Parameterized Picard–Vessiot extensions and Atiyah extensions

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Abstract

Generalizing Atiyah extensions, we introduce and study differential abelian tensor categories over differential rings. By a differential ring, we mean a commutative ring with an action of a Lie ring by derivations. In particular, these derivations act on a differential category. A differential Tannakian theory is developed. The main application is to the Galois theory of linear differential equations with parameters. Namely, we show the existence of a parameterized Picard–Vessiot extension and, therefore, the Galois correspondence for many differential fields with, possibly, non-differentially closed fields of constants, that is, fields of functions of parameters. Other applications include a substantially simplified test for a system of linear differential equations with parameters to be isomonodromic, which will appear in a separate paper. This application is based on differential categories developed in the present paper, and not just differential algebraic groups and their representations.

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Contents

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1. Introduction

Classical differential Galois theory studies symmetry groups of solutions of linear differential equations, or equivalently the groups of automorphisms of the corresponding extensions of differential fields; the groups that arise are linear algebraic groups over the field of constants. This theory, started in the 19th century by Picard and Vessiot, was put on a firm modern footing by Kolchin [43]. In [46], Landesman initiated a generalized differential Galois theory that uses Kolchin's axiomatic approach [44] and realizes differential algebraic groups as Galois groups. The parameterized Picard–Vessiot Galois theory considered by Cassidy and Singer in [10] is a special case of the Landesman generalized differential Galois theory and studies symmetry groups of the solutions of linear differential equations whose coefficients contain parameters. This is done by constructing a differential field containing the solutions and their derivatives with respect to the parameters, called a parameterized Picard–Vessiot (PPV) extension, and studying its group of differential symmetries, called a parameterized differential Galois group. The Galois groups that arise are linear differential algebraic groups, which are defined by polynomial differential equations in the parameters. Another approach to the Galois theory of systems of linear differential equations with parameters is given in [7], where the authors study Galois groups for generic values of the parameters.

The tradition in classical differential Galois theory has been to assume that the field of constants of the coefficient field is algebraically closed [43, 67, 49]. Cassidy and Singer follow the spirit of this tradition. For example, as in [10, Section 3], consider the differential equation $\partial_x f = \frac{t}{x} f$. The solutions of this equation will be functions of x, which also depend on the parameter t. If x and t are complex variables, these solutions are of the form $a \cdot x^t$, $a \in \mathbb{C}(t)$, and the field generated by the solutions together with their derivatives with respect to both x and t is $\mathbb{C}(x, t, x^t, \log(x))$. The automorphisms of this field over $\mathbb{C}(x,t)$ are given by non-zero elements a in $\mathbb{C}(t)$ that satisfy the differential equation $\partial_t\left(\frac{\partial_t(a)}{a}\right) = 0$. However, as explained in [10], this group does not have enough elements to give a Galois correspondence between subgroups of the group of automorphisms and intermediate differential fields. This leads Cassidy and Singer to require that the field of ∂_x-constants is a ∂_t-closed differential field (or, more generally, that the field of functions of the parameters is differentially closed).

Recall that a differential field is differentially closed if it contains solutions of consistent systems of polynomial differential equations with coefficients in the field. However, this requirement is an obstacle to the practical applicability of the methods of the parameterized theory. A similar phenomenon occurs in the classical differential Galois theory: if the field of constants is not algebraically closed, a Picard–Vessiot extension might not exist at all (see the famous counterexample of Seidenberg [69]); therefore, there are no differential Galois group and Galois correspondence if this happens. Since the beginning of the theory [43], it has been a major open problem in Picard–Vessiot theory to determine to what extent one can avoid taking the algebraic closure of the field of constants. In the present paper, we are able to remove the assumption that the field of constants has to be differentially closed in order to have a Galois correspondence in the parameterized case.

With this aim, following [13], we use the Tannakian approach to linear differential equations. In particular, in the usual non-parameterized case [67], we show in Theorem 2.2 that, under a relatively existentially closed assumption on the field of constants (which includes the case of formally real fields with real closed fields of constants, as well as fields that are purely transcendental extensions of the fields of constants), one can always construct a Picard–Vessiot extension for a system of linear differential equations. To treat the parameterized case, which is our main interest, we develop a theory of differential categories over differential rings and the corresponding theory of differential Tannakian categories. Here, by a differential ring, we mean a commutative ring together with a Lie ring acting on it by derivations (this is also often called a Lie algebroid). The theory of differential Tannakian categories allows us to show that a PPV extensions exists under a much milder assumption (relatively differentially closed) on the field of constants than being differentially closed, Theorem 2.5. This assumption is satisfied by many differential fields used in practice, Theorem 2.8.

The importance of the existence of a PPV extension is that it leads to a Galois correspondence, Section 8.1. The Galois group is a differential algebraic group [8, 9, 44, 62, 11, 55, 56, 71] defined over the field of constants, which, after passing to the differential closure, coincides with the parameterized differential Galois group from [10], Corollary 8.10. The Galois correspondence, as usual, can be used to analyze how one may build the extension, step-by-step, by adjoining solutions of differential equations of lower order, corresponding to taking intermediate extensions of the base field. For example, consider the special function known as the incomplete Gamma function γ, which is the solution of a second-order parameterized differential equation [10, Example 7.2] over $\mathbb{Q}(x,t)$. Knowing the relevant Galois correspondence, one could show how to build the differential field extension of $\mathbb{Q}(x,t)$ containing γ without taking the (unnecessary and unnatural) differential closure of $\mathbb{Q}(t)$.

The general nature of our approach will allow in the future to adapt it to the Galois theory of linear difference equations, which has numerous applications. Differential algebraic dependencies among solutions of difference equations were studied in [31, 32, 33, 18, 20, 19, 21, 17, 16, 26]. Among many applications of the Galois theory, one has an algebraic proof of the differential algebraic independence of the Gamma function over $\mathbb{C}(x)$, [33] (the Gamma function satisfies the difference equation $\Gamma(x+1) = x \cdot \Gamma(x)$). Moreover, such a method leads to algorithms, given in the above papers, that test differential algebraic dependency with applications to solutions of even higher order difference equations (hypergeometric functions, etc.). General results on the subject can be found in [1, 52, 65, 77, 78, 76].

It turns out that the results of the present paper, including the new theory of differential categories not restricted to the case of just one derivation, lead (see [25]) to a new understanding of isomonodromic systems of parameterized linear differential equations [59, 58, 10, 51, 50, 70] allowing one to substantially simplify the test for isomonodromicity generalizing the classical results [37, 38]. This, in turn, has become a part of a new algorithm [57] in the PPV theory.

Let us compare the present paper with some previously known results. The existence of a PV extension with a non-algebraically closed field of constants was considered by a number of authors. In particular, the case when the Galois group is GL_n, the base field is formally real, and the constants are real closed was solved positively in [72], while the case of the field $\mathbb{R}(z)$ has been also studied in [22]. In the case of one derivation, differential Tannakian categories were defined and studied in [64, 63, 41]. In the present paper, we define differential Tannakian categories over fields that may have many derivations. A similar approach in the case of one derivation was independently developed in [40]. Also, we do not choose a basis of the space of derivations, allowing us to give a functorial description of the constructions involved. One reason that this generalization is needed was explained in [5], in the context of Coleman integration. The paper [55] considers the case of several derivations but chooses a basis in the space of derivations and uses a fiber functor to give the axioms of a differential Tannakian category. On the contrary, the axioms in the present paper need to be and are given independently of the fiber functor.

It turns out that [81], in the case of one derivation, one can relax the differentially closed assumption, and just ask that the field of constants be algebraically closed in order to guarantee the existence of a PPV extension, by using the more straightforward method of differential kernels [47]. This approach was initiated by M. Wibmer who first applied difference kernels [12, 48] to study differential equations with difference parameters [80]. While not including all the cases from [81], the method presented in our paper gives the existence of PPV extensions in many other new situations important for applications. For instance, in the case of the incomplete Gamma function, if one used the differential kernels approach, one would have to take the algebraic closure $\overline{\mathbb{Q}(t)}$ instead of just $\mathbb{Q}(t)$.

Now we give more details about our method. To apply the Tannakian approach in the case of parameterized linear differential equations, one needs to develop a theory of differential Tannakian categories over differential fields. For this, one needs first to describe what a differential abelian tensor category is, Definition 4.6. In other words, one needs to define what it means for a Lie ring of derivations of a field k to act on an abelian k-linear category. The main subtlety here is that one cannot "subtract" functors in order to give a straightforward definition. There are two ingredients needed to overcome this difficulty. First, one uses the equivalence established by Illusie [35] between complete formal Hopf algebroids and differential rings, Section 3.7. Then, one uses the formalism of the extension of scalars for categories, Section 4.1 and [24], [74], in order to define the action of a complete formal Hopf algebroid over k on an abelian k-linear category. This leads to the notion of a differential category. For example, the category of all modules over a differential ring is a differential category. In this case, the differential structure is given by the Atiyah extension [2].

The approach to differential categories via the action of Hopf algebroids on categories can be generalized to many other situations, including the difference case, when the corresponding Hopf algebroid is given by the difference ring itself. For the purposes of this paper, it is in fact enough to consider only the degree two quotient of the formal Hopf algebroid. Having introduced differential categories, one defines differential Tannakian categories, Definition 4.22, and proves a differential version of the Tannaka duality between differential Hopf algebroids and differential Tannakian categories, Proposition 4.25 and Theorem 4.27.

The main non-trivial example of a differential category in this paper is the category formed by parameterized systems of linear differential equations, Section 5. In this case, the differential structure is given by what could be called a parameterized Atiyah extension. Based on this construction, one shows that the category of PPV extensions is equivalent to the category of differential fiber functors, Theorem 5.5. Thus, the problem of constructing a PPV extension is equivalent to the problem of constructing a differential fiber functor. For the latter, we use a geometric approach. The main technical difficulty here is to obtain flatness of a certain differential algebra over a differential ring after localizing this ring by a non-zero element. In general, this seems to be unknown, however we prove this result in the special case of a Hopf algebroid, Theorem 6.1, which is enough for our purpose. As an auxiliary result, we prove that a differentially finitely generated differential Hopf algebra is a quotient of the ring of differential polynomials by a differentially finitely generated ideal (one does not need to take a radical), Lemma 6.3. Besides, Theorem 6.1 implies the existence of a differential fiber functor for a differential Tannakian category over a differentially closed field. Finally, using simple algebro-geometric considerations, we construct a differential fiber functor, thus, providing a PPV extension in the case of Theorem 2.8.

The paper is organized as follows. We start by describing our main results in the non-parameterized case, Section 2.1, and the main parameterized case, Section 2.2. The proofs for the parameterized case are postponed until Section 7. In the intermediate sections, we develop our main technique as follows. In Section 3, we fix most notation used in the paper (Section 3.1) and introduce differential rings, algebras, modules, PPV extensions, and jet-rings using the invariant language convenient for the proofs of the main results. We then recall facts about extensions of scalars for categories and introduce differential abelian tensor categories and differential functors in Section 4. We use this to define parameterized Atiyah extensions in Section 5 and prove in Theorem 5.5 that the categories of PPV extensions and differential functors are equivalent. Section 6 contains the main technical ingredient, Theorem 6.1, needed for the proofs of the main results shown in Section 7. Finally, in Section 8, we discuss the parameterized differential Galois correspondence for arbitrary fields of constants and the behavior of the Galois group under the extensions of constants (see also [58]). For the convenience of the reader, we finish by giving the necessary background on Hopf algebroids and the usual Tannakian categories in the appendix, Section 8.2.

2. Statement of the main results

2.1. Non-parameterized case

Following P. Deligne [13], let us recall how Tannakian categories can be used to construct (non-parameterized) Picard–Vessiot extensions for systems of linear differential equations. For simplicity, we consider differential fields with only one derivation and we use a more common notation (K, ∂) instead of $(K, K \cdot \partial)$ as in Definition 3.1. So, let (K, ∂) be a differential field with a derivation ∂ and the field of constants $k := K^{\partial}$ of characteristic zero.

A system of linear ∂-differential equations over K is the same as a finite-dimensional differential module M over the differential field (K, ∂). A Picard–Vessiot extension for M is a differential field extension $(K, \partial) \subset (L, \partial)$ without new ∂-constants such that there is a basis of horizontal vectors in $L \otimes_K M$ over L and L is generated by their coordinates in a basis of M over K (see also Definition 3.25).

Definition 2.1. A field k is *existentially closed* in a field F over k if, for any finitely generated subalgebra R in F over k, there exists a morphism of k-algebras $R \to k$ (see [23, Proposition 3.1.1] for the equivalence with a more standard definition).

Note that, if $F = k(X)$ for an irreducible variety X over k, then k is existentially closed in F if and only if the set of k-rational points is Zariski dense in X. In particular, k is existentially closed in F in the following cases:

- the field k is algebraically closed and F is any field over k;
- the field k is pseudo algebraically closed and is algebraically closed in F;
- the field F is a subfield in a purely transcendental extension of k;
- the field F is real with k being real closed (in this case, one applies the Artin–Lang homomorphism theorem, [6, Theorem 4.1.2]).

Also, there is a range of non-trivial examples coming from various special geometrical considerations. In the case when K is real, k is real closed and the differential Galois group is GL_n, the following result is also proved in [72] by explicit methods.

Theorem 2.2. *Suppose that k is existentially closed in K. Then, for any finite-dimensional differential module (M, ∇_M) over (K, ∂), there exists a Picard–Vessiot extension.*

The construction of a Picard–Vessiot extension is based on the theory of Tannakian categories (Section A.2) and uses the following two results from [13].

Proposition 2.3. *[13, Proof of Corollaire 6.20] Let C be a Tannakian category over a field k such that C is tensor generated by one object and there is a fiber functor $C \to \mathbf{Vect}(K)$ for a field extension $K \supset k$. Then there exists a finitely generated subalgebra R in K over k and a fiber functor $C \to \mathbf{Mod}(R)$.*

According to the notation of Section A.2, $\langle M\rangle_\otimes$ is a full subcategory in the category of all differential modules over (K,∂) generated by subquotients of objects of type $M^{\otimes m}\otimes(M^\vee)^{\otimes n}$. The following statement uses that $\operatorname{char} k=0$, which implies that any algebraic group scheme over k is smooth.

Proposition 2.4. *[13, 9.5, 9.6] If there exists a fiber functor $\omega_0:\langle M\rangle_\otimes\to\mathbf{Vect}(k)$, then there exists a Picard–Vessiot extension for (M,∇_M).*

Proof of Theorem 2.2. We put $\mathcal{C}:=\langle M\rangle_\otimes$. By definition, the category $\mathcal{C}$ is tensor generated by the object (M,∇_M). Consider the fiber functor $\mathcal{C}\to\mathbf{Vect}(K)$ that forgets the differential structure on a differential module over (K,∂). By Proposition 2.3, there exist a finitely generated subalgebra R in K over k and a fiber functor $\omega:\mathcal{C}\to\mathbf{Mod}(R)$. Since k is existentially closed in K, there exists a homomorphism of k-algebras $R\to k$. As shown in [13, 1.9], for any object X in $\mathcal{C}$, the R-module $\omega(X)$ is finitely generated and projective. Hence,

$$\omega_0:\mathcal{C}\to\mathbf{Vect}(k),\qquad X\mapsto k\otimes_R\omega(X)$$

is a fiber functor on $\mathcal{C}$. We conclude the proof by Proposition 2.4. □

The main goal of the present paper is to make a parameterized analogue of the above reasoning. As an application, we obtain a construction of a parameterized Picard–Vessiot extension in a range of cases when the constants are not differentially closed.

2.2. Main results: parameterized case

The following is a parameterized analogue of Theorem 2.2. We use notions and notation from Section 3.

Theorem 2.5. *Let (K,D_K) be a parameterized differential field (Definition 3.14) over a differential field (k,D_k) (Definition 3.1) with $\operatorname{char} k=0$. Suppose that there is a splitting $\widetilde{D}_k$ (Definition 3.15) of (K,D_K) over (k,D_k) such that (k,D_k) is relatively differentially closed in $(K,K\otimes_k\widetilde{D}_k)$ (Definition 3.11, Remark 3.16).*

Then, for any finite-dimensional differential module (Definition 3.19) over $(K,D_{K/k})$ (Definition 3.14), there exists a parameterized Picard–Vessiot extension (Definition 3.27).

Remark 2.6. The existence of a PPV extension implies the existence of a parameterized differential Galois group, which is a linear differential algebraic group, together with the Galois correspondence (Section 8.1).

Remark 2.7. According to our definition of a parameterized differential field, derivations from D_k do not act on the field K. Having the splitting $\widetilde{D}_k$ from Theorem 2.5, we can replace the differential field (k,D_k) with the differential field $\left(k,\widetilde{D}_k\right)$ so that derivations from $\widetilde{D}_k$ act on K (Remark 3.16). This allows us to consider $\widetilde{D}_k$-Hopf algebroids of type (K,H) over k and produce an analogue of the proof of Theorem 2.2.

Theorem 2.5 is proved in Section 7.1. The following result describes two rather broad cases when the hypotheses of Theorem 2.5 are satisfied.

Theorem 2.8. *Let (K,D_K) be a parameterized differential field over a differential field (k,D_k) with $\operatorname{char} k=0$. Suppose that one of the following conditions is satisfied:*

1. *There exists a splitting $\widetilde{D}_k$ of (K,D_K) over (k,D_k) such that*

- *the structure map $D_K \to K \otimes_k D_k$ induces an isomorphism between $\widetilde{D}_k$ and $1 \otimes D_k$,*
- *The field K is generated as a field by $K_0 := K^{\widetilde{D}_k}$ and k,*
- *the field $k_0 := k^{D_k}$ is existentially closed in K_0 (Definition 2.1).*

2. *The field k is existentially closed in K and the map $D_{K/k} \to \mathrm{Der}_k(K, K)$ is an isomorphism.*

Then the parameterized differential field (K, D_K) over (k, D_k) satisfies the hypotheses of Theorem 2.5. Thus, for any finite-dimensional differential module over $(K, D_{K/k})$, there exists a PPV extension.

Theorem 2.8 is proved in Section 7.2.

Remark 2.9.

1. In general, fields generated by two subfields may have a complicated structure. However, condition 1 in Theorem 2.8 implies that $K_0 \otimes_{k_0} k$ is a domain and $K = \mathrm{Frac}(K_0 \otimes_{k_0} k)$. Indeed, by Lemma 8.7, the differential algebra $K_0 \otimes_{k_0} k$ over (k, D_k) is D_k-simple, that is, contains no D_k-ideals, whence the morphism $K_0 \otimes_{k_0} k \to K$ is injective, which yields the required statement.
2. Condition 2 in Theorem 2.8 is equivalent to requiring that k be existentially closed in K, $\dim_K(D_{K/k}) = \mathrm{tr.deg}(K/k)$, and map the $D_{K/k} \to \mathrm{Der}(K, K)$ be injective.

Here is a series of examples that satisfy the hypotheses of Theorem 2.8.

