbf{1}}, B \otimes X), \tag{22}$$

$X \in \mathcal{O}b(\mathcal{C})$, we will have:

(1) ω is a fibre functor over $\mathrm{Hom}(\underline{\mathbf{1}}, B)$ in the sense of [1, Section 1.9],
(2) there exists a functorial isomorphism $\omega\left(X^{(1)}\right) \to \omega(X)^{(1)}$, $X \in \mathcal{O}b(\mathcal{C})$.

PROOF. Define a "differential polynomial" ring object (cf. [31, Section 1.2])

$$B := \bigcup_{n \geqslant 1,\, p \geqslant 0} \mathrm{Sym}^n\left(F^p(A)\right).$$

To continue with our argument we need the following

Lemma 15 *We have:*

(1) there exists an isomorphism

$$d_B : \omega\left(B^{(1)}\right) \cong \omega(B)^{(1)} \tag{23}$$

functorial with respect to $\mathrm{End}(B)$,
(2) B is faithfully flat over A,
(3) tensoring with B splits short exact sequences from $\mathcal{C}$,
(4) for each $X \in \mathcal{O}b(\mathcal{C})$ there exists an isomorphism

$$B \otimes X \cong B^{\dim X}. \tag{24}$$

PROOF. There is a functorial isomorphism

$$(B \otimes B)^{(1)} \to B \otimes B^{(1)} \tag{25}$$

given by iterated applications of functorial injections $T_{.,.}$ defined in (14). Indeed, diagram (14) implies by induction an injection for each n and p

$$(\mathrm{Sym}^n(F^p(A)))^{(1)} \to \mathrm{Sym}^n(F^{p+1}(A)),$$

making the following diagram commutative:

$$\begin{array}{ccc} (\mathrm{Sym}^n(F^p(A)))^{(1)} & \longrightarrow & \mathrm{Sym}^n(F^{p+1}(A)) \\ \uparrow & & \uparrow \\ \mathrm{Sym}^n(F^p(A)) & = & \mathrm{Sym}^n(F^p(A)) \end{array}$$

which after taking the union with respect to all n and p induces a functorial isomorphism

$$B^{(1)} \to B. \tag{26}$$

A similar argument provides a functorial isomorphism

$$(B \otimes B)^{(1)} \to B \otimes B,$$

which combined with (26) gives isomorphism (25). Now, there is also a functorial isomorphism

$$\operatorname{Hom}(\underline{\mathbf{1}}, Y)^{(1)} \to \operatorname{Hom}\left(\underline{\mathbf{1}}, Y^{(1)}\right)$$

for any object Y. Setting $Y = B \otimes B$, this gives us (23). Let

$$0 \to X \to Y \to Z \to 0$$

be a short exact sequence in $\mathcal{C}$. By [1, Section 7.18], the sequence

$$0 \to A \otimes X \to A \otimes Y \to A \otimes Z \to 0$$

splits. Then, the functors

$$(F^p)^{\otimes n} \otimes \mathrm{id}_{\mathcal{C}}, \quad p \geqslant 0,\ n \geqslant 1, \tag{27}$$

give a splitting of the exact sequence

$$0 \to B \otimes X \to B \otimes Y \to B \otimes Z \to 0.$$

Using functors (27) and additivity of F, isomorphism (24) follows from (20). As in the proof of [1, Lemma 7.15], the ring object B is faithfully flat over A, and, therefore, we have constructed B in $\mathcal{I}\mathrm{nd}(\mathcal{C})$ with the desired properties. $\square$

Statement (2–4) of the lemma imply statement (1) of the theorem (see [1, Sections 1.9 and 7.18]). It follows from (24) that the functorial in X morphism of B-modules

$$\pi_1 : B^{(\oplus \operatorname{Hom}(B, B \otimes X))} \to B \otimes X \tag{28}$$

is surjective. Here, for a set I, $B^{(\oplus I)}$ denotes the direct sum of B with respect to I. The functorial isomorphism chosen in (23) induces (by taking direct sums) a functorial isomorphism

$$d_{B^{(\oplus I)}} : \omega\left(\left(B^{(\oplus I)}\right)^{(1)}\right) \to \omega\left(B^{(\oplus I)}\right)^{(1)}. \tag{29}$$