Example 2.10. Let the bar over a field denote the algebraic closure. All fields K below are subfields in the algebraic closure of the field $\mathbb{C}(x_1, \ldots, x_m, t_1, \ldots, t_n)$; all fields k below are subfields in the algebraic closure of the field $\mathbb{C}(t_1, \ldots, t_n)$, and, except for Examples 2 and 3, we put

$$D_K := K \cdot \partial_{x_1} + \ldots + K \cdot \partial_{x_m} + K \cdot \partial_{t_1} + \ldots + K \cdot \partial_{t_n}, \quad D_k := k \cdot \partial_{t_1} + \ldots + k \cdot \partial_{t_n}.$$

We obtain $D_{K/k} = K \cdot \partial_{x_1} + \ldots + K \cdot \partial_{x_m}$. In Examples 1, 2, 5, 3, and 4, we put

$$\widetilde{D}_k := k \cdot \partial_{t_1} + \ldots + k \cdot \partial_{t_n} \subset D_K.$$

The following parameterized differential fields (K, D_K) over (k, D_k) satisfy the hypotheses of Theorem 2.8:

1. If $K = \mathrm{Frac}\,(K_0 \otimes_{\overline{\mathbb{Q}}} k)$, where K_0 is a finite extension of $\overline{\mathbb{Q}}(x_1, \ldots, x_m)$ and k is an algebraic extension of $\overline{\mathbb{Q}}(t_1, \ldots, t_n)$, then (K, D_K) satisfies condition 1 with $k_0 = \overline{\mathbb{Q}}$ being algebraically closed.
2. If $K = \mathrm{Frac}\,(K_0 \otimes_{\overline{\mathbb{Q}}} k)$, where K_0 is a finite extension of $\overline{\mathbb{Q}}(x_1, x_2)$ and k is an algebraic extension of $\overline{\mathbb{Q}}(t_1, \ldots, t_n)$, then (K, D_K) satisfies condition 1 with

$$D_K := K \cdot (\partial_{x_1} + x_2 \partial_{x_2}) + K \cdot \partial_{t_1} + \ldots + K \cdot \partial_{t_n}, \quad D_{K/k} = K \cdot (\partial_{x_1} + x_2 \partial_{x_2}),$$

and with $k_0 = \overline{\mathbb{Q}}$ being algebraically closed.
3. If $K = k(x_1, \ldots, x_m)$, where k is an algebraic extension of $\mathbb{Q}(t_1, \ldots, t_n)$ such that $\mathbb{Q}$ is algebraically closed in k, then (K, D_K) satisfies condition 1 with $K_0 = \mathbb{Q}(x_1, \ldots, x_m)$, $k_0 = \mathbb{Q}$.

4. If $K = \mathrm{Frac}\,(K_0 \otimes_{\mathbb{R}} k)$, where K_0 is a finite extension of $\mathbb{R}(x_1, \ldots, x_m)$ such that K_0 a real field, and k is an algebraic extension of $\mathbb{R}(t_1, \ldots, t_n)$ such that $\mathbb{R}$ is algebraically closed in k, then (K, D_K) satisfies condition 1 with $k_0 = \mathbb{R}$.
5. If $K = \mathrm{Frac}\,(K_0 \otimes_{\mathbb{R}} k)$, where K_0 is a finite extension of $\mathbb{R}(x_1, \ldots, x_m)$ such that K_0 a real field, and k is an algebraic extension of $\mathbb{R}(t_1, t_2, t_3)$ such that $\mathbb{R}$ is algebraically closed in k, then (K, D_K) satisfies condition 1 with

$$D_K := K \cdot \partial_{x_1} + \ldots + K \cdot \partial_{x_m} + K \cdot (t_1 \partial_{t_1} + \sqrt{2} t_2 \partial_{t_2} + \sqrt{3} t_3 \partial_{t_3}),$$
$$D_k := k \cdot (t_1 \partial_{t_1} + \sqrt{2} t_2 \partial_{t_2} + \sqrt{3} t_3 \partial_{t_3}),$$

and with $k_0 = \mathbb{R}$, [68, Remark 4.9].
6. If k is an algebraic closure of $\mathbb{Q}(t_1, \ldots, t_n)$ and K is a finite extension of $k(x_1, \ldots, x_m)$, then (K, D_K) satisfies condition 2.
7. If k is a real closure of $\mathbb{Q}(t_1, \ldots, t_n)$ with respect to some ordering and K is a real finite extension of $k(x_1, \ldots, x_m)$, then (K, D_K) satisfies condition 2.

3. Differential rings and jet rings

We do not claim any originality of most of the definitions and constructions in this section, for example, see [35, Section 1.1], [3, §1], [13, 9.9], [34] for Section 3.2, see any standard reference about modules with connections for Section 3.4, see [10] for Section 3.5, [35, Section 1.2,1.3], [4, §2], [30, §16], [60, 61, 66] for Section 3.6 and Section 3.9, [35] for Section 3.7, and see any standard reference about the Lie derivative for Section 3.10. The definition of a differential object (Definition 3.35) generalizes the well-known notion of a stratification on a sheaf, [4].

Only the definition of a parameterized differential algebra (Definition 3.14) seems to be new. However, we have decided to fix the notation and notions concerning differential rings, differential modules over them, PPV extensions, and jet rings. Note that the more commonly used name for the notion from Definition 3.1 is a *Lie algebroid*, but we use the term *differential ring*, which seems to be more standard in differential algebra. There is a direct generalization of differential rings as defined below from rings to schemes replacing modules by quasi-coherent sheaves.

3.1. Notation

First let us fix the notation that we use in the paper.

- Given data D, we say that an object O associated with D is *canonical* if its construction does not depend on the choice of any additional structure on D (for example, the choice of a basis in a vector space). Usually, this implies that O is functorial in D in the reasonable sense.
- All rings are assumed to be commutative and having a unit element.
- Denote the category of sets by **Sets**.
- Given a non-zero element f in a ring R, denote the localization of R over the multiplicative set formed by all natural powers of f by R_f.
- Given two rings R and S, denote their tensor product over $\mathbb{Z}$ by $R \otimes S$.

- Given a ring R and two R-bimodules M and N, their tensor product is denoted by $M \otimes_R N$, where M and N are considered with the right and left R-module structures, respectively.

- For rings R and S, denote the set of all derivations from R to S, that is, additive homomorphisms that satisfy the Leibniz rule, by $\mathrm{Der}(R, S)$. If R and S are algebras over a ring κ, denote the set of all κ-linear derivations from R to S by $\mathrm{Der}_\kappa(R, S)$. Note that $\mathrm{Der}(R, S)$ and $\mathrm{Der}_\kappa(R, S)$ have canonical S-module structures. Also, $\mathrm{Der}(R, R)$ and $\mathrm{Der}_\kappa(R, R)$ are Lie rings.

- Given a ring homomorphism $R \to S$ and an R-module M, denote the extension of scalars $S \otimes_R M$ also by M_S. If only one R-module structure on S is considered, we put the new scalars on the left in the tensor product, that is, we use the notation $S \otimes_R M$. If two R-module structures on S are considered, then we usually refer to them as right and left structures and use the notations $S \otimes_R M$ and $M \otimes_R S$ for the corresponding extensions of scalars.

- Given a ring homomorphism $R \to S$ and a morphism $f : M \to N$ of R-modules, we denote the extension of scalars for f from R to S by $\mathrm{id}_S \otimes f$, $S \otimes_R f$, or f_S, that is, we have

$$S \otimes_R f : S \otimes_R M \to S \otimes_R N \quad \text{or} \quad f_S : M_S \to N_S.$$

- For a field K, denote the category of vector spaces over K by $\mathbf{Vect}(K)$. Denote the full subcategory of finite-dimensional K-vector spaces by $\mathbf{Vect}^{fg}(K)$.

- For a ring R, denote the category of R-modules by $\mathbf{Mod}(R)$. Denote the full subcategory of finitely generated R-modules by $\mathbf{Mod}^{fg}(R)$.

- For a ring R, denote the category of R-algebras by $\mathbf{Alg}(R)$.

- For a Hopf algebra A over a ring R, denote the category of comodules over A by $\mathbf{Comod}(A)$. Denote the full subcategory of comodules over A that are finitely generated as R-modules by $\mathbf{Comod}^{fg}(A)$.

- For an affine group scheme G over a field k, denote the category of algebraic representations of G over k by $\mathbf{Rep}(G)$ (they correspond to comodules over a Hopf algebra). Denote the full subcategory of finite-dimensional representations of G over k by $\mathbf{Rep}^{fg}(G)$.

- Given a category C and objects X, Y in C, denote the set of morphisms from X to Y by $\mathrm{Hom}_C(X, Y)$. Put $\mathrm{End}_C(X) := \mathrm{Hom}_C(X, X)$.

- Given exact sequences

$$\begin{array}{c} 0 \longrightarrow X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \longrightarrow 0 \\ 0 \longrightarrow X \xrightarrow{\alpha'} Y' \xrightarrow{\beta'} Z \longrightarrow 0 \end{array}$$

  in an abelian category, denote their Baer sum by $Y +_{\mathbf{B}} Y'$, that is, we have

$$Y +_{\mathbf{B}} Y' = \mathrm{Ker}\,(\beta - \beta' : Y \oplus Y' \to Z)/\,\mathrm{Im}\,(\alpha \oplus -\alpha' : X \to Y \oplus Y').$$

3.2. Differential rings

Definition 3.1. A *differential ring* is a triple (R, D_R, θ_R), where R is a ring, D_R is a finitely generated projective R-module together with a Lie bracket $[\,\cdot\,,\cdot\,] : D_R \times D_R \to D_R$, and $\theta_R : D_R \to \mathrm{Der}(R, R)$ is a morphism of both R-modules and Lie rings such that, for all $a \in R$ and $\partial_1, \partial_2 \in D_R$, we have

$$[\partial_1, a\partial_2] - a[\partial_1, \partial_2] = \theta_R(\partial_1)(a)\,\partial_2.$$

For short, we usually omit θ_R in the notation. Thus, a differential ring is denoted just by (R, D_R), and

$$\partial(a) := \theta_R(\partial)(a) \quad a \in R,\ \partial \in D_R.$$

Let R^{D_R} denote the subring of *D_R-constants*, that is, the set of all $a \in R$ such that, for any $\partial \in D_R$, we have $\partial(a) = 0$.

Remark 3.2. In most of the situations that we have here, it is enough to consider differential rings (R, D_R) with D_R being a finitely generated free R-module.

Recall that, for an R-module M, its second wedge power $\wedge^2_R M$ is the quotient of $M \otimes_R M$ over the submodule generated by all elements $m \otimes m$, where $m \in M$. Given $m, n \in M$, the image of $m \otimes n$ under the natural map

$$M \otimes_R M \to \wedge^2_R M$$

is denoted by $m \wedge n$. There is a canonical morphism of R-modules

$$\wedge^2_R\big(M^\vee\big) \to \big(\wedge^2_R M\big)^\vee, \quad p \wedge q \mapsto \{m \wedge n \mapsto p(m)q(n) - p(n)q(m)\},$$

where $M^\vee := \mathrm{Hom}_R(M, R)$. If M is finitely generated and projective, then $\wedge^2_R M$ is also finitely generated and projective and the above morphism $\wedge^2_R(M^\vee) \to \big(\wedge^2_R M\big)^\vee$ is an isomorphism.

Definition 3.3. For a differential ring (R, D_R), we put $\Omega_R := D_R^\vee$ and define additive maps

$$\mathbf{d} : R \to \Omega_R, \quad a \mapsto \{\partial \mapsto \partial(a)\}$$

$$\mathbf{d} : \Omega_R \to \wedge^2_R \Omega_R, \quad \omega \mapsto \{\partial_1 \wedge \partial_2 \mapsto \partial_1(\omega(\partial_2)) - \partial_2(\omega(\partial_1)) - \omega([\partial_1, \partial_2])\} \tag{1}$$

for all $a \in R$, $\omega \in \Omega_R$ and $\partial_1, \partial_2 \in D_R$.

In the notation of Definition 3.3, for all $a, b \in R$ and $\omega \in \Omega_R$, we have

$$\mathbf{d}(ab) = a\,\mathbf{d}b + b\,\mathbf{d}a, \quad \mathbf{d}(a\omega) = a\,\mathbf{d}\omega + \mathbf{d}a \wedge \omega, \quad \text{and} \quad \mathbf{d}(\mathbf{d}(a)) = 0.$$

Remark 3.4. The map $\mathbf{d}$ is well-defined for all wedge powers of Ω_R,

$$\mathbf{d} : \wedge^i_R \Omega_R \to \wedge^{i+1}_R \Omega_R,$$

and this defines a dg-ring structure on $\wedge^\bullet_R \Omega_R$. Actually, to define a differential ring structure on R with D_R being a finitely generated projective R-module is the same as to define a dg-ring structure on $\wedge^\bullet_R \Omega_R$ with the natural product structure and grading, where, as above, $\Omega_R = D_R^\vee$, [35, Remarques 1.1.9 b)]. Namely, given $\mathbf{d}$, we put

$$\partial(a) := (\mathbf{d}a)(\partial),$$

and define the Lie bracket $[\partial_1, \partial_2]$ such that it satisfies the condition

$$\omega([\partial_1, \partial_2]) = \partial_1(\omega(\partial_2)) - \partial_2(\omega(\partial_1)) - (\mathbf{d}\omega)(\partial_1 \wedge \partial_2)$$

for all $a \in R$, $\partial, \partial_1, \partial_2 \in D_R$, and $\omega \in \Omega_R$.

Example 3.5.

1. Let R be the coordinate ring of a smooth affine variety X over a field k and put $D_R := \mathrm{Der}_k(R,R)$. Then the pair (R, D_R) is a differential ring with Ω_R, $\wedge^2_R\Omega_R$, and $\mathbf{d}$ being the modules of differential 1-, 2-forms on X, and the de Rham differential, respectively.
2. Let $\partial_1, \ldots, \partial_n$ be formal symbols that denote commuting derivations from a ring R to itself (possibly, some of the ∂_i's correspond to the zero derivation). Then the pair
$$(R, R\cdot\partial_1 \oplus \ldots \oplus R\cdot\partial_n)$$
defines a differential ring.
3. The data
$$(K, K\cdot(z\partial_x + \partial_y) + K\cdot\partial_z)$$
with $K := \mathbb{C}(x,y,z)$ and natural θ_K do not define a differential ring because of the lack of a Lie bracket.
4. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field K. Then $(K, \mathfrak{g})$ is a differential field with the zero θ_K.
5. Let $R \hookrightarrow S$ be an embedding of rings and D_R be a finitely generated projective R-submodule and a Lie subring in the R-module of all derivations $\partial : S \to S$ with $\partial(R) \subset R$. Let
$$\theta_R : D_R \to \mathrm{Der}(R,R)$$
be defined by the restriction to R of derivations from S to itself. Then (R, D_R, θ_R) is a differential ring with, possibly, non-trivial kernel and image of θ_R.
6. Let (R, A) be a Hopf algebroid (Section A.1). Put
$$I := \mathrm{Ker}(e : A \to R) \quad \text{and} \quad \Omega_R := I/I^2.$$
Then the cosimplicial ring structure on the tensor powers of A as an R-bimodule defines a dg-ring structure on $\wedge^\bullet_R\Omega_R$. Explicitly, for any $a \in R$, the element $\mathbf{d}a \in \Omega_R = I/I^2$ is the class of
$$r(a) - l(a) \in I.$$
For any $\omega \in \Omega_R$, the element
$$\mathbf{d}\omega \in \wedge^2_R\Omega_R$$
is defined as follows. Let $\tilde{\omega} \in I$ be such that its class in Ω_R equals ω. One takes the class of the element
$$\tilde{\omega}\otimes 1 - \Delta(\tilde{\omega}) + 1\otimes\tilde{\omega} \in I\otimes_R I$$
in the quotient $\Omega_R\otimes_R\Omega_R$ and then one applies the canonical map
$$\Omega_R\otimes_R\Omega_R \to \wedge^2_R\Omega_R$$
to obtain $\mathbf{d}\omega$. By Remark 3.4, the dg-ring structure on $\Omega^\bullet_R$ defines a differential ring (R, D_R) with $D_R = \Omega_R^\vee$. See more details about this example in [35, Proposition 1.2.8].

Note that, for any differential field (K, D_K) with char $K = 0$ and injective $\theta_K : D_K \to \mathrm{Der}(K,K)$, there exists a commuting basis for D_K as shown in [44, p. 12, Proposition 6] and Proposition 3.18. However, we prefer not to choose such a basis and to give coordinate-free definitions and constructions. In particular, here is a definition of a morphism between differential rings.

Definition 3.6. A *morphism between differential rings* $(R, D_R) \to (S, D_S)$ is a pair (φ, φ_*), where $\varphi : R \to S$ is a ring homomorphism and $\varphi_* : \Omega_R \to \Omega_S$ is an R-linear map such that φ_* commutes with $\mathbf{d}$, that is, for all $a \in R$, $\omega \in \Omega_R$, we have

$$\mathbf{d}(\varphi(a)) = \varphi_*(\mathbf{d}a) \in \Omega_S \quad \text{and} \quad \mathbf{d}(\varphi_*(\omega)) = \varphi_*(\mathbf{d}\omega) \in \wedge^2_S \Omega_S\,,$$

where we denote the R-linear map

$$\wedge^2_R \Omega_R \to \wedge^2_S \Omega_S$$

induced by φ_* for short also by φ_*. The second condition,

$$\mathbf{d}(\varphi_*(\omega)) = \varphi_*(\mathbf{d}\omega),$$

is called the *integrability condition*. For short, we sometimes omit φ_* in the notation. A morphism (φ, φ_*) is *strict* if the S-linear morphism

$$S \otimes_R \Omega_R \to \Omega_S$$

induced by φ_* is an isomorphism.

Taking the dual modules, one obtains an explicit definition of a morphism between differential rings in terms of derivations. The pair (φ, φ_*) from Definition 3.6 corresponds to a pair (φ, D_φ), where $\varphi : R \to S$ is a ring homomorphism and

$$D_\varphi : D_S \to S \otimes_R D_R$$

is a morphism of S-modules. Sometimes we refer to D_φ as a *structure map* associated with a morphism between differential rings. The first condition,

$$\mathbf{d}(\varphi(a)) = \varphi_*(\mathbf{d}a),$$

is equivalent to the equality

$$\partial(\varphi(a)) = \sum_i b_i \cdot \varphi(\partial_i(a)) \tag{2}$$

for all $a \in R$ and $\partial \in D_S$, where

$$D_\varphi(\partial) = \sum_i b_i \otimes \partial_i, \quad b_i \in S,\ \partial_i \in D_R.$$

The integrability condition is equivalent to the equality

$$D_\varphi([\partial, \delta]) = \sum_j \partial(c_j) \otimes \delta_j - \sum_i \delta(b_i) \otimes \partial_i + \sum_{i,j} b_i c_j \otimes \left[\partial_i, \delta_j\right] \tag{3}$$

for all $\partial, \delta \in D_S$, where

$$D_\varphi(\delta) = \sum_j c_j \otimes \delta_j, \quad c_j \in S,\ \delta_j \in D_R.$$

The morphism (φ, φ_*) is strict if and only if D_φ is an isomorphism.