By taking a functorial in X free resolution of (28):

$$B^{(\oplus I_X)} \xrightarrow{\pi_2} B^{(\oplus \operatorname{Hom}(B, B \otimes X))} \xrightarrow{\pi_1} B \otimes X \longrightarrow 0, \tag{30}$$

where $I_X := \operatorname{Hom}\left(B, \ker\left(B^{(\oplus \operatorname{Hom}(B, B\otimes X))} \to B \otimes X\right)\right)$, one gets a functorial (in X) isomorphism

$$d_X : \omega\left(X^{(1)}\right) \to \omega(X)^{(1)}. \tag{31}$$

Indeed, the functor F is exact on the right and, therefore, transforms free resolution (30) of $B \otimes X$ to a free resolution of $(B \otimes X)^{(1)}$ as follows:

$$\begin{array}{ccccc}
B^{(\oplus I_X)} & \longrightarrow & \left(B^{(\oplus I_X)}\right)^{(1)} & \longrightarrow & B^{(\oplus I_X)} \\
{\scriptstyle \pi_2}\big\downarrow & & {\scriptstyle F(\pi_2)}\big\downarrow & & {\scriptstyle \pi_2}\big\downarrow \\
B^{(\oplus \operatorname{Hom}(B, B\otimes X))} & \longrightarrow & \left(B^{(\oplus \operatorname{Hom}(B, B\otimes X))}\right)^{(1)} & \longrightarrow & B^{(\oplus \operatorname{Hom}(B, B\otimes X))} \\
{\scriptstyle \pi_1}\big\downarrow & & {\scriptstyle F(\pi_1)}\big\downarrow & & {\scriptstyle \pi_1}\big\downarrow \\
B \otimes X & \longrightarrow & (B \otimes X)^{(1)} & \longrightarrow & B \otimes X \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
0 & & 0 & & 0
\end{array}$$

Applying the exact functor ω and using the isomorphism given in (29), we have:

$$\begin{array}{ccc}
\omega\left(\left(B^{(\oplus I_X)}\right)^{(1)}\right) & \xrightarrow{d_{B^{(\oplus I_X)}}} & \omega\left(B^{(\oplus I_X)}\right)^{(1)} \\
\big\downarrow & & \big\downarrow \\
\omega\left(\left(B^{(\oplus \operatorname{Hom}(B, B\otimes X))}\right)^{(1)}\right) & \xrightarrow{d_{B^{(\oplus \operatorname{Hom}(B, B\otimes X))}}} & \omega\left(B^{(\oplus \operatorname{Hom}(B, B\otimes X))}\right)^{(1)} \\
\big\downarrow & & \big\downarrow \\
\omega\left((B \otimes X)^{(1)}\right) & & \omega(B \otimes X)^{(1)} \\
\big\downarrow & & \big\downarrow \\
0 & & 0
\end{array}$$

where the upper square is commutative because the isomorphism d_B is functorial. Therefore, one can define a functorial isomorphism

$$d_{B\otimes X} : \omega\left((B \otimes X)^{(1)}\right) \to \omega(B \otimes X)^{(1)} \tag{32}$$

By recalling (21) and restricting in (32) to

$$\mathcal{I}_X : X \to \underline{\mathbf{1}} \otimes X \to B \otimes X,$$

employing the morphism

$$\mu \otimes \mathrm{id}_X : B \otimes B \otimes X \to B \otimes X$$

with $(\mu \otimes \mathrm{id}_X) \circ (\mathrm{id}_B \otimes \mathcal{I}_X) = \mathrm{id}_{B\otimes X}$, the isomorphism $d_{B\otimes X}$ induces a functorial in X isomorphism d_X in (31). Indeed, for