Remark 3.7.

1. In the notation of Definition 3.6, assume the injectivity of the canonical map

$$S \otimes_R D_R \to \mathrm{Der}(R, S)$$

induced by the ring homomorphism $\varphi : R \to S$. Then it follows from (2) that the morphism D_φ, as well as φ_*, is unique if it exists. In particular, the above injectivity assumption holds if R is a field and θ_R is injective.
2. In the notation of Definition 3.6, it follows from (3) that the S-submodule $D_{S/R} := \mathrm{Ker}(D_\varphi)$ in D_S is closed under the Lie bracket, that is, if $D_\varphi(\partial) = D_\varphi(\delta) = 0$, then

$$D_\varphi([\partial, \delta]) = 0.$$

Therefore, we obtain a differential ring $(S, D_{S/R})$ with the map $D_{S/R} \to \mathrm{Der}(S, S)$ induced by θ_S.
3. If (R, ∂_R) and (S, ∂_S) are two rings with derivations, then a morphism of differential rings

$$(R, R \cdot \partial_R) \to (S, S \cdot \partial_S)$$

is given by a ring homomorphism $\varphi : R \to S$ and an element $b \in S$ such that, for any $a \in R$, we have

$$\partial_S(\varphi(a)) = b \cdot \varphi(\partial_R(a)).$$

Thus, up to a rescaling, this is the usual definition of a morphism between differential rings with one derivation.

Example 3.8. For a field k, consider the rings $R := k[x, y, z]$, $S := k[x, y]$, the modules

$$D_R := R \cdot \partial_x + R \cdot \partial_y + R \cdot z\partial_z, \quad D_S := S \cdot \partial_x + S \cdot \partial_y,$$

and the ring homomorphism $\varphi : R \to S$ being the quotient by the ideal $(z) \subset R$. Then we have

$$\Omega_R = R \cdot \mathbf{d}x + R \cdot \mathbf{d}y + R \cdot (1/z)\, \mathbf{d}z, \quad \Omega_S = S \cdot \mathbf{d}x + S \cdot \mathbf{d}y.$$

Given polynomials $f, g \in S$, consider the morphism of R-modules

$$\varphi_* : \Omega_R \to \Omega_S, \quad \mathbf{d}x \mapsto \mathbf{d}x, \quad \mathbf{d}y \mapsto \mathbf{d}y, \quad (1/z)\, \mathbf{d}z \mapsto f\mathbf{d}x + g\mathbf{d}y.$$

Then (φ, φ_*) satisfies

$$\varphi_*(\mathbf{d}(a)) = \mathbf{d}(\varphi_*(a))$$

for all $a \in R$. Further, (φ, φ_*) satisfies the integrability condition if and only if $\partial_y f = \partial_x g$, because

$$\mathbf{d}((1/z)\mathbf{d}z) = 0, \quad \mathbf{d}(\varphi_*((1/z)\mathbf{d}z)) = \left(-\partial_y f + \partial_x g\right) \cdot \mathbf{d}x \wedge \mathbf{d}y.$$

3.3. Differential algebras

In the present paper, we consider several types of algebras over differential rings. The first type is the most general one.

Definition 3.9.

- Given a morphism of differential rings $(R, D_R) \to (S, D_S)$, we say that (S, D_S) is a *differential algebra* over (R, D_R).

- A *morphism between differential algebras* over (R, D_R) is a morphism between differential rings that commutes with the given morphisms from (R, D_R).

Definition 3.10. Given a differential ring (S, D_S) and a morphism of rings $R \to S$, we say that (S, D_S) is *differentially finitely generated* over R if there are finite subsets $\Sigma \subset S$ and $\Delta \subset D_S$ such that any element in S can be represented as a polynomial with coefficients from $\mathrm{Im}(R \to S)$ in elements of the form

$$(\partial_1 \cdot \ldots \cdot \partial_n)\, a, \quad \partial_i \in \Delta,\ a \in \Sigma,$$

and the product stands for the composition of derivations.

The following is a differential version of Definition 2.1.

Definition 3.11. Let $(k, D_k) \to (K, D_K)$ be a morphism between differential fields. We say that (k, D_k) is *relatively differentially closed* in (K, D_K) if, for any differential subalgebra (R, D_R) in (K, D_K) over (k, D_k) such that (R, D_R) is differentially finitely generated over k and the morphism $(R, D_R) \to (K, D_K)$ is strict, there is a morphism $(R, D_R) \to (k, D_k)$ of differential algebras over (k, D_k).

The following type of algebras corresponds to the usual notion of a differential algebra.

Definition 3.12.

- Given a strict morphism of differential rings $(R, D_R) \to (S, D_S)$, we say that (S, D_S) is a D_R*-algebra* over (R, D_R) (or simply over R).

- Denote the category of D_R-algebras over (R, D_R) by $\mathbf{DAlg}(R, D_R)$.

- If a D_R-algebra (S, D_S) over a differential ring (R, D_R) is differentially finitely generated over R, then we say that S is D_R*-finitely generated* over R.

- Denote the D_R-algebra freely D_R-generated over R by the finite set $T_1, \ldots, T_n$, that is, the ring of D_R-polynomials in the differential indeterminates $T_1, \ldots, T_n$, by $R\{T_1, \ldots, T_n\}$.

For short, we usually omit D_S in the notation of a D_R-algebra over R, because it is reconstructed by the isomorphism

$$D_\varphi : D_S \xrightarrow{\sim} S \otimes_R D_R.$$

Given $\partial \in D_R$ and $b \in S$, we put

$$\partial(b) := \theta_S\Big(D_\varphi^{-1}(1 \otimes \partial)\Big)(b).$$

We have that S is D_R-finitely generated if and only if there is a finite subset $\Sigma \subset S$ such that any element in S can be represented as a polynomial with coefficients from $\mathrm{Im}(R \to S)$ in elements of the form

$$(\partial_1 \ldots \partial_n)\, a, \quad \partial_i \in D_R,\ a \in \Sigma.$$

Equivalently, there is no smaller D_R-subalgebra over R in S containing Σ.

Definition 3.13. A D_R-algebra S over a differential ring (R, D_R) is *of D_R-finite presentation over R* if there is an isomorphism of D_R-algebras over R

$$S \cong R\{T_1, \ldots, T_n\}/I,$$

where I is a D_R-finitely generated ideal.

The following type of algebras is needed to work with parameterized differential equations.

Definition 3.14. A differential algebra (R, D_R) over a differential field (k, D_k) is called *parameterized* if the structure map $D_R \to R \otimes_k D_k$ is surjective and we have $k = R^{D_{R/k}}$, where $D_{R/k}$ is the kernel of the structure map.

Given a parameterized differential algebra (R, D_R) over (k, D_k), one has the differential ring $(R, D_{R/k})$ (Remark 3.7(2)).

Definition 3.15. A *splitting* of a parameterized differential algebra (R, D_R) over a differential field (k, D_k) is a finite-dimensional k-subspace $\widetilde{D}_k$ in D_R closed under the Lie bracket on D_R such that the structure map $D_R \to R \otimes_k D_k$ induces a surjection

$$\widetilde{D}_k \to D_k \cong 1 \otimes D_k.$$

Remark 3.16. In the notation of Definition 3.15, put $\widetilde{D}_R := R \otimes_k \widetilde{D}_k$ and consider the differential field $(k, \widetilde{D}_k)$, where $\widetilde{D}_k \to \mathrm{Der}(k, k)$ is defined as the composition

$$\widetilde{D}_k \to D_k \to \mathrm{Der}(k, k).$$

We obtain a commutative diagram of differential rings with the bottom horizontal morphism being strict:

$$\begin{array}{ccc} (k, D_k) & \longrightarrow & (R, D_R) \\ \downarrow & & \downarrow \\ (k, \widetilde{D}_k) & \longrightarrow & (R, \widetilde{D}_R). \end{array}$$

Example 3.17.

1. Let

   $$\{\partial_{x,1}, \ldots \partial_{x,m}, \tilde{\partial}_{t,1}, \ldots, \tilde{\partial}_{t,n}\}$$

   be formal symbols that denote commuting derivations from a field K to itself and let k be the field of $\{\partial_{x,1}, \ldots, \partial_{x,m}\}$-constants. Denote the restriction of $\tilde{\partial}_{t,i}$ from K to k by $\partial_{t,i}$, $1 \leqslant i \leqslant n$. Then (K, D_K) is a parameterized differential field over (k, D_k) with

   $$D_K := K \cdot \partial_{x,1} \oplus \ldots \oplus K \cdot \partial_{x,m} \oplus K \cdot \tilde{\partial}_{t,1} \oplus \ldots \oplus K \cdot \tilde{\partial}_{t,n},$$
   $$D_k := k \cdot \partial_{t,1} \oplus \ldots \oplus k \cdot \partial_{t,n}, \quad D_{K/k} = K \cdot \partial_{x,1} \oplus \ldots \oplus K \cdot \partial_{x,n}.$$

2. Let $(k, \widetilde{D}_k)$ be a differential field and D_k be the image of the map

   $$\theta_k : \widetilde{D}_k \to \mathrm{Der}(k, k).$$

   Then $(k, \widetilde{D}_k)$ is a parameterized differential field over (k, D_k).

Actually, Example 3.17(1) is quite general as the following statement shows.

Proposition 3.18. *Let (K, D_K) be a parameterized differential field over a differential field (k, D_k) with* $\operatorname{char} k = 0$ *and injective θ_K and θ_k. Then we are in the case of Example 3.17(1), that is, there exists a commuting basis*

$$\{\partial_{x,1}, \ldots \partial_{x,m}, \tilde{\partial}_{t,1}, \ldots, \tilde{\partial}_{t,n}\}$$

of D_K over K such that

$$D_k = k \cdot \partial_{t,1} + \ldots + k \cdot \partial_{t,n}, \quad D_{K/k} = K \cdot \partial_{x,1} + \ldots + K \cdot \partial_{x,n} \quad \textit{where} \quad \partial_{t,i} := \tilde{\partial}_{t,i}|_k.$$

Proof. We follow the idea of the proof of [44, p. 12, Proposition 6]. First, there are sets of formal variables $\{x_\alpha\}$ and $\{t_\beta\}$ such that K is an algebraic extension of the field $k(\{x_\alpha\})$ and k is an algebraic extension of the field $\mathbb{Q}(\{t_\beta\})$. Since $\operatorname{char} k = 0$, these algebraic extensions are separable, whence there are uniquely defined commuting derivations $\{\partial_{x_\alpha}\}$ and $\left\{\partial_{t_\beta}\right\}$ from K to itself. Note that we have

$$\operatorname{Der}_k(K, K) = \prod_\alpha K \cdot \partial_{x_\alpha}, \quad \operatorname{Der}(K, K) = \prod_\alpha K \cdot \partial_{x_\alpha} \oplus \prod_\beta K \cdot \partial_{t_\beta}.$$

In what follows, by a *coordinate subspace* in $\operatorname{Der}(K, K)$, we mean a product of some of (possibly, infinitely many) $(K \cdot \partial_{x_\alpha})$'s and $(K \cdot \partial_{t_\beta})$'s.

Since θ_K is injective, we can consider $D_{K/k}$ and D_K as K-subspaces in $\operatorname{Der}_k(K, K)$ and $\operatorname{Der}(K, K)$, respectively. Let $U \subset \operatorname{Der}_k(K, K)$ be a maximal coordinate subspace such that

$$U \cap D_{K/k} = 0.$$

Explicitly, U is spanned by some of ∂_{x_α}'s in the sense of infinite products. Since $D_{K/k}$ is a finite-dimensional K-vector space, the composition

$$D_{K/k} \to \operatorname{Der}_k(K, K) \to \operatorname{Der}_k(K, K)/U$$

is an isomorphism of K-vector spaces (finite-dimensionality of $D_{K/k}$ is important here, because we allow U to be only a coordinate subspace in $\operatorname{Der}_k(K, K)$, not an arbitrary one).

Further, let $V \subset \operatorname{Der}(K, K)$ be a maximal coordinate subspace such that $V \cap D_K = 0$ and $V \supset U$. Explicitly, the basis of V in the sense of infinite products, as above, is obtained by adding some ∂_{t_β}'s to the basis of U. Since θ_k is injective, we have

$$D_{K/k} = \operatorname{Der}_k(K, K) \cap D_K \subset \operatorname{Der}(K, K).$$

Together with the finite-dimensionality of D_K over K, this implies that the composition

$$D_K \to \operatorname{Der}(K, K) \to \operatorname{Der}(K, K)/V$$

is an isomorphism of K-vector spaces. Denote this isomorphism by α.

Let π be the composition

$$\operatorname{Der}(K, K) \to \operatorname{Der}(K, K)/V \xrightarrow{\alpha^{-1}} D_K.$$

Since $V \cap \mathrm{Der}_k(K,K) = U$, we have the following commutative diagram with injective vertical maps

$$\begin{array}{ccccc} D_{K/k} & \longrightarrow & \mathrm{Der}_k(K,K) & \longrightarrow & \mathrm{Der}_k(K,K)/U \\ \downarrow & & \downarrow & & \downarrow \\ D_K & \longrightarrow & \mathrm{Der}(K,K) & \longrightarrow & \mathrm{Der}(K,K)/V. \end{array}$$

Therefore,

$$\pi(\mathrm{Der}_k(K,K)) \subset D_{K/k}.$$

Finally, consider the finite sets of all indices $\{\alpha_i\}$ and $\{\beta_j\}$ such that the corresponding derivations $\partial_{x_{\alpha_i}}$ and $\partial_{t_{\beta_j}}$ do not belong to V. Then the elements

$$\partial_{x,i} := \pi(\partial_{\alpha_i}), \tilde{\partial}_{t,j} := \pi(\partial_{\beta_j})$$

form a basis in D_K. Since the subspaces U and V are coordinate, this is a commuting basis, as required. □

3.4. Differential modules

We define differential modules as follows.

Definition 3.19.

- A *D_R-module* over a differential ring (R, D_R) (or simply over R) is a pair (M, ∇_M), where M is an R-module and
$$\nabla_M : M \to \Omega_R \otimes_R M$$
is an additive map such that, for all $a \in R$ and $m \in M$, we have
$$\nabla_M(am) = \mathbf{d}a \otimes m + a \cdot \nabla_M(m)$$
and the composition
$$M \xrightarrow{\nabla_M} \Omega_R \otimes_R M \xrightarrow{\nabla_M} \wedge^2_R \Omega_R \otimes_R M$$
is zero, where $\nabla_M : \Omega_R \otimes_R M \to \wedge^2_R \Omega_R \otimes_R M$ is defined by
$$\nabla_M(\omega \otimes m) := \mathbf{d}\omega \otimes m - \omega \wedge \nabla_M(m)$$
for all $m \in M$ and $\omega \in \Omega_R$.
- The condition $\nabla_M \circ \nabla_M = 0$ is called the *integrability condition*.
- We put $M^{D_R} := \mathrm{Ker}\, \nabla_M$.
- A *morphism between D_R-modules* $\Psi : (M, \nabla_M) \to (N, \nabla_N)$ is a morphism of R-modules $\Psi : M \to N$ that commutes with ∇. For short, we sometimes omit ∇_M in the notation. Denote the category of D_R-modules over R by $\mathbf{DMod}(R, D_R)$. Denote the full subcategory of D_R-modules over R that are finitely generated as R-modules by $\mathbf{DMod}^{fg}(R, D_R)$.

Equivalently, a D_R-module over a differential ring (R, D_R) is a pair (M, ρ_M), where M is an R-module and $\rho_M : D_R \to \mathrm{End}_{\mathbb{Z}}(M)$ is an R-linear morphism of Lie rings such that for all $\partial \in D_R$, $a \in R$, and $m \in M$, we have

$$\rho_M(\partial)(am) = a \cdot \rho_M(\partial)(m) + \partial(a) \cdot m.$$

Further, an R-linear map $\Psi : M \to N$ is a morphism of differential modules if and only if, for all $m \in M$ and $\partial \in D_R$, we have

$$\Psi\left(\rho_M(\partial)(m)\right) = \rho_N(\partial)\left(\Psi(m)\right).$$

We sometimes omit ρ_M and use just $\partial(m)$ to denote $\rho_M(\partial)(m)$.

Remark 3.20. If M and D_R are finitely generated free R-modules, then a choice of bases in D_R and M over R gives an equivalent definition of the D_R-module structure on M in terms of connection matrices.

Definition 3.21. Given D_R-modules (M, ∇_M) and (N, ∇_N) over R, the D_R-module structures on the *tensor product* $M \otimes_R N$ and on the *internal Hom module* $\mathrm{Hom}_R(M, N)$ are defined by

$$\nabla_{M\otimes N}(m \otimes n) := m \otimes \nabla_N(n) + \nabla_M(m) \otimes n \in \Omega_R \otimes_R M \otimes_R N,$$

$$\nabla_{\mathrm{Hom}(M,N)}(\Psi)(m) := \nabla_N(\Psi(m)) - \Psi(\nabla_M(m)) \in \Omega_R \otimes_R N$$

for all $m \in M$, $n \in N$, and $\Psi \in \mathrm{Hom}_R(M, N)$ (we use the canonical isomorphism $\Omega_R \otimes_R M \otimes_R N \cong M \otimes_R \Omega_R \otimes_R N$ and write Ψ instead of $\mathrm{id}_{\Omega_R} \otimes_R \Psi$ to be short).

Note that

$$(M \otimes_R N, \nabla_{M\otimes N}) \quad \text{and} \quad (\mathrm{Hom}_R(M, N), \nabla_{\mathrm{Hom}(M,N)})$$

are well-defined as D_R-modules over R, namely, the integrability condition holds for them. The tensor product on D_R-modules defines a tensor category structure on $\mathbf{DMod}(R, D_R)$ with the internal Hom object being defined as above (Section A.2).

Remark 3.22. A D_R-algebra S over R is the same as an R-algebra S with a D_R-module structure ∇_S on S over R such that the unit and multiplication maps are morphisms of D_R-modules over R. Given D_R-algebras S and T, we obtain a D_R-algebra structure on $S \otimes_R T$ following Definition 3.21.

The extension of scalars for differential modules is defined as follows.

Definition 3.23. Let $\varphi : (R, D_R) \to (S, D_S)$ be a morphism of differential rings and (M, ∇_M) be a D_R-module over R. Then the *extension of scalars of* (M, ∇_M) *from* (R, D_R) *to* (S, D_S) is the D_S-module

$$(M_S := S \otimes_R M, \nabla_{M_S}),$$

where, for all $m \in M$ and $a \in S$, we have:

$$\nabla_{M_S}(a \otimes m) := a \cdot (\varphi_* \otimes \mathrm{id}_M)(\nabla_M(m)) + \mathbf{d}a \otimes m \in \Omega_S \otimes_R M = \Omega_S \otimes_S M_S.$$

Equivalently, for all $\partial \in D_S$, $m \in M$, and $a \in S$, we have

$$\rho_{M_S}(\partial)(a \otimes m) := \sum_i (ab_i) \otimes \rho_M(\partial_i)(m) + \partial(a) \otimes m \in M_S,$$

where

$$D_\varphi(\partial) = \sum_i b_i \otimes \partial_i, \quad b_i \in S,\ \partial_i \in D_R.$$

Note that (M_S, ∇_{M_S}) is well-defined as a D_S-module over S, namely, the integrability condition holds for it.

Example 3.24. In the notation of Example 3.8, consider the rank one R-module $M = R \cdot e$ with the D_R-module structure over R defined by

$$\nabla_M(e) := (1/z)\,\mathbf{d}z \otimes e.$$

Then the pair (M_S, ∇_{M_S}), with

$$\nabla_{M_S}(e) = \mathbf{d}x \otimes fe + \mathbf{d}y \otimes ge,$$

satisfies the integrability condition if and only if $\partial_y f = \partial_x g$, that is, if and only if φ_* satisfies the integrability condition.

3.5. Parameterized Picard–Vessiot extensions

First, let us give the definition of a non-parameterized Picard–Vessiot extension in terms of differential fields as defined above.

Definition 3.25. Let (K, D_K) be a differential field and M be a finite-dimensional D_K-module over K. A *Picard–Vessiot extension for M*, or, shortly, *a PV extension for M*, is a D_K-field (L, D_L) over K (in particular, we have a field extension $K \subset L$ and $D_L \cong L \otimes_K D_K$) such that the following conditions are satisfied:

1. We have $K^{D_K} = L^{D_L}$.
2. There is a basis $\{m_1, \ldots, m_n\}$ of M_L over L such that all the m_i's belong to $(M_L)^{D_L}$.
3. There is no smaller D_K-subfield over K in L containing the coordinates of the m_i's in a basis of M over K.

In particular, the canonical morphism

$$L \otimes_k M_L^{D_L} \to M_L$$

is an isomorphism, where $k := K^{D_K} = L^{D_L}$.

Example 3.26. Consider the differential field $(K, \mathfrak{g})$ with zero θ_K, where $\mathfrak{g}$ is a finite-dimensional Lie algebra over K. Let V be a finite-dimensional $\mathfrak{g}$-module over K, that is, V is a finite-dimensional representation of $\mathfrak{g}$ over K. Let G be the smallest algebraic subgroup in $\mathrm{GL}(V)$ such that its Lie algebra contains the image of the representation map $\rho_V : \mathfrak{g} \to \mathfrak{gl}(V)$. The field L of rational functions on G is a $\mathfrak{g}$-field over K: $\mathfrak{g}$ acts on L by translation invariant vector fields on G through ρ_V. The $\mathfrak{g}$-field L is a Picard–Vessiot extension for V.

Let (K, D_K) be a parameterized differential field over a differential field (k, D_k) and let (L, D_L) be a D_K-field over K. Then we obtain a morphism of differential fields

$$(k, D_k) \to (L, D_L)$$

as the composition of the morphisms $(k, D_k) \to (K, D_K)$ and $(K, D_K) \to (L, D_L)$. The isomorphism $D_L \cong L \otimes_K D_K$ induces an isomorphism

$$D_{L/k} \cong L \otimes_K D_{K/k},$$

where, as in Definition 3.14,

$$D_{L/k} := \operatorname{Ker}(D_L \to L \otimes_k D_k).$$

Thus, $(L, D_{L/k})$ is a $D_{K/k}$-field over K.

The following definition of a parameterized Picard–Vessiot extension essentially repeats the corresponding definition from [10].

Definition 3.27. Let (K, D_K) be a parameterized differential field over a differential field (k, D_k) and M be a finite-dimensional $D_{K/k}$-module over K.

- A *parameterized Picard–Vessiot extension for* M, or, shortly, *a PPV extension for* M, is a D_K-field (L, D_L) over K such that the following conditions are satisfied:
  1. We have $K^{D_{K/k}} = L^{D_{L/k}}$.
  2. There is a basis $\{m_1, \ldots, m_n\}$ of M_L over L such that all the m_i's belong to $(M_L)^{D_{L/k}}$, where M_L is a $D_{L/k}$-module over the $D_{L/k}$-field L (see the discussion preceding the definition).
  3. There is no smaller D_K-subfield over K in L containing the coordinates of the m_i's in a basis of M over K.

- A *morphism between PPV extensions* is an isomorphism between the corresponding D_K-fields over K. Let $\mathbf{PPV}(M)$ denote the category of all PPV extensions for M.

Note that, in the notation of Definition 3.27, we have $L^{D_{L/k}} = k$, that is, (L, D_L) is a parameterized differential field over (k, D_k). If $\operatorname{char} k = 0$ and (k, D_k) is differentially closed, then all PPV extensions for a given $D_{K/k}$-module are isomorphic, [10, Theorem 3.5] (see examples of PPV extensions therein).

In the case of Example 3.17(1), Definition 3.27 becomes equivalent to the definition of a PPV extension as given in [10]. It makes sense to consider PPV extensions, because they lead to a reasonable Galois theory for integrable systems of differential equations with parameters. Namely, as shown in [10], a PPV extension defines a parameterized differential Galois group, which is a differential algebraic group over (k, D_k).

In addition, there is a Galois correspondence between differential algebraic subgroups and PPV subextensions, see Section 8.1 for the case when (k, D_k) is not necessarily differentially closed. To investigate the parameterized differential Galois theory, one also needs the notion of a PPV ring.

Definition 3.28. Let (K, D_K) be a parameterized differential field over a differential field (k, D_k), M be a finite-dimensional $D_{K/k}$-module over K, and L be a PPV extension for M. Let $m_i \in M_L$ be as in Definition 3.27. A *parameterized Picard–Vessiot ring associated with* L is a D_K-subalgebra in L generated by the coordinates a_{ij} of the m_i's in a basis of M over K and the inverse of the determinant $1/\det(a_{ij})$.

3.6. Jet rings

The proof of the main result, Theorem 2.5, requires an appropriate notion of a differential Tannakian category over a differential field that goes along with the notion of a differential module. Since it seems not possible to give a direct analogue of Definition 3.19 in a more general setting, one needs another approach to differential modules. Actually, D_R-modules over a differential ring (R, D_R) turn out to be comodules over an object similar to a Hopf algebroid, namely, the 2-jet ring P^2_R (Definition 3.30). This approach has a natural version with R-modules replaced by other objects over R, e.g., Hopf algebras over R or abelian R-linear tensor categories. The latter leads to the notion of a differential object (Definition 3.35).

Let (R, D_R) be a differential ring.

Definition 3.29. A 1-*jet ring* is the abelian group $P^1_R := R \oplus \Omega_R$ with the following commutative ring structure:

$$a \cdot b = ab, \quad a \cdot \omega = a\omega, \quad \text{and} \quad \omega \cdot \eta = 0, \quad a \in R,\ \omega, \eta \in \Omega_R$$

(recall that $\Omega_R = D_R^\vee$).

Consider two ring homomorphisms $l, r : R \to P^1_R$ given by

$$l(a) := a \quad \text{and} \quad r(a) := a + \mathbf{d}a, \quad a \in R.$$

Thus, P^1_R is an algebra over $R \otimes R$. Explicitly, for all $a, b \in R$ and $\omega \in \Omega_R$, we have

$$l(a) \cdot (b + \omega) := ab + a\omega \quad \text{and} \quad (b + \omega) \cdot r(a) := ab + a\omega + b\,\mathbf{d}a.$$

It follows that Ω_R is an $(R\otimes R)$-ideal in P^1_R. The homomorphism $r : R \to P^1_R$ provides a canonical right R-linear splitting

$$P^1_R \cong R \oplus \Omega_R,$$

which differs from the left R-linear splitting. It follows that P^1_R is a finitely generated projective R-module with respect to both R-module structures.

Definition 3.30. A 2-*jet ring* P^2_R is the subset in $P^1_R \otimes_R P^1_R$ that consists of all elements

$$a \otimes 1 + 1 \otimes \omega + \omega \otimes 1 - \eta,$$

where $a \in R$, $\omega \in \Omega_R$, and $\eta \in \Omega_R \otimes_R \Omega_R$ are such that $\mathbf{d}\omega$ equals the image of η under the natural map

$$\Omega_R \otimes_R \Omega_R \to \wedge^2_R \Omega_R.$$

Put I_R to be the set of elements in P^2_R with $a = 0$.

Remark 3.31. Note that, according to our notation, the tensor product $P^1_R \otimes_R P^1_R$ involves both left and right R-module structures on P^1_R.

Example 3.32. Let (R, D_R) be as in Example 3.5 (1). Then

$$P^1_R = (R \otimes_k R)/J^2,$$

where J is the kernel of the multiplication homomorphism $R \otimes_k R \to R$. If 2 is invertible in R, then $P^2_R = (R \otimes_k R)/J^3$, [4].

Let us list some important properties of the 2-jet ring. One can show that P^2_R is an $(R \otimes R)$-subalgebra in $P^1_R \otimes_R P^1_R$ with respect to the "external" R-modules structures. This defines two ring homomorphisms from R to P^2_R, which we denote also by l and r. Explicitly, we have

$$l(a) = a \otimes 1 \quad \text{and} \quad r(a) = a \otimes 1 + 1 \otimes \mathbf{d}a + \mathbf{d}a \otimes 1.$$

Denote the natural embedding by

$$\Delta : P^2_R \to P^1_R \otimes_R P^1_R, \tag{4}$$

and put also

$$e : P^1_R \to R, \quad a + \omega \mapsto a.$$

Note that Δ and e are morphisms of algebras over $R \otimes R$. Both compositions

$$\begin{tikzcd} P^2_R \arrow[r, "\Delta"] & P^1_R \otimes_R P^1_R \arrow[r, shift left, "e\cdot\mathrm{id}"] \arrow[r, shift right, "\mathrm{id}\cdot e"'] & P^1_R \end{tikzcd}$$

coincide with the surjective morphism of $(R \otimes R)$-algebras

$$P^2_R \to P^1_R, \quad (a \otimes 1 + 1 \otimes \omega + \omega \otimes 1 - \eta) \mapsto a + \omega. \tag{5}$$

The kernel of this homomorphism equals the kernel of the natural map

$$\Omega_R \otimes_R \Omega_R \to \wedge^2_R \Omega_R,$$

which is, by definition, the second symmetric power $\mathrm{Sym}^2_R \Omega_R$ of Ω_R. Since $\mathrm{Sym}^2_R \Omega_R$ is a finitely generated projective R-module, P^2_R is a finitely generated projective R-module with respect to both R-module structures, being an extension of P^1_R by $\mathrm{Sym}^2_R \Omega_R$. We also denote the map $P^2_R \to R$ defined as the composition

$$P^2_R \to P^1_R \xrightarrow{e} R$$

by e. Explicitly, we have

$$e(a \otimes 1 + 1 \otimes \omega + \omega \otimes 1 - \eta) = a.$$

Thus, we have $I_R = \mathrm{Ker}(e)$.

A morphism between differential rings $\varphi : (R, D_R) \to (S, D_S)$ defines a homomorphism of $(R \otimes R)$-algebras

$$\left(P^1_\varphi := \varphi \oplus \varphi_*\right) : P^1_R \to P^1_S.$$

The integrability condition for φ is equivalent to the fact that the ring homomorphism

$$P^1_\varphi \otimes P^1_\varphi : P^1_R \otimes_R P^1_R \to P^1_S \otimes_S P^1_S$$

sends P^2_R to P^2_S. Indeed, for any element

$$1 \otimes \omega + \omega \otimes 1 - \eta \in P^2_R,$$

the element

$$\left(P^1_\varphi \otimes P^1_\varphi\right)(1 \otimes \omega + \omega \otimes 1 - \eta) = 1 \otimes \varphi_*(\omega) + \varphi_*(\omega) \otimes 1 - \varphi_*(\eta)$$

belongs to P^2_S if and only if $\mathbf{d}(\varphi_*(\omega))$ equals the image of $\varphi_*(\eta)$ under the natural map

$$\Omega_S \otimes_S \Omega_S \to \wedge^2_S \Omega_S,$$

while the latter coincides with $\varphi_*(\mathbf{d}\omega)$. Thus, we obtain a morphism of $(R \otimes R)$-algebras

$$P^2_\varphi : P^2_R \to P^2_S.$$

One can show that P^2_φ commutes with the morphisms l, r, Δ, and e.

Remark 3.33. Assume that 2 is invertible in R (in particular, $\operatorname{char} R \neq 2$). Then there is a section

$$\wedge^2_R \Omega_R \hookrightarrow \Omega_R \otimes_R \Omega_R, \quad \omega_1 \wedge \omega_2 \mapsto \frac{1}{2}(\omega_1 \otimes \omega_2 - \omega_2 \otimes \omega_1) \tag{6}$$

of the natural quotient map

$$\Omega_R \otimes_R \Omega_R \to \wedge^2_R \Omega_R,$$

and P^2_R is generated as a subring and a left (or right) submodule in $P^1_R \otimes_R P^1_R$ by all elements of type

$$\langle\omega\rangle := 1 \otimes \omega + \omega \otimes 1 - \mathbf{d}\omega, \quad \omega \in \Omega_R.$$

In addition, I_R is generated by $\langle\omega\rangle$ for all $\omega \in \Omega_R$ and $\operatorname{Sym}^2_R \Omega_R = I_R \cdot I_R$.

3.7. *Differential rings vs. Hopf algebroids*

Let us cite some relations between differential rings and Hopf algebroids from [35]. The content of this section is not needed for the rest of the text, but we have decided to include it for the convenience of the reader.

Given a differential ring (R, D_R), we have defined a 2-jet ring P^2_R in Section 3.6. Actually, the construction depends only on the *2-truncated de Rham complex*

$$R \xrightarrow{\mathbf{d}} \Omega_R \xrightarrow{\mathbf{d}} \wedge^2_R \Omega_R$$

associated with (R, D_R) (Definition 3.3).

Conversely, let (R, A^2) satisfy the similar properties as (R, P^2_R) does. Namely, call a *2-truncated Hopf algebroid with divided powers* a pair of rings (R, A^2) (A^2 is just a notation for a ring) together with the following data: two ring homomorphisms $l, r : R \to A^2$, a morphism of $(R \otimes R)$-algebras $e : A^2 \to R$, a map of sets $\gamma : I \to A^2$, where $I := \operatorname{Ker}(e)$, and a morphism of $(R \otimes R)$-algebras

$$\Delta : A^2 \to A^1 \otimes_R A^1,$$

where

$$A^1 := A^2 / I^{[2]}, \quad I^{[2]} := I \cdot I + \gamma(I).$$

We require that, for all $a \in R$, $x, y \in I$, we have

$$\gamma(ax) = a^2\gamma(x), \quad \gamma(x + y) = \gamma(x) + xy + \gamma(y), \quad I \cdot I^{[2]} = 0,$$

and the compositions

$$A^2 \xrightarrow{\Delta} A^1 \otimes_R A^1 \underset{\mathrm{id}\otimes e}{\overset{e\otimes \mathrm{id}}{\rightrightarrows}} A^1$$

are equal to the canonical surjection $A^2 \to A^1$, where e also denotes the canonical morphism $A^1 \to R$. In particular, for P^2_R, we put

$$\gamma(1 \otimes \omega + \omega \otimes 1 - \eta) := \omega \otimes \omega.$$

It follows that there is an antipode map $A^2 \to (A^2)^s$ that satisfies the usual properties. An analogous construction to the one from Example 3.5(6) provides a 2-truncated de Rham complex associated with $\left(R, A^2\right)$. This implies that the category of 2-truncated Hopf algebroids with divided powers is equivalent to the category of 2-truncated de Rham complexes.

Further, as shown in Remark 3.4, there is a way to construct a differential ring based on a 2-truncated de Rham complex with finitely generated projective Ω_R. The Jacobi identity for the Lie bracket is equivalent to the vanishing of the composition

$$\mathbf{d} \circ \mathbf{d} : \Omega_R \to \wedge^3_R \Omega_R.$$

This gives an auxiliary condition on the corresponding 2-truncated Hopf algebroids with divided powers, which can be explicitly written in terms of a certain ring A^3 (which is a 3-jet ring in the case of P^2_R), [35, 1.3.5]. This condition is similar to the associativity condition for a Hopf algebroid (Section A.1).

It follows from [35, Proposition 1.2.8] that the category of 2-truncated Hopf algebroids with divided powers, with $I/I^{[2]}$ being a flat R-module, and with the associativity condition is equivalent to the category of 2-truncated de Rham complexes with Ω_R being a flat R-module and with vanishing

$$\mathbf{d} \circ \mathbf{d} : \Omega_R \to \wedge^3_R \Omega_R.$$

Also, the category of 2-truncated de Rham complexes with Ω_R being a finitely generated projective R-module and with vanishing $\mathbf{d} \circ \mathbf{d} : \Omega_R \to \wedge^3_R \Omega_R$ is equivalent to the category of differential rings.

Recall that a *formal Hopf algebroid* is defined similarly to a Hopf algebroid with A being a pro-object in the category of $(R \otimes R)$-algebras. A formal Hopf algebroid $(R, \widehat{A})$ with divided powers on

$$I = \mathrm{Ker}\,(e : \widehat{A} \to R)$$

is *complete* if the natural map

$$\widehat{A} \to \text{“}\varprojlim\text{”} A/I^{[i]}$$

is an isomorphism. It follows from [35, Théorème 1.3.6] that the category of 2-truncated Hopf algebroids with divided powers, with $I/I^{[2]}$ being a flat R-module, and with the associativity condition is equivalent to the category of complete formal Hopf algebroids with divided powers and with $I/I^{[2]}$ being a flat R-module.

In particular, the category of differential rings over $\mathbb{Q}$ is equivalent to the category of complete formal Hopf algebroids $(R, \widehat{A})$ with R being a $\mathbb{Q}$-algebra and I/I^2 being a finitely generated projective R-module. For example, if $D_R = R \cdot \partial$ and R is a $\mathbb{Q}$-algebra, then, for the corresponding complete formal Hopf algebroid $(R, \widehat{A})$, the ring $\widehat{A}$ equals the ring of formal Taylor series $R[[t]]$, and we have

$$l(a) = a, \quad r(a) = \sum_{i=0}^{\infty} \partial^i(a)/i!, \quad \text{and} \quad \Delta(t) = 1 \otimes t + t \otimes 1.$$

In other words, the formal Hopf algebroid $(R, \widehat{A})$ is given by the action of the formal additive group $\widehat{\mathbb{G}}_a$ on $\mathrm{Spec}(R)$.

It seems that, in general, formal Hopf algebroids that are complete with respect to the usual powers I^i correspond to iterative Hasse–Schmidt derivations on the differential side.