$$\varphi \in \omega\left(X^{(1)}\right) = \operatorname{Hom}\left(\underline{\mathbf{1}}, B \otimes X^{(1)}\right)$$

we have

$$\omega(F(\mathrm{id}_B \otimes \mathcal{I}_X)) \circ \varphi \in \mathrm{Hom}\left(\underline{\mathbf{1}}, B \otimes (B \otimes X)^{(1)}\right) = \omega\left((B \otimes X)^{(1)}\right) \to$$
$$\to \omega(B \otimes X)^{(1)} = \mathrm{Hom}(\underline{\mathbf{1}}, B \otimes B \otimes X)^{(1)} \xrightarrow{\circ\omega(F(\mu \otimes \mathrm{id}_X))}$$
$$\to \mathrm{Hom}(\underline{\mathbf{1}}, B \otimes X)^{(1)} = \omega(X)^{(1)}.$$

□

4.2 Additional assumption

Assume now that

$$\mathrm{Hom}(\underline{\mathbf{1}}, B) \cong \mathbf{k}\,. \tag{33}$$

It then follows from isomorphism (24) that $\omega(X)$ is a finite dimensional vector space over $\mathbf{k}$ for all $X \in \mathcal{O}b(\mathcal{C})$ making ω a fibre functor over $\mathbf{k}$; and we have a functorial isomorphism

$$d_X : \omega\left(X^{(1)}\right) \cong \omega(X)^{(1)},$$

such that the following diagram is commutative:

$$\begin{tikzcd} \omega(X) \arrow[r, "\omega(i_X)"] \arrow[d, equal] & \omega\left(X^{(1)}\right) \arrow[r, "\omega(\varphi_X)"] \arrow[d, "d_X"'] & \omega(X) \arrow[d, equal] \\ \omega(X) \arrow[r, "i_{\omega(X)}"] & \omega(X)^{(1)} \arrow[r, "\varphi_{\omega(X)}"] & \omega(X) \end{tikzcd}$$

4.3 Equivalence of categories

Recall (from [15, Definition 8]) that the functor $\mathbf{Aut}^{\otimes,\partial}(\omega)$ from ∂-$\mathbf{k}$-algebras to groups is defined as follows. For a ∂-$\mathbf{k}$-algebra K the group $\mathbf{Aut}^{\otimes,\partial}(\omega)(K)$ is the set of sequences

$$\lambda(K) = (\lambda_X \mid X \in \mathcal{O}b(\mathcal{C})) \in \mathbf{Aut}^{\otimes,\partial}(\omega)(K)$$

such that λ_X is a K-linear automorphism of $\omega(X) \otimes K$ for each object X, that is, $\lambda_X \in \mathbf{Aut}_K(\omega(X) \otimes K)$, such that

- for all X_1, X_2 we have

$$\lambda_{X_1 \otimes X_2} = \lambda_{X_1} \otimes \lambda_{X_2}, \tag{34}$$

- $\lambda_{\underline{1}}$ is the identity map on $\omega(\underline{1}) \otimes K$,
- for every $\alpha \in \mathrm{Hom}(X, Y)$ we have

$$\lambda_Y \circ (\alpha \otimes \mathrm{id}_K) = (\alpha \otimes \mathrm{id}_K) \circ \lambda_X : \omega(X) \otimes K \to \omega(Y) \otimes K, \qquad (35)$$

- for every X we have

$$F(\lambda_X) = \lambda_{X^{(1)}}, \qquad (36)$$

- the group operation $\lambda_1(K) \cdot \lambda_2(K)$ is defined by composition in each $\mathbf{Aut}_K(\omega(X) \otimes K)$.

Theorem 16 *Let $\mathcal{C}$ be a differential Tannakian category and $\omega : \mathcal{C} \to \mathbf{Vect_k}$ be the fibre functor defined by* (22) *with additional assupmtion* (33)*. Then*

$$(\mathcal{C}, \omega) \cong \mathbf{Rep}_G$$

for the differential group scheme

$$G = \mathbf{Aut}^{\otimes,\partial}(\omega),$$

which is a pro-linear differential algebraic group, that is, G as a functor is represented by a direct limit of finitely ∂-generated ∂-$\mathbf{k}$-Hopf algebras.

PROOF. The proof will consist of several steps, lasting until the end of the paper finishing with Proposition 23, which combined with [18, Corollary 2] gives the result.

First, we will recall the constructions given in [18, Section 3.4]. Let X be an object of $\mathcal{C}$ and $\{\{X\}\}$ (respectively, $\mathcal{C}_X$) be the full abelian (respectively, full abelian tensor) subcategory of $\mathcal{C}$ generated by X (respectively, containing X and closed under the functor F). Consider

$$F_X = \bigoplus_{V \in \mathcal{O}b(\{\{X\}\})} V \otimes \omega(V)^*.$$

This tensor product is understood in the sense of [3, page 131]. For an object V of the category $\{\{X\}\}$ we have the canonical injections:

$$j_V : V \otimes \omega(V)^* \to F_X.$$

Consider the minimal subobject R_X of F_X with subobjects

$$\left\{ (j_V(\mathrm{id} \otimes \phi^*) - j_W(\phi \otimes \mathrm{id}))(V \otimes \omega(W)^*) \;\middle|\; V, W \in \mathcal{O}b(\{\{X\}\}), \phi \in \mathrm{Hom}(V, W) \right\}.$$

We let

$$P_X = F_X \big/ R_X.$$

We now put

$$A_X = \omega(P_X),$$

which is a **k**-vector space. Let V be an object of $\{\{X\}\}$. For

$$v \in \omega(V),\ u \in \omega(V)^*$$

we denote by

$$a_V\,(v \otimes u)$$

the image in A_X of

$$\omega(j_V)\,(v \otimes u)\,.$$

So, for any $\phi \in \mathrm{Hom}(V, W)$ we have

$$a_V\,(v \otimes \omega(\phi)^*(u)) = a_W(\omega(\phi)(v) \otimes u). \tag{37}$$

for all

$$v \in \omega(V),\ u \in \omega(W)^*.$$

Let us define a **comultiplication** on A_X. Let $\{v_i\}$ be a basis of $\omega(V)$ with the dual basis $\{u_j\}$ of $\omega(V)^*$. We let

$$\Delta : a_V(v \otimes u) \mapsto \sum_i a_V(v_i \otimes u) \otimes a_V(v \otimes u_i), \tag{38}$$

giving the same as (10). The **counit** is defined in the following way:

$$\varepsilon : a_V(v \otimes u) \mapsto \omega(\mathrm{ev}_V)(v \otimes u) = u(v). \tag{39}$$

The **coinverse** is defined as follows. We let

$$S : a_V(v \otimes u) \mapsto a_{V^*}(\omega(\psi_{V,V^*})(v \otimes u)) = a_{V^*}(u \otimes v). \tag{40}$$

We shall introduce a **differential algebra** structure on

$$\mathcal{A}_X = \varinjlim_{Y \in \mathcal{O}b(\mathcal{C}_X)} A_Y$$

as follows. Let $V, W \in \mathcal{O}b(\mathcal{C}_X)$, $v \in \omega(V)$, $w \in \omega(W)$, $u \in \omega(V)^*$, and $t \in \omega(W)^*$. Define a **multiplication** on $\mathcal{A}_X$ by

$$a_V(v \otimes u) \cdot a_W(w \otimes t) := a_{V \otimes W}((v \otimes w) \otimes (u \otimes t)). \tag{41}$$