Assume that all D_R-modules over R that are finitely generated over R are projective R-modules (for example, this holds if R is a field). Then the category $\mathbf{DMod}^{fg}(R, D_R)$ is a Tannakian category with the forgetful fiber functor

$$\mathbf{DMod}^{fg}(R, D_R) \to \mathbf{Mod}(R).$$

It seems to be a non-trivial problem to give an explicit description of the corresponding Hopf algebroid (R, A) in terms of D_R, whose formal completion is the complete formal Hopf algebroid associated with (R, D_R).

3.8. Differential objects

The pair (R, P^2_R) resembles a Hopf algebroid (Section A.1). The main difference with a Hopf algebroid is that Δ does not send P^2_R to the tensor square of itself. However, one can define a comodule over (R, P^2_R) in the same way as one defines a comodule over a Hopf algebroid. In the present paper, we use a generalization of this notion.

Let $\mathcal{M}$ be a category *cofibred* over commutative rings, that is, for each commutative ring R, there is a category $\mathcal{M}(R)$ and, given a ring homomorphism $R \to S$, there is a functor

$$S \otimes_R - : \mathcal{M}(R) \to \mathcal{M}(S),$$

called an *extension of scalars*, compatible with taking composition of ring homomorphisms (for more details, see [27]).

Example 3.34. $\mathcal{M}(R)$ can be the category of R-modules, R-algebras, Hopf algebras over R, Hopf algebroids over R, etc.

We will now define differential objects, generalizing stratifications on sheaves from [4].

Definition 3.35.

- A *D_R-object in $\mathcal{M}$* over (R, D_R) (or simply over R) is a pair (X, ϵ^2_X), where X is an object in $\mathcal{M}(R)$ and
$$\epsilon^2_X : X \otimes_R P^2_R \to P^2_R \otimes_R X$$
is a morphism in the category $\mathcal{M}(P^2_R)$ such that the following two conditions are satisfied. First, we have
$$R \otimes_{P^2_R} \epsilon^2_X = \mathrm{id}_X,$$
where the P^2_R-module structure on R is defined by the ring homomorphism $e : P^2_R \to R$. Put
$$\left(\epsilon^1_X := P^1_R \otimes_{P^2_R} \epsilon^2_X\right) \colon X \otimes_R P^1_R \to P^1_R \otimes_R X,$$
where the P^2_R-module structure on P^1_R is given by the canonical surjection $P^2_R \to P^1_R$. The second condition says that the composition of morphisms in $\mathcal{M}(P^1_R \otimes_R P^1_R)$
$$X \otimes_R P^1_R \otimes_R P^1_R \xrightarrow{\epsilon^1_X \otimes_{P^1_R} (P^1_R \otimes_R P^1_R)} P^1_R \otimes_R X \otimes_R P^1_R \xrightarrow{(P^1_R \otimes_R P^1_R) \otimes_{P^1_R} \epsilon^1_X} P^1_R \otimes_R P^1_R \otimes_R X$$

is equal to the extension of scalars

$$\left(P^1_R \otimes_R P^1_R\right) \otimes_{P^2_R} \epsilon^2_X,$$

where the P^2_R-module structure on $P^1_R \otimes_R P^1_R$ is given by the ring homomorphism Δ.

- The morphism ϵ^2_X is called a *D_R-structure* on X.
- A *morphism between D_R-objects* in $\mathcal{M}$ over (R, D_R) is a morphism between objects in $\mathcal{M}(R)$ that commutes with ϵ^2.

Remark 3.36. Perhaps, a more conceptually proper way to define a differential object would also involve the 3-jet ring to encode the associativity condition (Section 3.7), but the present definition will be enough for our purposes. However, all examples that arise in the paper satisfy the associativity condition.

Definition 3.37. We say that a cofibred category $\mathcal{M}$ over rings has *restrictions of scalars* if, for any ring homomorphism $R \to S$, there is a functor $\mathcal{M}(S) \to \mathcal{M}(R)$, called a *restriction of scalars*, that is right adjoint to the extension of scalars. We usually denote the value of the restriction of scalars functor in the same way as its argument.

Thus, for all objects X in $\mathcal{M}(R)$ and Y in $\mathcal{M}(S)$, there is a functorial isomorphism

$$\mathrm{Hom}_{\mathcal{M}(R)}(X, Y) \cong \mathrm{Hom}_{\mathcal{M}(S)}(S \otimes_R X, Y).$$

Also, the restriction of scalars defines an object $S \otimes_R X$ in $\mathcal{M}(R)$, which is functorial in S and X: given a homomorphism of R-algebras $\varphi : S \to T$ and a morphism $f : X \to Y$ in $\mathcal{M}(R)$, we have the morphism in $\mathcal{M}(R)$

$$\varphi \otimes f : S \otimes_R X \to T \otimes_R Y.$$

In particular, we have a canonical morphism $X \to S \otimes_R X$ in $\mathcal{M}(R)$ given by the ring homomorphism $R \to S$.

Example 3.38. The cofibred categories of modules and algebras have restrictions of scalars, while the cofibred categories of Hopf algebras and Hopf algebroids do not have restrictions of scalars.

Given an object X in $\mathcal{M}(R)$, by ${}_R(P^2_R \otimes_R X)$, denote the object in $\mathcal{M}(R)$ defined as follows: first one takes the extension of scalars $P^2_R \otimes_R X$ with respect to the right morphism $r : R \to P^2_R$ and then applies the restriction of scalars with respect to the left morphism $l : R \to P^2_R$. The proof of the following proposition is a direct application of the adjointness between the extension and restriction of scalars.

Proposition 3.39. *Suppose that a cofibred category $\mathcal{M}$ over rings has restrictions of scalars. Then a D_R-object in $\mathcal{M}$ over (R, D_R) is the same as a pair (X, ϕ^2_X), where X is an object in $\mathcal{M}(R)$ and*

$$\phi^2_X : X \to {}_R\left(P^2_R \otimes_R X\right)$$

is a morphism in $\mathcal{M}(R)$ such that

$$(e \otimes \mathrm{id}_X) \circ \phi^2_X = \mathrm{id}_X$$

and the following diagram commutes in $\mathcal{M}(R)$:

$$\begin{array}{ccc} X & \xrightarrow{\phi_X^2} & {}_R\big(P_R^2 \otimes_R X\big) \\ {\scriptstyle \phi_X^1}\big\downarrow & & \big\downarrow{\scriptstyle \Delta\otimes \mathrm{id}_X} \\ {}_R\big(P_R^1 \otimes_R X\big) & \xrightarrow{\mathrm{id}_P \otimes \phi_X^1} & {}_R\big(P_R^1 \otimes_R P_R^1 \otimes_R X\big), \end{array}$$

where ϕ_X^1 is the composition of ϕ_X^2 with the morphism

$${}_R\big(P_R^2 \otimes_R X\big) \to {}_R\big(P_R^1 \otimes_R X\big).$$

In Section 4.3, we use the following statement.

Proposition 3.40. *Suppose that a cofibred category $\mathcal{M}$ over rings has restrictions of scalars. Then, for any D_R-object X in $\mathcal{M}$ over R, the morphism ϵ_X^1 in $\mathcal{M}(P_R^1)$ is an isomorphism.*

Proof. The proof is similar to that for a Hopf algebra or a Hopf algebroid. The idea is that (R, P_R^1) corepresents a groupoid in the category of R-algebras with a two-step filtration, where the filtration on P_R^1 is given by $P_R^1 \supset \Omega_R$. More precisely, put

$$P_R^1 \otimes_R^1 P_R^1 := \big(P_R^1 \otimes_R P_R^1\big)/(\Omega_R \otimes_R \Omega_R), \quad \iota : P_R^1 \to \big(P_R^1\big)^s, \quad a + \omega \mapsto a - \omega,$$

where, as in Section A.1, the superscript s denotes the interchange of the left and right R-module structures. The homomorphism

$$\Delta : P_R^2 \to P_R^1 \otimes_R P_R^1$$

induces a homomorphism

$$P_R^1 \to P_R^1 \otimes_R^1 P_R^1,$$

also denoted by Δ.

Let us construct an inverse to ϵ_X^1 explicitly. Denote the composition in $\mathcal{M}(R)$

$$X \xrightarrow{\phi_X^1} {}_R\big(P_R^1 \otimes_R X\big) \xrightarrow{\iota\otimes \mathrm{id}_X} \big(X \otimes_R P_R^1\big)_R$$

by ψ. We will prove that $\epsilon_X^1 \circ \psi$ equals the morphism

$$X \to \big(P_R^1 \otimes_R X\big)_R$$

given by the ring homomorphism $r : R \to P_R^1$. This would imply that ϵ_X^1 is inverse to the morphism in $\mathcal{M}(P_R^1)$ from $P_R^1 \otimes_R X$ to $X \otimes_R P_R^1$ that corresponds by adjunction to ψ, thus, ϵ_X^1 is an isomorphism.

By the adjunction relation between ϵ and ϕ, the composition

$${}_R\big(P_R^1 \otimes_R X\big) \xrightarrow{\iota\otimes \mathrm{id}_X} \big(X \otimes_R P_R^1\big)_R \xrightarrow{\epsilon_X^1} \big(P_R^1 \otimes_R X\big)_R$$

is equal to the composition

$${}_R\big(P_R^1 \otimes_R X\big) \xrightarrow{\mathrm{id}_P \otimes \phi_X^1} {}_R\big(P_R^1 \otimes_R^1 P_R^1 \otimes_R X\big) \xrightarrow{\iota\cdot \mathrm{id}_P \otimes \mathrm{id}_X} \big(P_R^1 \otimes_R X\big)_R.$$

Since X is a D_R-object, we have

$$(\Delta \otimes \mathrm{id}_X) \circ \phi_X^2 = \left(\mathrm{id}_P \otimes \phi_X^1\right) \circ \phi_X^1 : X \to {}_R\left(P_R^1 \otimes_R P_R^1 \otimes_R X\right).$$

Applying the ring homomorphism

$$P_R^1 \otimes_R P_R^1 \to P_R^1 \otimes_R^1 P_R^1,$$

we obtain that both compositions

$$X \xrightarrow{\phi_X^1} {}_R\left(P_R^1 \otimes_R X\right) \underset{\Delta \otimes \mathrm{id}_X}{\overset{\mathrm{id}_P \otimes \phi_X^1}{\rightrightarrows}} {}_R\left(P_R^1 \otimes_R^1 P_R^1 \otimes_R X\right)$$

are the same. Further, as for Hopf algebroids, we have

$$(\iota \cdot \mathrm{id}_P) \circ \Delta = r \circ e : P_R^1 \to P_R^1,$$

where we consider

$$\Delta : P_R^1 \to P_R^1 \otimes_R^1 P_R^1.$$

Finally, the composition $e \circ \phi_X^1 : X \to X$ is the identity. All together, this implies the needed statement about $\epsilon_X^1 \circ \psi$. □

Remark 3.41. It is not clear whether the morphism ϕ_X^2 must be an isomorphism in the general case.

3.9. Examples of differential objects

Definition 3.35 is motivated by the following statement.

Proposition 3.42. *Given an R-module M, a D_R-module structure on M over R is the same as a D_R-structure on M as an object in the cofibred category of modules.*

Proof. The cofibred category of modules has restrictions of scalars. Hence, by Proposition 3.39, a D_R-structure on M as an object in the cofibred category of modules is given by an R-linear morphism

$$\phi_M^2 : M \to {}_R\left(P_R^2 \otimes_R M\right)$$

that satisfies the conditions therein. Assume that ∇_M is a D_R-module structure on M. Consider the map

$$\phi_M^1 : M \to P_R^1 \otimes_R M, \quad m \mapsto 1 \otimes m - \nabla_M(m). \tag{7}$$

The Leibniz rule for ∇_M is equivalent to the left R-linearity of ϕ_M^1. Also, we have

$$(e \otimes \mathrm{id}_M) \circ \phi_M^1 = \mathrm{id}_M .$$

Note that the cokernel of the injective map

$$\Delta : P_R^2 \to P_R^1 \otimes_R P_R^1$$

is a projective R-module, being an extension of Ω_R by $\wedge^2_R\Omega_R$. Therefore, the map

$$\Delta \otimes \mathrm{id}_M : P^2_R \otimes_R M \to P^1_R \otimes_R P^1_R \otimes_R M$$

is injective. The integrability condition for ∇_M is equivalent to the fact that the image of the composition

$$M \xrightarrow{\phi^1_M} P^1_R \otimes_R M \xrightarrow{\mathrm{id}\otimes\phi^1_M} P^1_R \otimes_R P^1_R \otimes_R M$$

is contained in $P^2_R \otimes_R M$. To see this, take any $m \in M$ and set

$$\nabla_M(m) = \sum_i \omega_i \otimes m_i, \quad \omega_i \in \Omega_R,\ m_i \in M.$$

Then the element

$$\begin{aligned}\left(\mathrm{id}\otimes\phi^1_M\right)\left(\phi^1_M(m)\right) &= \left(\mathrm{id}\otimes\phi^1_M\right)\left(1 \otimes m - \sum_i \omega_i \otimes m_i\right) = \\ &= 1 \otimes 1 \otimes m - \sum_i 1 \otimes \omega_i \otimes m_i - \sum_i \omega_i \otimes 1 \otimes m_i + \sum_i \omega_i \otimes \nabla_M(m_i)\end{aligned}$$

belongs to $P^2_R \otimes_R M$ if and only if

$$\sum_i \mathbf{d}\omega_i \otimes m_i = \sum_i \omega_i \wedge \nabla_M(m_i) \in \wedge^2_R\Omega_R \otimes_R M.$$

Finally, put

$$\phi^2_M := \left(\mathrm{id}\otimes\phi^1_M\right) \circ \phi^1_M$$

to be the obtained map from M to $P^2_R \otimes_R M$.

Conversely, assume that ϕ^2_M is a D_R-structure on M. Then the formula

$$\nabla_M(m) := 1 \otimes m - \phi^1_M(m)$$

defines a D_R-module structure on M over R. □

Example 3.43.

1. A D_R-object over R in the cofibred category of algebras is the same as a D_R-algebra over R.
2. A D_R-Hopf algebra over R is a Hopf algebra A over R such that A is a D_R-algebra over R and the coproduct, the counit, and the antipode maps are morphisms of D_R-algebras.
3. Given a differential ring (κ, D_κ), a D_κ-Hopf algebroid over κ is a Hopf algebroid (R, A) over κ such that R and A are D_κ-algebras over κ and (l, r, Δ, e, ι) are morphisms of D_κ-algebras over κ.

Here is an application of this approach to differential structures.

Proposition 3.44. *Let A be a D_R-algebra over R such that A is also a Hopf algebra over R. Suppose that the coproduct map is a morphism of D_R-algebras over R. Then the counit and antipode maps are also morphisms of D_R-algebras over R, that is, A is a D_R-Hopf algebra over R.*

Proof. Since the coproduct map is differential, the morphism

$$\epsilon_A^2 : P_R^2 \otimes_R A \to A \otimes_R P_R^2$$

commutes with the coproduct maps in the corresponding Hopf algebras $P_R^2 \otimes_R A$ and $A \otimes_R P_R^2$ over P_R. Therefore, it commutes with the counit and the antipode maps (for example, see [79, Section 2.1]). $\square$

In Section 4.2 we use the following statement.

Lemma 3.45. *Let (R, A) be a D_κ-Hopf algebroid over a differential ring (κ, D_κ). Then the composition of the isomorphisms of abelian groups*

$$A \otimes_R P_R^2 \xrightarrow{\sim} P_A^2 \xrightarrow{\sim} P_R^2 \otimes_R A$$

is an isomorphism of R-bimodules.

Proof. Let $\varphi : R \to A$ denote the left homomorphism. The left R-module structure on $A \otimes_R P_R^2$ corresponds to the R-module structure on P_A^2 given by the composition

$$R \xrightarrow{\varphi} A \xrightarrow{l} P_A^2.$$

The left R-module structure on $P_R^2 \otimes_R A$ corresponds to the R-module structure on P_A^2 given by the composition

$$R \xrightarrow{\quad l \quad} P_R^2 \xrightarrow{\quad P_\varphi^2 \quad} P_A^2.$$

Since φ is a morphism of differential rings, P_φ^2 is a morphism of R-bimodules. In particular, the compositions above coincide. Therefore, the left R-modules structures on $A \otimes_R P_R^2$ and $P_R^2 \otimes_R A$ are the same. The proof for the right R-module structures in analogous. $\square$

3.10. *Lie derivative*

In Section 5, we use the Lie derivatives defined on jet rings. Let (R, D_R) be a differential ring.

Definition 3.46. A *weak D_R-module* is an R-module M together with a morphism of Lie rings

$$\rho_M : D_R \to \mathrm{End}_{\mathbb{Z}}(M)$$

that satisfies the Leibniz rule with respect to the multiplication by scalars from R (thus, a D_R-module is a weak D_R-module such that ρ_M is R-linear). Morphisms between weak D_R-modules are defined similarly to morphisms between D_R-modules. As with differential modules, we sometimes omit ρ_M and use just $\partial(m)$ to denote $\rho_M(\partial)(m)$.

Remark 3.47. As in Definition 3.21, given two weak D_R-modules, one can show that the Leibniz rule defines a weak D_R-structure on their tensor product.

Definition 3.48. Given $\partial \in D_R$ and $\omega \in \Omega_R$, define the *Lie derivative* as follows:

$$L_\partial(\omega) := \mathbf{d}(\omega(\partial)) + (\mathbf{d}\omega)(\partial \wedge -) \in \Omega_R,$$

where $(\mathbf{d}\omega)(\partial \wedge -)$ denotes the element in $\Omega_R = D_R^\vee$ that sends any $\delta \in D_R$ to $(\mathbf{d}\omega)(\partial \wedge \delta)$.

The notation $L_\partial(\omega)$ instead of $\partial(\omega)$ avoids confusing the Lie derivative with the result of the pairing between D_R and Ω_R. It follows from the definition of $\mathbf{d}\omega$ that, for any $\xi \in D_R$, we have

$$L_\partial(\omega)(\xi) = \partial(\omega(\xi)) - \omega([\partial, \xi]). \tag{8}$$

Also, one can show that, for any $a \in R$, we have

$$L_{a\partial}(\omega) = aL_\partial(\omega) + \omega(\partial)\mathbf{d}a. \tag{9}$$

The Lie derivative defines a weak D_R-structure on Ω_R. By linearity, we obtain a weak D_R-structure on $P^1_R \cong R \oplus \Omega_R$:

$$\partial(a + \omega) := \partial(a) + L_\partial(\omega), \quad a \in R,\ \omega \in \Omega_R\,, \partial \in D_R.$$

It follows that $r : R \to P^1_R$ is a morphism of weak D_R-modules. Since

$$\mathbf{d}(\partial(a)) = L_\partial(\mathbf{d}a) \quad \text{for all } a \in R,\ \partial \in D_R,$$

we have that $l : R \to P^1_R$ is a morphism of weak D_R-modules. The Leibniz rule for L_∂ on Ω_R implies that the multiplication morphism

$$P^1_R \otimes_R P^1_R \to P^1_R$$

is also a morphism of weak D_R-modules.