Before we define a derivation on $\mathcal{A}_X$ we will do some preparation. Since $T_{\cdot,\cdot}$ is functorial in each argument, the following diagram is commutative:

$$\begin{array}{ccc} ((V\otimes V^*)\otimes(W\otimes W^*))^{(1)} & \xrightarrow{T_{V\otimes V^*,W\otimes W^*}} & (V\otimes V^*)^{(1)}\otimes(W\otimes W^*)^{(1)} \\ \Big\downarrow{\scriptstyle F(\mathrm{ev}_V\otimes\mathrm{ev}_W)} & & \Big\downarrow{\scriptstyle F(\mathrm{ev}_V)\otimes F(\mathrm{ev}_W)} \\ (\underline{\mathbf{1}}\otimes\underline{\mathbf{1}})^{(1)} & \xrightarrow{T_{\underline{\mathbf{1}},\underline{\mathbf{1}}}} & \underline{\mathbf{1}}^{(1)}\otimes\underline{\mathbf{1}}^{(1)} \end{array} \tag{42}$$

It follows from (14) that

$$\begin{aligned} \omega\left(T_{V\otimes V^*,W\otimes W^*}\right)(\partial\otimes(v\otimes u\otimes w\otimes t)) = \partial\otimes(v\otimes u)\otimes 1\otimes(w\otimes t)+ \\ +1\otimes(v\otimes u)\otimes\partial\otimes(w\otimes t)+ \\ +1\otimes(\Phi_{V\otimes V^*}\otimes\mathrm{id}+\mathrm{id}\otimes\Phi_{W\otimes W^*})((v\otimes u)\otimes(w\otimes t)), \end{aligned} \tag{43}$$

where Φ is a functorial $\mathbf{k}$-liner map.

Remark 17 *According to* (43) *the morphism $T_{V\otimes V^*,W\otimes W^*}$ gives us the product rule for the tensor structure up to the linear term. We will take this linear term given by Φ into account defining the differential structure on $\mathcal{A}_X$ in* (45).

Lemma 18 *We have*

$$\omega(\mathrm{ev}_V)(\Phi_{V\otimes V^*}(v\otimes u)) = 0. \tag{44}$$

PROOF. Indeed, diagram (14) with $X = Y = \underline{\mathbf{1}}$ together with splitting (12) give us

$$\omega(T_{\underline{\mathbf{1}},\underline{\mathbf{1}}})(\partial\otimes(u(v)\otimes t(w))) = \partial\otimes u(v)\otimes\omega(i_{\underline{\mathbf{1}}})(t(w))+\omega(i_{\underline{\mathbf{1}}})(u(v))\otimes\partial\otimes t(w).$$

Therefore,

$$\omega(\mathrm{ev}_V\otimes\mathrm{ev}_W)\left((\Phi_{V\otimes V^*}\otimes\mathrm{id}+\mathrm{id}\otimes\Phi_{W\otimes W^*})((v\otimes u)\otimes(w\otimes t))\right) = 0.$$

Since Φ is functorial, taking $W := V$, $w := v$, $t := u$, we obtain the result. □

We define a **derivation** on $\mathcal{A}_X$ as follows:

$$\partial(a_V(v\otimes u)) := a_{V^{(1)}}\left(\omega(S_V)(\partial\otimes(v\otimes u)+2\otimes\Phi_{V\otimes V^*}(v\otimes u))\right). \tag{45}$$

Lemma 19 *Let $V\in\mathcal{O}b(\mathcal{C}_X)$, $v\in\omega(V)$, and $u\in\omega(V)^*$. Then, we have*

$$\begin{aligned} a_{(V^*)^{(1)}}\left(\omega\left(S_{V^*}\circ F\left(\psi_{V,V^*}\right)\right)(\partial\otimes(v\otimes u)+2\otimes\Phi_{V\otimes V^*}(v\otimes u))\right) = \\ = a_{\left(V^{(1)}\right)^*}\left(\omega\left(\psi_{V^{(1)},\left(V^{(1)}\right)^*}\circ S_V\right)(\partial\otimes(v\otimes u)+2\otimes\Phi_{V\otimes V^*}(v\otimes u))\right) \end{aligned} \tag{46}$$

and the "differential evaluation" is

$$\omega\left(\mathrm{ev}_{V^{(1)}}\circ S_V\right)\left(\partial\otimes(v\otimes u)+2\otimes\Phi_{V\otimes V^*}(v\otimes u)\right)=\partial(u(v)). \tag{47}$$