Remark 3.49. By the Leibniz rule, the Lie derivative extends to a weak D_R-structure on $\wedge^2\Omega_R$, which we also denote by L_∂. One can show that L_∂ commutes with the map $\mathbf{d} : \Omega_R \to \wedge^2_R\Omega_R$. This implies that the subring

$$P^2_R \subset P^1_R \otimes_R P^1_R$$

is preserved under the action of D_R via the weak D_R-module structure on $P^1_R \otimes_R P^1_R$.

4. Differential categories

4.1. Extension of scalars for abelian tensor categories

Our aim is to apply Definition 3.35 of a differential object with X being an abelian R-linear tensor category. For this, we need to use extension of scalars for such categories associated with homomorphisms of rings. Let us briefly describe this. One can find more details, for example, in [15, p. 155], [54, p. 407], and more recent papers [24] and [74].

We use the terminology from Section A.2. We fix a commutative ring R, a commutative R-algebra S, and an abelian R-linear tensor category C. According to our definitions, this means that, in particular, the tensor product in C is right-exact and R-linear in both arguments.

Definition 4.1. The *extension of scalars of C from R to S* is an abelian S-linear tensor category $S \otimes_R C$ together with a right-exact R-linear tensor functor

$$S \otimes_R - : C \to S \otimes_R C$$

that is universal from the left among all such data, that is, for any abelian S-linear tensor category $\mathcal{D}$, taking the composition with $S \otimes_R -$ defines an equivalence of categories:

$$\mathbf{Fun}^{r,\otimes}_S(S \otimes_R C, \mathcal{D}) \xrightarrow{\sim} \mathbf{Fun}^{r,\otimes}_R(C, \mathcal{D}), \quad F \mapsto F \circ (S \otimes_R -),$$

where $\mathbf{Fun}^{r,\otimes}_S$ denotes the category of right-exact S-linear tensor functors (similarly, for $\mathbf{Fun}^{r,\otimes}_R$).

We usually denote the extension of scalars just by $S \otimes_R \mathcal{C}$ (keeping in mind that we also fix the functor $S \otimes_R -$). Let us describe some general properties of the extension of scalars for categories. First, consider a homomorphism of R-algebras $S \to T$ and assume that the extensions of scalars $S \otimes_R \mathcal{C}$ and $T \otimes_S (S \otimes_R \mathcal{C})$ exist. Then $T \otimes_S (S \otimes_R \mathcal{C})$ is equivalent to the extension of scalars $T \otimes_R \mathcal{C}$.

Further, the category $S \otimes_R \mathcal{C}$ is functorial in S and $\mathcal{C}$ in the following way. Let $\varphi : S \to T$ be a homomorphism of R-algebras, $\mathcal{D}$ be an abelian R-linear tensor category, and $F : \mathcal{C} \to \mathcal{D}$ be a right-exact R-linear tensor functor. Assume that both $S \otimes_R \mathcal{C}$ and $T \otimes_R \mathcal{D}$ exist. Then we obtain a right-exact S-linear tensor functor

$$\varphi \otimes F : S \otimes_R \mathcal{C} \to T \otimes_R \mathcal{D}$$

defined by the universal property of $S \otimes_R \mathcal{C}$ applied to the right-exact R-linear tensor functor

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{T \otimes_R -} T \otimes_R \mathcal{D}.$$

The assignment $F \mapsto \varphi \otimes F$ is functorial in F. If $\psi : T \to U$ is a homomorphism of R-algebras, $\mathcal{E}$ is an abelian R-linear tensor category, $G : \mathcal{D} \to \mathcal{E}$ is a right-exact R-linear tensor functor, and $U \otimes_R \mathcal{E}$ exists, then there is a canonical isomorphism between tensor functors:

$$(\psi \otimes G) \circ (\varphi \otimes F) \cong (\psi \circ \varphi) \otimes (G \circ F).$$

Sometimes, we also denote $\mathrm{id}_S \otimes F$ by $S \otimes_R F$. Also, we have that $\varphi \otimes \mathrm{id}_{\mathcal{C}} = S \otimes_R -$ for $\varphi : R \to S$. We hope that this coincidence will not make any confusion.

In Definition 4.9, we will need a slight generalization of the previous functor $\varphi \otimes F$. Namely, let

$$\begin{array}{ccc} R & \longrightarrow & S \\ \big\downarrow & & \big\downarrow{\scriptstyle \varphi} \\ U & \longrightarrow & T \end{array}$$

be a commutative diagram of rings, $\mathcal{D}$ be an abelian U-linear tensor category, and $F : \mathcal{C} \to \mathcal{D}$ be a right-exact R-linear tensor functor. Assume that both $S \otimes_R \mathcal{C}$ and $T \otimes_U \mathcal{D}$ exist. Then, similarly to the above, we obtain a right-exact S-linear tensor functor

$$\varphi \otimes F : S \otimes_R \mathcal{C} \to T \otimes_U \mathcal{D}.$$

If $(S \otimes_R U) \otimes_U \mathcal{D}$ exists, then we have

$$(\varphi \otimes F)(X) = T \otimes_{(S \otimes_R U)} F(X). \tag{10}$$

The following important result is proved in [74, Theorem 1.4.1] (see also [15, p. 155] and [54, p. 407]).

Theorem 4.2. *Let $\mathcal{C}$ be a Tannakian category over a field k and $k \subset K$ be a field extension. Then there exists the extension of scalars $K \otimes_k \mathcal{C}$.*

Further, recall that an *S-module in $\mathcal{C}$* is a pair (X, α_X), where X is an object in $\mathcal{C}$ and

$$\alpha_X : S \to \mathrm{End}_{\mathcal{C}}(X)$$

is a homomorphism of R-algebras. Morphisms between S-modules in C are naturally defined. Given an R-module M and an object X in C, define

$$M \otimes_R X$$

to be an object in C such that there is a functorial isomorphism of R-modules

$$\mathrm{Hom}_C(M \otimes_R X, Y) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_C(X, Y)). \tag{11}$$

The object $M \otimes_R X$ is well-defined up to a unique isomorphism if it exists. If an R-module M is of finite presentation, that is, there is a right-exact sequence of R-modules

$$R^{\oplus m} \xrightarrow{\varphi} R^{\oplus n} \longrightarrow M \longrightarrow 0,$$

then, $M \otimes_R X$ exists for any X. By (11), for an S-module (X, α_X) in C, the homomorphism α_X defines a morphism

$$a_X : S \otimes_R X \to X.$$

The following result is extensively used in what follows. Its proof can be found in [13, 5.11], where an equivalent approach to the extension of scalars for categories is used (see also [74] and [24]).

Proposition 4.3. *Let C be an abelian R-linear tensor category. Suppose that S is of finite presentation as an R-module. Then the abelian S-linear tensor category of S-modules in C is equivalent to the extension of scalars $S \otimes_R C$ and the functor $S \otimes_R -$ sends X to $S \otimes_R X$.*

Example 4.4. If S is of finite presentation as an R-module, then the extension of scalars category $S \otimes_R \mathbf{Mod}(R)$ is equivalent to the category $\mathbf{Mod}(S)$ and the functor $S \otimes_R -$ coincides with the usual tensor product functor.

Example 4.5. Let M be an R-module of finite presentation. Put $S := R \oplus M$, where an R-algebra structure on S is uniquely defined by the condition $M \cdot M = 0$. An S-module in C is the same as an exact sequence

$$0 \to X' \to X \to X'' \to 0$$

together with a morphism

$$f_X : M \otimes_R X'' \to X'.$$

Namely, with an S-module (X, α_X), we associate $X' := M \cdot X$ and $X'' := X/(M \cdot X)$, where $M \cdot X$ is the image of the morphism

$$a_X : M \otimes_R X \to X.$$

If S-modules (X, α_X) and (Y, α_Y) correspond to the data

$$\begin{aligned} 0 \to X' \to X \to X'' \to 0, &\quad f_X : M \otimes_R X'' \to X'; \\ 0 \to Y' \to Y \to Y'' \to 0, &\quad f_Y : M \otimes_R Y'' \to Y', \end{aligned}$$

then their tensor product $(X, \alpha_X) \otimes (Y, \alpha_Y)$ in $S \otimes_R C$ is defined as the cokernel of the morphism

$$b_X \otimes \mathrm{id}_Y - \mathrm{id}_X \otimes b_Y : M \otimes_R (X \otimes Y) \to X \otimes Y,$$

where b_X is defined as the composition

$$M \otimes_R X \longrightarrow M \otimes_R X'' \xrightarrow{f_X} X' \longrightarrow X,$$

and similarly for b_Y. In particular, if the tensor product in C is exact in both arguments and the morphisms f_X, f_Y are isomorphisms, then (see also [41, 5.1.3]) the tensor product $(X, \alpha_X) \otimes (Y, \alpha_Y)$ corresponds to the Baer sum of the exact sequences

$$0 \to M \otimes_R (X'' \otimes Y'') \to X \otimes Y'' \to X'' \otimes Y'' \to 0,$$
$$0 \to X'' \otimes (M \otimes_R Y'') \to X'' \otimes Y \to X'' \otimes Y'' \to 0.$$

4.2. Differential abelian tensor categories

Throughout this subsection, we fix a differential ring (R, D_R). We use constructions from Sections 3.8 and 4.1. Recall that the jet rings P^1_R, P^2_R, and $P^1_R \otimes_R P^1_R$ (Definition 3.30) are finitely generated projective R-modules with respect to both left and right R-module structures. Hence, they are of finite presentation as R-modules and there is an extension of scalars from R to P^2_R for abelian tensor categories (Definition 4.1 and Proposition 4.3).

Consequently, Definition 3.35 gives the notion of a D_R-object over R in the cofibred 2-category of abelian tensor categories, or a *D_R-category* over R for short. Here "morphisms" between tensor categories are tensor functors. The main difference with the case of a usual cofibred category as in Definition 3.35 is that, instead of considering equalities between morphisms, one should fix isomorphisms between tensor functors.

Further, there are also restrictions of scalars between P^2_R and R for abelian tensor categories (this follows from the definition of the extension of scalars for categories, Definition 4.1). Proposition 3.39 remains valid in the case of a cofibred 2-category instead of a cofibred (1-)category. Thus, one has an equivalent definition of a D_R-category over R in terms of ϕ's instead of ϵ's. We prefer to use the definition in terms of ϕ's. Note that Definitions 4.6 and 4.9 below have a more explicit equivalent form, see Section 4.3 (also, compare with [24, Example 12], where the case of the coaction of a Hopf algebra on a category is considered).

Similarly to Section 3.8,

$${}_R\big(P^2_R \otimes_R C\big)$$

denotes the abelian tensor category $P^2_R \otimes_R C$ considered with the R-linear structure obtained by the left ring homomorphism $l : R \to P^2_R$.

Definition 4.6. A *D_R-category* over (R, D_R) (or simply over R) is a collection $\big(C, \phi^2_C, \Phi_C, \Psi_C\big)$, where C is an abelian R-linear tensor category,

$$\phi^2_C : C \to {}_R\big(P^2_R \otimes_R C\big)$$

is a right-exact R-linear tensor functor,

$$\Phi_C : (e \otimes \mathrm{id}_C) \circ \phi^2_C \xrightarrow{\sim} \mathrm{id}_C$$

is an isomorphism of tensor functors from C to itself (recall that $e : P^2_R \to R$ is a ring homomorphism), and

$$\Psi_C : (\Delta \otimes \mathrm{id}_C) \circ \phi^2_C \xrightarrow{\sim} (\mathrm{id}_{P^1_R} \otimes \phi^1_C) \circ \phi^1_C$$

is an isomorphism between tensor functors from C to $P^1_R \otimes_R P^1_R \otimes_R C$, where ϕ^1_C is the composition of ϕ^2_C with the functor

$$P^2_R \otimes_R C \to P^1_R \otimes_R C.$$

For short, we usually denote a D_R-category $\left(C, \phi^2_C, \Phi_C, \Psi_C\right)$ just by C. We call the collection $\left(\phi^2_C, \Phi_C, \Psi_C\right)$ a *D_R-structure on C.*

In other words, Φ_C is an isomorphism between the composition

$$C \xrightarrow{\phi^2_C} P^2_R \otimes_R C \xrightarrow{e \otimes \mathrm{id}_C} C$$

and the identity functor from C to itself, while the isomorphism Ψ_C makes the following diagram of categories to commute:

$$\begin{tikzcd} C \arrow[r, "\phi^2_C"] \arrow[d, "\phi^1_C"'] & P^2_R \otimes_R C \arrow[d, "\Delta \otimes \mathrm{id}_C"] \\ P^1_R \otimes_R C \arrow[r, "\mathrm{id}_P \otimes \phi^1_C"] & P^1_R \otimes_R P^1_R \otimes_R C. \end{tikzcd}$$

Example 4.7. The category $\mathbf{Mod}(R)$ of R-modules has a canonical D_R-structure given by the composition of R-linear tensor functors (see also Example 4.4)

$$\mathbf{Mod}(R) \xrightarrow{- \otimes_R P^2_R} {}_R\mathbf{Mod}\left(P^2_R\right) \cong {}_R\left(P^2_R \otimes_R \mathbf{Mod}(R)\right).$$

Explicitly, for an R-module M, we put

$$\phi^2_R(M) := \left(\left(M \otimes_R P^2_R\right)_R, \alpha\right)$$

in $P^2_R \otimes_R \mathbf{Mod}(R)$, where

$$\alpha : P^2_R \to \mathrm{End}_R\left(M \otimes_R P^2_R\right)$$

is the natural homomorphism. In other words, $\phi^2_R(M)$ is the *Atiyah extension* of M (see also Proposition 4.15 and Remark 4.16 (1)).

Example 4.8. The κ-linear category $\mathbf{Comod}(R, A)$ of comodules over a D_κ-Hopf algebroid (R, A) over a differential ring (κ, D_κ) (Example 3.43(3)) has a canonical D_κ-structure given by the composition of κ-linear tensor functors

$$\begin{aligned} \mathbf{Comod}(R, A) \xrightarrow{- \otimes_\kappa P^2_\kappa} {}_\kappa\mathbf{Comod}\left(R \otimes_\kappa P^2_\kappa, A \otimes_\kappa P^2_\kappa\right) &\cong {}_\kappa\mathbf{Comod}\left(P^2_\kappa \otimes_\kappa R, P^2_\kappa \otimes_\kappa A\right) \\ &\cong {}_\kappa\left(P^2_\kappa \otimes_\kappa \mathbf{Comod}(R, A)\right). \end{aligned}$$

Explicitly, given a comodule M over A, we define an A-comodule structure on $\phi^2_R(M)$ as the composition

$$\phi^2_R(M) \to \phi^2_R(M \otimes_R A) = \left(M \otimes_R A \otimes_R P^2_R\right)_R \cong \left(M \otimes_R P^2_R \otimes_R A\right)_R = \left(\phi^2_R(M) \otimes_R A\right)_R,$$

where the non-trivial isomorphism in the middle is defined as in Lemma 3.45. Thus, the functor ϕ^2_R extends to a D_κ-structure on the category $\mathbf{Comod}(R, A)$. If one does an explicit calculation in the case when (κ, D_κ) is a differential field with one derivation, $R = \kappa$, and A is a D_κ-Hopf algebra over κ, then one recovers the formula from [62, Theorem 1].

We will define differential functors now.

Definition 4.9.

- Let $\varphi : (R, D_R) \to (S, D_S)$ be a morphism of differential rings, C be a D_R-category over R, and $\mathcal{D}$ be a D_S-category over S. A *differential functor* from C to $\mathcal{D}$ is a pair (F, Π_F), where $F : C \to \mathcal{D}$ is a right-exact R-linear tensor functor and

$$\Pi_F : \left(P^2_\varphi \otimes F\right) \circ \phi^2_C \xrightarrow{\sim} \phi^2_{\mathcal{D}} \circ F$$

  is a isomorphism between tensor functors from C to $P^2_S \otimes_S \mathcal{D}$ such that Φ_C commutes with $\Phi_{\mathcal{D}}$ via Π_F and Ψ_C commutes with $\Psi_{\mathcal{D}}$ via Π_F. For short, we usually denote a differential functor (F, Π_F) just by F. We call Π_F a *differential structure* on F.

- A *morphism between differential functors* is a morphism between tensor functors $\Phi : F \to G$ that commutes with the Π's.

Denote the category of differential functors from C to $\mathcal{D}$ by $\mathbf{Fun}^D_R(C, \mathcal{D})$.

In other words, the isomorphism Π_F makes the following diagram of categories commutative:

$$\begin{tikzcd}
C \arrow[r, "F"] \arrow[d, "\phi^2_C"'] & \mathcal{D} \arrow[d, "\phi^2_{\mathcal{D}}"'] \\
P^2_R \otimes_R C \arrow[r, "P^2_\varphi \otimes F"] & P^2_S \otimes_S \mathcal{D}.
\end{tikzcd}$$

Example 4.10.

1. Given a morphism of differential rings $(R, D_R) \to (S, D_S)$, the extension of scalars functor

$$S \otimes_R - : \mathbf{Mod}(R) \to \mathbf{Mod}(S)$$

   is canonically a differential functor.
2. Given a D_κ-Hopf algebroid (R, A) over a differential ring (κ, D_κ), the forgetful functor

$$\mathbf{Comod}(R, A) \to \mathbf{Mod}(R)$$

   is canonically a differential functor, where we consider the D_R-structure on $\mathbf{Mod}(R)$ with $D_R := R \otimes_\kappa D_\kappa$.
3. Given a D_κ-Hopf algebroid (R, A) over a differential ring (κ, D_κ) and a morphism of D_κ-algebras $R \to S$, the extension of scalars functor

$$S \otimes_R - : \mathbf{Comod}(R, A) \to \mathbf{Comod}(S, {}_SA_S)$$

   is canonically a differential functor.

The following statement is needed in the proof of Theorem 5.5.