PROOF. To show (46) note that using (37) we obtain that

$$\begin{aligned}
&a_{(V^*)^{(1)}}\left(\omega\left(S_{V^*}\circ F\left(\psi_{V,V^*}\right)\right)\left(\partial\otimes(v\otimes u)+2\otimes\Phi_{V\otimes V^*}(v\otimes u)\right)\right)=\\
&=a_{(V^*)^{(1)}}(\omega(S_{V^*})(\partial\otimes(u\otimes v)+2\otimes\Phi_{V\otimes V^*}(v\otimes u)))=\\
&=a_{\left(V^{(1)}\right)^*}\left(\omega\left(\left(D_V^{-1}\otimes {D_V}^*\right)\circ S_{V^*}\right)\left(\partial\otimes(u\otimes v)+2\otimes\Phi_{V\otimes V^*}(v\otimes u)\right)\right),
\end{aligned}$$

since D_V is an isomorphism. Moreover, from (17) we conclude that

$$\begin{aligned}
&a_{\left(V^{(1)}\right)^*}\left(\omega\left(\psi_{V^{(1)},\left(V^{(1)}\right)^*}\circ S_V\right)\left(\partial\otimes(v\otimes u)+2\otimes\Phi_{V\otimes V^*}(v\otimes u)\right)\right)=\\
&=a_{\left(V^{(1)}\right)^*}\left(\omega\left(\left(D_V^{-1}\otimes {D_V}^*\right)\circ S_{V^*}\right)\left(\partial\otimes(u\otimes v)+2\otimes\Phi_{V\otimes V^*}(v\otimes u)\right)\right),
\end{aligned}$$

which proves (46). Let us show (47) now. According to (13) and (44) we have

$$\begin{aligned}
&\omega\left(\mathrm{ev}_{V^{(1)}}\circ S_V\right)\left(\partial\otimes(v\otimes u)+2\otimes\Phi_{V\otimes V^*}(v\otimes u)\right)=\\
&\quad=\omega\left(i_{\underline{1}}^{-1}\circ F(\mathrm{ev}_V)\right)\left(\partial\otimes(v\otimes u)+2\otimes\Phi_{V\otimes V^*}(v\otimes u)\right)=\\
&\quad=\omega\left(i_{\underline{1}}^{-1}\right)(\partial\otimes u(v)+0)=\omega\left(i_{\underline{1}}^{-1}\right)(\partial(u(v))\otimes 1+u(v)\cdot\partial\otimes 1)=\\
&\quad=\partial(u(v))
\end{aligned}$$

because $\omega\left(i_{\underline{1}}^{-1}\right)$ is the projection to the first copy of $\mathbf{k}$ in splitting (6). $\square$

Remark 20 *Now, it follows from [15, Lemma 11] that our coinverse S is a ∂-$\mathbf{k}$-algebra homomorphism. Formula* (47) *shows that the counit ε is a ∂-$\mathbf{k}$-algebra homomorphism as well. It remains to show that the comultiplication Δ is a ∂-$\mathbf{k}$-algebra homomorphism and that the differential structure we defined satisfies the product rule. We will do this in Lemmas 21 and 22. Correctness of Δ is shown in [15, Lemma 9]. The Hopf algebra commutative diagrams for Δ, S, and ε are justified in [15, Lemmas 9–11].*

Lemma 21 *The comultiplication Δ defined in* (38) *is a ∂-$\mathbf{k}$-algebra homomorphism of $\mathcal{A}_X$.*

PROOF. Follows from diagram (15) and the observations given in Remark 20. $\square$

Lemma 22 *Formula* (45) *defines a differential algebra structure on $\mathcal{A}_X$, that is,*

$$\partial\left(a_V(v\otimes u)\cdot a_W(w\otimes t)\right)=\partial(a_V(v\otimes u))\cdot a_W(w\otimes t)+a_V(v\otimes u)\cdot\partial(a_W(w\otimes t))$$

for all $V, W \in \mathcal{O}b(\mathcal{C})$, $v \in \omega(V)$, $u \in \omega(V)^*$, $w \in \omega(W)$, and $t \in \omega(W)^*$.