Lemma 4.11. *Let C, $\mathcal{D}$, and $\mathcal{E}$ be D_R-categories, $F : C \to \mathcal{D}$ be a functor, and $G : \mathcal{D} \to \mathcal{E}$ be a fully faithful differential functor. Then there is a bijection between differential structures on F and $G \circ F$.*

Proof. If F is a differential functor, then $G \circ F$ is also a differential functor, being a composition of differential functors. Conversely, suppose that $G \circ F$ is a differential functor. Consider the diagram of categories:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "\phi^2_{\mathcal{C}}"'] & \mathcal{D} \arrow[r, "G"] \arrow[d, "\phi^2_{\mathcal{D}}"'] & \mathcal{E} \arrow[d, "\phi^2_{\mathcal{E}}"'] \\ P^2_R \otimes_R \mathcal{C} \arrow[r, "\mathrm{id}_P \otimes F"] & P^2_R \otimes_R \mathcal{D} \arrow[r, "\mathrm{id}_P \otimes G"] & P^2_R \otimes_R \mathcal{E}. \end{tikzcd}$$

Since $G \circ F$ is a differential functor, we obtain an isomorphism between tensor functors

$$(\mathrm{id}_{P^2_R} \otimes G) \circ (\mathrm{id}_{P^2_R} \otimes F) \circ \phi^2_{\mathcal{C}} \xrightarrow{\sim} \phi^2_{\mathcal{E}} \circ G \circ F.$$

Further, since G is a differential functor, we obtain an isomorphism between tensor functors

$$\phi^2_{\mathcal{E}} \circ G \circ F \xrightarrow{\sim} (\mathrm{id}_{P^2_R} \otimes G) \circ \phi^2_{\mathcal{D}} \circ F.$$

Taking the composition, we obtain an isomorphism between tensor functors

$$(\mathrm{id}_{P^2_R} \otimes G) \circ (\mathrm{id}_{P^2_R} \otimes F) \circ \phi^2_{\mathcal{C}} \xrightarrow{\sim} (\mathrm{id}_{P^2_R} \otimes G) \circ \phi^2_{\mathcal{D}} \circ F.$$

Since G is fully faithful, the functor $\mathrm{id}_{P^2_R} \otimes G$ is also fully faithful. Therefore, we obtain an isomorphism between tensor functors

$$\Pi_F : (\mathrm{id}_{P^2_R} \otimes F) \circ \phi^2_{\mathcal{C}} \xrightarrow{\sim} \phi^2_{\mathcal{D}} \circ F.$$

It follows that this indeed defines a differential structure on F. □

We also use extensions of scalars for differential categories in the proof of Theorem 5.5.

Proposition 4.12. *Let $\varphi : (R, D_R) \to (S, D_S)$ be a morphism between differential rings, $\mathcal{C}$ be a D_R-category over R, and suppose that the extension of scalars category $S \otimes_R \mathcal{C}$ exists (Definition 4.1).*

1. *There is a canonical D_S-structure on $S \otimes_R \mathcal{C}$ such that the functor*

$$S \otimes_R - : \mathcal{C} \to S \otimes_R \mathcal{C}$$

   is canonically a differential functor.
2. *Let $\mathcal{D}$ be a D_S-category over S. Then taking the composition with $S \otimes_R -$ defines an equivalence of categories:*

$$\mathbf{Fun}^D_S(S \otimes_R \mathcal{C}, \mathcal{D}) \xrightarrow{\sim} \mathbf{Fun}^D_R(\mathcal{C}, \mathcal{D}), \quad F \mapsto F \circ (S \otimes_R -).$$

Proof. To prove statement 1, define the functor

$$\phi^2_{S \otimes_R \mathcal{C}} : S \otimes_R \mathcal{C} \to P^2_S \otimes_S (S \otimes_R \mathcal{C}) \cong P^2_S \otimes_R \mathcal{C}$$

by the universal property of $S \otimes_R \mathcal{C}$ applied to the right-exact R-linear tensor functor

$$\mathcal{C} \xrightarrow{\phi^2_{\mathcal{C}}} P^2_R \otimes_R \mathcal{C} \xrightarrow{P^2_\varphi \otimes \mathrm{id}_{\mathcal{C}}} P^2_S \otimes_R \mathcal{C}.$$

This also defines a differential structure on the functor $S \otimes_R -$. To prove statement 2, one applies the universal property of $S \otimes_R \mathcal{C}$ directly. □

Remark 4.13.

1. Applying Proposition 4.12 to $C = \mathbf{Mod}(R)$, we obtain that the canonical functor
$$S \otimes_R \mathbf{Mod}(R) \to \mathbf{Mod}(S)$$
is a differential functor between D_S-categories (Example 4.7) provided that $S \otimes_R \mathbf{Mod}(R)$ exists.
2. Given D_R-categories C and $\mathcal{D}$ over R and a differential functor $F : C \to \mathcal{D}$, the functor
$$S \otimes_R F : S \otimes_R C \to S \otimes_R \mathcal{D}$$
is canonically a differential functor between D_S-categories over S provided that $S \otimes_R C$ and $S \otimes_R \mathcal{D}$ exist.
3. Both Definition 3.23 and the construction from Proposition 4.12(1) are particular cases of extensions of scalars for differential objects.

4.3. *Definitions in the explicit form*

The following technical result provides an explicit information about objects of type $\phi^2_C(X)$, where X is an object in a D_R-category C. Recall that we have a decreasing filtration by ideals in P^2_R (see Definition 3.30 for I_R)
$$P^2_R \supset I_R \supset \mathrm{Sym}^2_R \Omega_R \supset 0$$
and canonical isomorphisms
$$P^2_R/I_R \cong R, \quad I_R/\mathrm{Sym}^2_R \Omega_R \cong \Omega_R.$$

Lemma 4.14. *Let C be a D_R-category over R. Then, for any object X in C, there are functorial isomorphisms (see Section 4.1 for $\Omega_R \otimes_R X$):*
$$\phi^2_C(X)/I_R\cdot\phi^2_C(X) \cong X, \quad I_R\cdot\phi^2_C(X)/\mathrm{Sym}^2_R \Omega_R\cdot\phi^2_C(X) \cong \Omega_R\otimes_R X, \quad \mathrm{Sym}^2_R \Omega_R\cdot\phi^2_C(X) \cong \mathrm{Sym}^2_R \Omega_R\otimes_R X.$$

Proof. The isomorphism Φ_C yields the first isomorphism, because
$$(e \otimes \mathrm{id}_C)\left(\phi^2_C(X)\right) \cong \phi^2_C(X)/I_R \cdot \phi^2_C(X).$$
Hence, the P^2_R-module structure on $\phi^2_C(X)$ defines surjective morphisms
$$\Omega_R \otimes_R X \xrightarrow{\alpha} I_R \cdot \phi^2_C(X)/\mathrm{Sym}^2_R \Omega_R \cdot \phi^2_C(X), \quad \mathrm{Sym}^2_R \Omega_R \otimes_R X \xrightarrow{\beta} \mathrm{Sym}^2_R \Omega_R \cdot \phi^2_C(X),$$
where we use that
$$I_R \cdot I_R \subset \mathrm{Sym}^2_R \Omega_R \quad \text{and} \quad I_R \cdot \mathrm{Sym}^2_R \Omega_R = 0.$$
Let us prove that α is injective and, thus, is an isomorphism. By the definition of ϕ^1_C (Definition 4.6), we have
$$\phi^1_C(X) \cong \phi^2_C(X)/\mathrm{Sym}^2_R \Omega_R \cdot \phi^2_C(X).$$
Thus, we need to show that the corresponding morphism
$$\Omega_R \otimes_R X \xrightarrow{\gamma} \phi^1_C(X)$$

is injective (see also Example 4.5). By Proposition 4.3, the right-exact R-linear tensor functor

$$\phi^1_C : C \to {}_R\big(P^1_R \otimes_R C\big)$$

defines a right-exact P^1_R-linear tensor functor

$$\epsilon^1_C : C \otimes_R P^1_R \to P^1_R \otimes_R C$$

such that, for any object X in C, we have a functorial isomorphism

$$\phi^1_C(X) \cong \epsilon^1_C\big(X \otimes_R P^1_R\big).$$

Further, the proof of Proposition 3.40 remains valid in the case of a cofibred 2-category instead of a cofibred (1-)category. Thus, ϵ^1_C is an equivalence of categories and, in particular, is exact. On the other hand, since Φ_C is an isomorphism, we have an isomorphism of tensor functors

$$R \otimes_{P^1_R} \epsilon^1_C \cong \mathrm{id}_C$$

from C to itself, where we consider the ring homomorphism $e : P^1_R \to R$. Explicitly, this means that, for a P^1_R-module Y in C such that Ω_R acts trivially on Y, there is a functorial isomorphism $\epsilon^1_C(Y) \cong Y$ (Proposition 4.3). Therefore, applying ϵ^1_C to the injective morphism in $C \otimes_R P^1_R$

$$X \otimes_R \Omega_R \longrightarrow X \otimes_R P^1_R$$

given by the split embedding $\Omega_R \subset P^1_R$ (and using that $X \otimes_R \Omega_R = \Omega_R \otimes_R X$), we show the injectivity of γ. Now let us prove that β is injective and, thus, it is an isomorphism. Consider the object

$$Z := \big(\mathrm{id}_{P^1_R} \otimes \phi^1_C\big)\big(\phi^1_C(X)\big)$$

in $P^1_R \otimes_R P^1_R \otimes_R C$. We have a commutative diagram in C

$$\begin{tikzcd}
\mathrm{Sym}^2_R\,\Omega_R \otimes_R X \arrow[r, "\beta"] \arrow[d, "f"'] & \mathrm{Sym}^2_R\,\Omega_R \cdot \phi^2_C(X) \arrow[d, "g"'] \\
\Omega_R^{\otimes 2} \otimes_R X \arrow[r, "h"] & \Omega_R^{\otimes 2} \cdot Z,
\end{tikzcd}$$

where h is given by the action of $P^1_R \otimes_R P^1_R$ on Z (we use that $X \cong Z/(I_R \cdot Z)$ and $\Omega_R^{\otimes 2} \cdot I_R = 0$), the morphism f is defined by the embedding

$$\mathrm{Sym}^2_R\,\Omega_R \to \Omega_R^{\otimes 2},$$

and the morphism g is induced by the isomorphism Ψ_C. Using the injectivity of γ for X and for $\phi^1_C(X)$, we obtain that h is injective. Since

$$\Omega_R^{\otimes 2} / \mathrm{Sym}^2_R\,\Omega_R \cong \wedge^2_R \Omega_R$$

is a projective R-module, f is also injective, which implies the injectivity of β. $\square$

Now let us give a more explicit (though, a longer) definition of a D_R-category. First, consider only the functor ϕ^1_C. In this case, the situation is similar to the previously known differential Tannakian category over a field with one derivation [63, Definition 3], and [41, Definition 5.2.1]. For simplicity, we assume that the tensor product is *exact* in C.

Proposition 4.15. *Let C be an abelian R-linear tensor category such that the tensor product is exact. Then to define a right-exact R-linear tensor functor*

$$\phi^1_C : C \to {}_R\left(P^1_R \otimes_R C\right)$$

together with an isomorphism between tensor functors

$$\Phi_C : (e \otimes \mathrm{id}_C) \circ \phi^1_C \xrightarrow{\sim} \mathrm{id}_C$$

is the same as to define the following data:

1. *A functor $\mathrm{At}^1_C : C \to C$ together with a functorial exact sequence*
$$0 \longrightarrow \Omega_R \otimes_R X \longrightarrow \mathrm{At}^1_C(X) \longrightarrow X \longrightarrow 0 \tag{12}$$
for any object X in C;
2. *An isomorphism*
$$\mathrm{At}^1_C(\mathbb{1}) \xrightarrow{\sim} P^1_R \otimes_R \mathbb{1},$$
where we consider the right R-module structure on P^1_R, such that exact sequence (12) coincides with the natural exact sequence for $X = \mathbb{1}$:
$$0 \longrightarrow \Omega_R \otimes_R \mathbb{1} \longrightarrow P^1_R \otimes_R \mathbb{1} \longrightarrow \mathbb{1} \longrightarrow 0,$$
and, for any $a \in R \to \mathrm{End}_C(\mathbb{1})$, we have
$$\mathrm{At}^1_C(a) = l(a),$$
where we denote elements of R (respectively, in P^1_R) and their images under the morphisms to $\mathrm{End}(\mathbb{1}_C)$ (respectively, to $\mathrm{End}(P^1_R \otimes_R \mathbb{1})$) in the same way;
3. *A functorial isomorphism with the Baer sum*
$$\mathrm{At}^1_C(X \otimes Y) \xrightarrow{\sim} \left(\mathrm{At}^1_C(X) \otimes Y\right) +_{\mathbf{B}} \left(X \otimes \mathrm{At}^1_C(Y)\right) \tag{13}$$
for all objects X and Y in C that respects commutativity and associativity constraints in C and the splitting of
$$\mathrm{At}^1_C(\mathbb{1}) \cong P^1_R \otimes_R \mathbb{1}$$
given by the canonical right R-linear splitting $P^1_R \cong R \oplus \Omega_R$.

Proof. Given ϕ^1_C, let At^1_C be the composition of ϕ^1_C with the forgetful functor $P^1_R \otimes_R C \to C$ (Proposition 4.3). Then Example 4.5 and Lemma 4.14 (namely, its part that concerns the first two adjoint quotients) imply the needed statement. □

Remark 4.16.

1. The notation At is explained by an analogy with the case $C = \mathbf{Mod}(R)$ (Example 4.7), when the corresponding functor coincides with the Atiyah extension:
$$\mathrm{At}^1_C(M) = \left(M \otimes_R P^1_R\right)_R$$
for an R-module M. In particular, for a D_k-Hopf algebra A or a D_k-Hopf algebroid (R, A) over k, we have that
$$\mathrm{At}^1_C(M) = \left(M \otimes_R P^1_R\right)_R,$$
where $C = \mathbf{Comod}(A)$ or $C = \mathbf{Comod}(R, A)$ (Example 4.8) and M is an A-comodule.
2. To give the functor At_C is the same as to give an object of type $(\Omega_R[1], \alpha)$ in the category of Kähler differentials for the derived category of C as defined in [53].
3. To be strict, we distinguish between a P^1_R-module (Y, α_Y) in C and the corresponding object Y in C, which makes the difference between $\phi^1_C(X)$ and $\mathrm{At}^1_C(X)$.

To define a D_R-category in these terms, let us first discuss several properties of the functor
$$\mathrm{At}^1_C : C \to C\,.$$
It is not tensor and is not R-linear. For any object X in C, $\mathrm{At}^1_C(X)$ is canonically a P^1_R-module in C with respect to the right R-module structure on P^1_R (Example 4.5). For any $a \in R$, we have
$$\mathrm{At}^1_C(a) = a - \mathbf{d}a,$$
where a acts on objects in C, being a scalar from R, and
$$\mathbf{d}a \in \Omega_R \subset P^1_R$$
acts on $\mathrm{At}^1_C(X)$ as the composition
$$\mathrm{At}^1_C(X) \to X \xrightarrow{\mathbf{d}a\otimes \mathrm{id}_X} \Omega_R \otimes_R X \to \mathrm{At}^1_C(X).$$
Further, for any X in C, the object $\mathrm{At}^1_C\left(\mathrm{At}^1_C(X)\right)$ is a $(P^1_R \otimes_R P^1_R)$-module in C. Consider the filtration by ideals:
$$P^1_R \otimes_R P^1_R \supset \left(\Omega_R \otimes_R P^1_R + P^1_R \otimes_R \Omega_R\right) \supset \Omega_R \otimes_R \Omega_R \supset 0. \tag{14}$$
This defines a decreasing filtration on $\mathrm{At}^1_C\left(\mathrm{At}^1_C(X)\right)$. By exact sequence (12), the corresponding adjoint quotients are as follows:
$$X, \quad (\Omega_R \otimes_R X) \oplus (\Omega_R \otimes_R X)\,, \quad \Omega_R \otimes_R \Omega_R \otimes_R X.$$
In addition, the Baer sum isomorphism (13) (or, equivalently, the tensor property of ϕ^1_C) implies that there is a product map
$$m : \mathrm{At}^1_C\left(\mathrm{At}^1_C(X)\right) \otimes \mathrm{At}^1_C\left(\mathrm{At}^1_C(Y)\right) \to \mathrm{At}^1_C\left(\mathrm{At}^1_C(X \otimes Y)\right).$$

We will use the following technical result. By a filtered ring, we mean a ring A together with a decreasing filtration
$$A = F^0A \supset F^1A \supset \ldots \quad \text{such that} \quad F^iA \cdot F^jA \subset F^{i+j}A.$$

Lemma 4.17. *Let A be a finitely filtered ring, $f : M \to N$ be a morphism between A-modules (possibly, between A-modules in an appropriate abelian tensor category). Suppose that*

$$\mathrm{gr}^0 f : \mathrm{gr}^0 M \to \mathrm{gr}^0 N$$

is an isomorphism and, for any i, the canonical morphism

$$\mathrm{gr}^i A \otimes_{\mathrm{gr}^0 A} \mathrm{gr}^0 N \to \mathrm{gr}^i N$$

is an isomorphism. Then f is an isomorphism.

Proof. We have surjective morphisms

$$\mathrm{gr}^i A \otimes_{\mathrm{gr}^0 A} \mathrm{gr}^0 M \to \mathrm{gr}^i M.$$

By the hypotheses of the lemma, their compositions with $\mathrm{gr}^i f$ is an isomorphism. Thus, $\mathrm{gr}^i f$ is an isomorphism. Since A is finitely filtered, we conclude that f is an isomorphism. □

Proposition 4.18. *Let (C, At^1_C) be as in Proposition 4.15. Then to define a D_R-structure on C with ϕ^1_C being given by At^1_C is the same as to define a functorial P^2_R-submodule*

$$\mathrm{At}^2_C(X) \subset \mathrm{At}^1_C\left(\mathrm{At}^1_C(X)\right)$$

such that, for all X and Y in C, the following is satisfied:

1. *The map m sends $\mathrm{At}^2_C(X) \otimes \mathrm{At}^2_C(Y)$ to $\mathrm{At}^2_C(X \otimes Y)$;*
2. *The adjoint quotients of the intersection of $\mathrm{At}^2_C(X)$ with the above filtration on $\mathrm{At}^1_C(\mathrm{At}^1_C(X))$ are contained in*
$$X, \quad \Omega_R \otimes_R X, \quad \mathrm{Sym}^2_R\, \Omega_R \otimes_R X,$$
where we consider the diagonal embedding
$$\Omega_R \otimes_R X \hookrightarrow (\Omega_R \otimes_R X) \oplus (\Omega_R \otimes_R X)$$
and the natural embedding
$$\mathrm{Sym}^2_R\, \Omega_R \otimes_R X \hookrightarrow \Omega_R \otimes_R \Omega_R \otimes_R X;$$
3. *The induced map from $\mathrm{At}^2_C(X)$ to $X = \mathrm{gr}^0\, \mathrm{At}^1_C(\mathrm{At}^1_C(X))$ is surjective.*

Proof. Given a D_R-structure ϕ^2_C, let At^2_C be the composition of ϕ^2_C with the forgetful functor

$$P^2_R \otimes_R C \to C\,.$$

Since ϕ^2_C is a tensor functor and we have an isomorphism of tensor functors Ψ_C, At^2_C satisfies statement 1. Also, by Lemma 4.14, we have statements 2 and 3.