PROOF. By definition,

$$\begin{aligned}
&\partial\left(a_V(v\otimes u)\cdot a_W(w\otimes t)\right) = \partial\left(a_{V\otimes W}((v\otimes w)\otimes(u\otimes t))\right) = \\
&= a_{(V\otimes W)^{(1)}}(\omega(S_{V\otimes W})(\partial\otimes((v\otimes w)\otimes(u\otimes t))+ \\
&\qquad + 2\otimes\Phi_{(V\otimes W)\otimes(V\otimes W)^*}((v\otimes w)\otimes(u\otimes t)))).
\end{aligned}$$

We have

$$\begin{aligned}
&(\partial a_V(v\otimes u))\cdot a_W(w\otimes t) + a_V(v\otimes u)\cdot(\partial a_W(w\otimes t)) = \\
&= a_{V^{(1)}}(\omega(S_V)(\partial\otimes(v\otimes u)+2\otimes\Phi_{V\otimes V^*}(v\otimes u)))\cdot a_W(w\otimes t)+ \\
&\quad + a_V(v\otimes u)\cdot a_{W^{(1)}}(\omega(S_W)(\partial\otimes(w\otimes t)+2\otimes\Phi_{W\otimes W^*}(w\otimes t))) = \\
&= a_{V^{(1)}\otimes W^{(1)}}(\omega(S_V\otimes S_W)((\partial\otimes(v\otimes u)+ \\
&\qquad\qquad + 2\otimes\Phi_{V\otimes V^*}(v\otimes u))\otimes\omega(i_W)(w\otimes t)+ \\
&\qquad\qquad + \omega(i_V)(v\otimes u)\otimes(\partial\otimes(w\otimes t)+2\otimes\Phi_{W\otimes W^*}(w\otimes t)))) = \\
&= a_{V^{(1)}\otimes W^{(1)}}(\omega((S_V\otimes S_W)\circ T_{V\otimes V^*, W\otimes W^*})(\partial\otimes((v\otimes u)\otimes(w\otimes t))+ \\
&\qquad\qquad + 2\otimes\Phi_{(V\otimes W)\otimes(V\otimes W)^*}((v\otimes w)\otimes(u\otimes t)))).
\end{aligned}$$

The last equality follows from (43) and the facts that Φ is functorial and the category $\mathcal{C}$ has an associativity morphism for the tensor product:

$$\Phi_{V\otimes V^*}\otimes\mathrm{id}_{W\otimes W^*}+\mathrm{id}_{V\otimes V^*}\otimes\Phi_{W\otimes W^*} = (\mathrm{id}_V\otimes\psi_{W,V^*}\otimes\mathrm{id}_{W^*})\circ\Phi_{(V\otimes W)\otimes(V\otimes W)^*}.$$

Now, the product rule we are showing follows from diagram (16), formula (37), and the condition that $T_{V,W}$ is injective. $\square$

We have all ingredients by now to recover a (pro-)linear differential algebraic group from $(\mathcal{C}, \omega)$.

Proposition 23 *The group G_X, defined by $G_X(R) = \mathrm{Hom}(\mathcal{A}_X, R)$ for each ∂-$\mathbf{k}$-algebra R, is a linear differential algebraic group.*

PROOF. It follows from Lemmas 19, 21, and 22 and Remark 20 combined with [18, Proposition 2] and [18, Corollary 1] that $\mathcal{A}_X$ is a finitely ∂-generated ∂-$\mathbf{k}$-Hopf algebra. In particular, our Lemmas 19 and 22 are used to substitute [18, Lemma 6] and [15, Lemmas 7 and 11] employed in [18, Proposition 2]. $\square$

Then, $G = \varprojlim G_X$ is represented by $\varinjlim \mathcal{A}_X$ and this finishes the proof of Theorem 16. $\square$

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