Conversely, let At^2_C satisfy statements 1, 2, and 3. To construct the isomorphism Ψ_C, we need to show that the natural morphism

$$\mu : \left(P^1_R \otimes_R P^1_R\right) \otimes_{P^2_R} \mathrm{At}^2_C(X) \to \mathrm{At}^1_C\left(\mathrm{At}^1_C(X)\right)$$

is an isomorphism. Note that μ is a morphism between $(P^1_R \otimes_R P^1_R)$-modules in C. Consider the filtration on the source and target of μ given by filtration (14). By statements 2 and 3, the natural morphism

$$\mathrm{At}^2_C(X)/I_R \cdot \mathrm{At}^2_C(X) \to X$$

is an isomorphism. Therefore, the first adjoint quotient of the source of μ is isomorphic to X and $\mathrm{gr}^0\mu$ is an isomorphism (being the identity from X to itself). By Lemma 4.17 applied to the finitely filtered ring $(P^1_R \otimes_R P^1_R)$, μ is an isomorphism.

The tensor structure on the functor ϕ^2_C is given by the product map m. The fact that we obtain an isomorphism follows from Lemma 4.17 applied to the finitely filtered ring P^2_R. Finally,

$$\phi^2_C : C \to P^2_R \otimes_R C$$

is R-linear with respect to the left homomorphism $l : R \to P^2_R$, because so is the functor ϕ^1_C, and, hence, $(\mathrm{id}_{P^1_R} \otimes \phi^1_C) \circ \phi^1_C$. □

Definition 4.19. Given an object X in a rigid D_k-category C, let $\langle X\rangle_{\otimes,D}$ denote the minimal full rigid D_k-subcategory in C that contains X and is an closed under taking subquotients. We say that the category $\langle X\rangle_{\otimes,D}$ is D_k-*tensor generated* by the object X.

Remark 4.20. In the notation of Definition 4.19, C is D_k-tensor generated by X if and only if there is no smaller full subcategory in C containing X and closed under taking direct sums, tensor products, duals, subquotients, and applying the functor At^1_C (Section 4.3), because At^2_C is a subobject in $\mathrm{At}^1_C(\mathrm{At}^1_C(X))$. In addition, the category $\langle X\rangle_{\otimes,D}$ is the union of all C_i's, where C_i is the subcategory in C tensor generated by $(\mathrm{At}^1_C)^{\circ i}(X)$.

Remark 4.21.

1. Definition 4.6 is analogous to the definition of a group action on a category (for example, see [14]) so that the isomorphisms Φ and Ψ correspond to the unit and associativity constraints, respectively. We do not require the pentagon condition for Ψ in Definition 4.6 as we are not considering P^3_R (Section 3.7).
   On the contrary, the compatibility condition between Φ and Ψ makes sense in our set-up and means that, for any object X in C, the following compositions coincide:
   $$\mathrm{At}^2_C(X) \to \mathrm{At}^1_C\left(\mathrm{At}^1_C(X)\right) \xrightarrow{\mathrm{At}^1(\pi_X)} \mathrm{At}^1_C(X), \quad \mathrm{At}^2_C(X) \to \mathrm{At}^1_C\left(\mathrm{At}^1_C(X)\right) \xrightarrow{\pi_{\mathrm{At}^1(X)}} \mathrm{At}^1_C(X),$$
   where $\pi_X : \mathrm{At}^1_C(X) \to X$ is the morphism given by exact sequence (12). We do not require this condition in Definition 4.6 as well. However, it holds for Examples 4.7, 4.8 and for the differential category constructed in Theorem 5.1.
2. Suppose that D_R is of rank one over R and the compatibility condition from part 1 holds for a D_R-category C. Then we have
   $$\mathrm{Sym}^2_R\,\Omega_R = \Omega_R \otimes_R \Omega_R$$
   and, by a dimension argument, the embedding
   $$\mathrm{At}^2_C(X) \to \mathrm{At}^1_C\left(\mathrm{At}^1_C(X)\right)$$

identifies $\mathrm{At}^2_{\mathcal{C}}(X)$ with the kernel of the morphism

$$\mathrm{At}^1(\pi_X) - \pi_{\mathrm{At}^1(X)} : \mathrm{At}^1_{\mathcal{C}}\left(\mathrm{At}^1_{\mathcal{C}}(X)\right) \to \mathrm{At}^1_{\mathcal{C}}(X).$$

Therefore, $\mathrm{At}^2_{\mathcal{C}}$ is uniquely defined by $\mathrm{At}^1_{\mathcal{C}}$, or, equivalently, $\phi^1_{\mathcal{C}}$ is uniquely defined up to a canonical isomorphism by $\phi^2_{\mathcal{C}}$.

3. Suppose that $F : \mathcal{C} \to \mathcal{D}$ is a faithful differential functor between D_R-categories and the compatibility condition from part 1 holds for $\mathcal{D}$. Then this condition also holds for $\mathcal{C}$. In particular, if $\mathcal{C}$ is a D_k-Tannakian category (Definition 4.22) over a differential field (k, D_k), then the compatibility condition holds for $\mathcal{C}$ by the end of part 1. If, in addition, $\dim_k(D_k) = 1$, then, by part 2, we see that Definition 4.22 is equivalent to the definitions of a differential Tannakian category over a field with one derivation from [63, Definition 3] and [41, Definition 5.2.1].

Let us discuss the relation between Definition 4.6 and the definition of a neutral differential Tannakian category with several commuting derivations given in [55, Definition 3.1]. Suppose that D_R is a free R-module generated by commuting derivations $\partial_1, \ldots, \partial_d$. Let $\omega_1, \ldots, \omega_d$ be the dual basis in $\Omega_R = D_R^\vee$. There is an involution σ of the $(R\otimes R)$-algebra $P^1_R \otimes_R P^1_R$ uniquely defined by the condition $\sigma(\omega_i \otimes 1) = 1 \otimes \omega_i$ for all i. For example, for any

$$\omega = \sum_i a_i \omega_i \in \Omega_R, \quad a_i \in R,$$

we have

$$\sigma(1 \otimes \omega) = \omega \otimes 1 + \sum_i \omega_i \otimes \mathbf{d}a_i.$$

The subring of invariants under the involution σ coincides with P^2_R, because $\mathbf{d}\omega_i = 0$ for all i. Further, for any i, the morphism of differential rings $(R, D_R) \to (R, R\cdot\partial_i)$ induces the ring homomorphism $P^1_R \to P^1_i$, where P^1_i denotes the 1-jet ring associated with the differential ring $(R, R\cdot\partial_i)$. It follows that σ induces a collection of ring isomorphisms

$$P^1_i \otimes_R P^1_j \cong P^1_j \otimes_R P^1_i$$

that commute with σ via the homomorphisms

$$P^1_R \otimes_R P^1_R \to P^1_i \otimes_R P^1_j.$$

Next, let $\mathcal{C}$ be a D_R-category over R. Then, for any object X in $\mathcal{C}$, the isomorphism

$$\mu : \left(P^1_R \otimes_R P^1_R\right) \otimes_{P^2_R} \mathrm{At}^2_{\mathcal{C}}(X) \xrightarrow{\sim} \mathrm{At}^1_{\mathcal{C}}\left(\mathrm{At}^1_{\mathcal{C}}(X)\right)$$

induces an involution σ_X on $\mathrm{At}^1_{\mathcal{C}}\left(\mathrm{At}^1_{\mathcal{C}}(X)\right)$ such that the invariants of σ_X coincide with $\mathrm{At}^2_{\mathcal{C}}(X)$. For any i, the ring homomorphism $P^1_R \to P^1_i$ induces a morphism

$$\mathrm{At}^1_{\mathcal{C}}(X) \to \mathrm{At}^1_i(X),$$

where we have a functorial exact sequence

$$0 \longrightarrow X \longrightarrow \mathrm{At}^1_i(X) \longrightarrow X \longrightarrow 0.$$

Since the ring homomorphism

$$P^1_R \otimes_R P^1_R \to \bigoplus_{i,j} \left(P^1_i \otimes_R P^1_j\right)$$

is injective, the natural morphism

$$\mathrm{At}^1_C\left(\mathrm{At}^1_C(X)\right) \to \bigoplus_{i,j} \mathrm{At}^1_i\left(\mathrm{At}^1_j(X)\right)$$

is also injective. It follows that to define σ_X it is enough to specify a collection of isomorphisms

$$S_{i,j} : \mathrm{At}^1_i\left(\mathrm{At}^1_j(X)\right) \xrightarrow{\sim} \mathrm{At}^1_j\left(\mathrm{At}^1_i(X)\right)$$

that should satisfy certain compatibility conditions. If, in addition, $R = k$ is a field, C is a neutral Tannakian category, and the fiber functor commutes with At^1 and sends the isomorphisms $S_{i,j}$ to the corresponding isomorphisms in $\mathbf{Vect}(k)$, then $S_{i,j}$ satisfy the compatibility conditions and define correctly At^2_C as the equalizer in $\mathrm{At}^1_C\left(\mathrm{At}^1_C(X)\right)$ of all the isomorphisms $S_{i,j}$. Also, one needs to require the Baer sum isomorphisms for At^1_i to obtain the Baer sum isomorphism for At^1_C, which would preserve At^2_C. The latter coincides with the definition of a neutral differential Tannakian category as given in [55, Definition 3.1].

Finally, let us perform a calculation that we use in Section 4.4. Let (R, D_R) be a differential ring with free D_R. Choose a basis $\partial_1, \ldots, \partial_d$ in D_R over R and let $\omega_1, \ldots, \omega_d$ be the dual basis in Ω_R. Consider free R-modules

$$M = R \cdot e_1 \oplus \ldots \oplus R \cdot e_m \quad \text{and} \quad N = R \cdot f_1 \oplus \ldots \oplus R \cdot f_n$$

and a morphism of R-modules $\phi : M \to N$ given by a matrix T. Then the morphism

$$\mathrm{At}^1_R(\phi) : \mathrm{At}^1_R(M) \to \mathrm{At}^1_R(N)$$

is given by the matrix

$$\begin{pmatrix} T & 0 & \ldots & 0 & 0 \\ -\partial_1(T) & T & \ldots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ -\partial_{d-1}(T) & 0 & \ldots & T & 0 \\ -\partial_d(T) & 0 & \ldots & 0 & T \end{pmatrix},$$

where we consider the basis

$$\left\{e_1 \otimes 1, \ldots, e_m \otimes 1, e_i \otimes \omega_j\right\}, \quad 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant d, \quad \text{in } \mathrm{At}^1_R(M) = \left(M \otimes_R P^1_R\right)_R$$

with respect to the right R-module structure (Remark 4.16(1)) and the analogous basis in $\mathrm{At}^1_R(N)$.

4.4. *Differential Tannakian categories*

Throughout this subsection, we fix a differential field (k, D_k) and use the notions and notation from Section A.2. Let us define differential Tannakian categories.

Definition 4.22.

- A D_k-*Tannakian category* over (k, D_k) (or simply over k) is a D_k-category C over k (Definition 4.6) such that C is rigid, the homomorphism $k \to \mathrm{End}_C(\mathbb{1})$ is an isomorphism, and there exists a D_k-algebra R over k together with a differential functor $\omega : C \to \mathbf{Mod}(R)$ (Definition 4.9).

- Given two differential functors $\omega, \eta : C \to \mathbf{Mod}(R)$, denote the set of isomorphisms between ω and η as differential functors by $\mathrm{Isom}^{\otimes,D}(\omega, \eta)$.

- A *neutral* D_k-*Tannakian category* over k is a D_k-Tannakian category over k with a fixed differential functor to $\mathbf{Vect}(k)$.

Remark 4.23. We use notation from Definition 4.22. Since the category C is rigid and any differential functor is right-exact (Definition 4.9), we see that the functor ω is exact [13, 2.10(i)]. In particular, ω is a fiber functor from C to $\mathbf{Mod}(R)$.

Example 4.24. Let (R, A) be a D_k-Hopf algebroid over k (Example 3.43(3)) such that A is faithfully flat over $R \otimes_k R$. Since the forgetful functor

$$\mathbf{Comod}^{fg}(R, A) \to \mathbf{Mod}(R)$$

is a fiber functor (Section A.1) and differential (Example 4.10(2)), the category $\mathbf{Comod}^{fg}(R, A)$ is a D_k-Tannakian category over k. In particular, if $R = k$ and A is a D_k-Hopf algebra over k (Example 3.43(2)), then the category $\mathbf{Comod}^{fg}(A)$ is a neutral D_k-Tannakian category over k.

Given a D_k-Tannakian category C over k, a differential functor $\omega : C \to \mathbf{Mod}(R)$, and a morphism of D_k-algebras $R \to S$, we put

$$\omega_S : C \to \mathbf{Mod}(S), \quad X \mapsto S \otimes_R \omega(X).$$

Proposition 4.25. *Let R be a D_k-algebra over k, C be a D_k-Tannakian category over k,*

$$\omega, \eta : C \to \mathbf{Mod}(R)$$

be differential functors, and let A be the R-algebra that corepresents the functor (Section A.2):

$$\underline{\mathrm{Isom}}^{\otimes}(\omega, \eta) : \mathbf{Alg}(R) \to \mathbf{Sets}, \quad S \mapsto \mathrm{Isom}^{\otimes}(\omega_S, \eta_S).$$

Then A has a canonical structure of a D_R-algebra over R such that A corepresents the functor

$$\underline{\mathrm{Isom}}^{\otimes,D}(\omega, \eta) : \mathbf{DAlg}(R, D_R) \to \mathbf{Sets}, \quad S \mapsto \mathrm{Isom}^{\otimes,D}(\omega_S, \eta_S).$$

Proof. First let us construct a D_R-structure on A. The idea is as follows. The collection (C, R, ω, η) is a D_k-object in the (2-)category of collections that consist of a Tannakian category over k, a k-algebra, and two fiber functors to modules over this algebra. On the other hand, the (pseudo-)functor that assigns A to such a collection commutes with extensions and restrictions of scalars between k and P^2_k. This defines a D_k-structure on A. Let us give more details. By the definition of A, the P^2_R-algebra $A \otimes_R P^2_R$ corepresents the functor

$$\underline{\mathrm{Isom}}^{\otimes}\left(\omega_{P^2_R}, \eta_{P^2_R}\right) : \mathbf{Alg}\left(P^2_R\right) \to \mathbf{Sets},$$

where, as above,

$$\omega_{P^2_R} : C \to \mathbf{Mod}\left(P^2_R\right), \quad X \mapsto \omega(X) \otimes_R P^2_R, \quad \text{and} \quad \eta_{P^2_R} : C \to \mathbf{Mod}\left(P^2_R\right), \quad X \mapsto \eta(X) \otimes_R P^2_R.$$

The functors $\omega_{P_R^2}$ and $\eta_{P_R^2}$ are exact k-linear tensor functors. Moreover, $\omega_{P_R^2}$ is the composition of the functor

$$- \otimes_k P_k^2 : C \to C \otimes_k P_k^2$$

and the functor

$$\omega \otimes_k P_k^2 : C \otimes_k P_k^2 \to \mathbf{Mod}(R) \otimes_k P_k^2 \cong \mathbf{Mod}\left(P_R^2\right).$$

The analogous relations hold for $\eta_{P_R^2}$ and $\eta \otimes_k P_k^2$. Hence, by Proposition 4.3, there is a canonical isomorphism of functors from $\mathbf{Alg}\,(P_R^2)$ to $\mathbf{Sets}$:

$$\underline{\mathrm{Isom}}^{\otimes}\left(\omega_{P_R^2}, \eta_{P_R^2}\right) \cong \underline{\mathrm{Isom}}^{\otimes}\left(\omega \otimes_k P_k^2, \eta \otimes_k P_k^2\right). \tag{15}$$

Similarly, the P_R^2-algebra $P_R^2 \otimes_R A$ corepresents the functor

$$\underline{\mathrm{Isom}}^{\otimes}\left({}_{P_R^2}\omega, {}_{P_R^2}\eta\right) : \mathbf{Alg}\left(P_R^2\right) \to \mathbf{Sets}$$

and we have an isomorphism of functors

$$\underline{\mathrm{Isom}}^{\otimes}\left({}_{P_R^2}\omega, {}_{P_R^2}\eta\right) \cong \underline{\mathrm{Isom}}^{\otimes}\left(P_k^2 \otimes_k \omega, P_k^2 \otimes_k \eta\right). \tag{16}$$

Again, by Proposition 4.3, the right-exact k-linear tensor functor

$$\phi_C^2 : C \to P_k^2 \otimes_k C$$

defines a right-exact P_k^2-linear tensor functor

$$\epsilon_C^2 : C \otimes_k P_k^2 \to P_k^2 \otimes_k C\,.$$

In addition, the isomorphism Π_ω defines an isomorphism of tensor functors

$$\omega \otimes_k P_k^2 \xrightarrow{\sim} \left(P_k^2 \otimes_k \omega\right) \circ \epsilon_C^2$$

from $C \otimes_k P_k^2$ to $\mathbf{Mod}\left(P_R^2\right)$. Analogously, Π_η defines an isomorphism of tensor functors

$$\eta \otimes_k P_k^2 \xrightarrow{\sim} \left(P_k^2 \otimes_k \eta\right) \circ \epsilon_C^2.$$

This leads to a morphism of functors

$$\underline{\mathrm{Isom}}^{\otimes}\left(P_k^2 \otimes_k \omega, P_k^2 \otimes_k \eta\right) \to \underline{\mathrm{Isom}}^{\otimes}\left(\omega \otimes_k P_k^2, \eta \otimes_k P_k^2\right).$$

Hence, by isomorphisms (15) and (16), we obtain a morphism of functors

$$\Lambda : \underline{\mathrm{Isom}}^{\otimes}\left({}_{P_R^2}\omega, {}_{P_R^2}\eta\right) \to \underline{\mathrm{Isom}}^{\otimes}\left(\omega_{P_R^2}, \eta_{P_R^2}\right).$$

By the corepresentability properties of $A \otimes_R P_R^2$ and $P_R^2 \otimes_R A$, the morphism of functors Λ corresponds to a morphism of P_R^2-algebras

$$\epsilon_A^2 : A \otimes_R P_R^2 \to P_R^2 \otimes_R A.$$

Since the isomorphisms Ψ_C and Φ_C commute with Ψ_R and Φ_R via ω and η, the morphism ϵ_A^2 satisfies the required properties (Example 3.43(1)) to define a D_R-structure on A.

Now let us prove the corepresentability property of A in the category of D_R-algebras. Let S be a D_R-algebra, $\alpha : \omega_S \to \eta_S$ be an isomorphism of tensor functors, and let $f : A \to S$ be the corresponding morphism of R-algebras. We need to show that α is differential if only if f is differential. Note that α is differential if and only if the map

$$\Lambda_S : \mathrm{Isom}^{\otimes}\left({}_{P_S^2}\omega, {}_{P_S^2}\eta\right) \to \mathrm{Isom}^{\otimes}\left(\omega_{P_S^2}, \eta_{P_S^2}\right)$$

sends ${}_{P_S^2}\alpha$ to $\alpha_{P_S^2}$. This is equivalent to the equality between the morphism

$$f \otimes \mathrm{id}_{P_R^2} : A \otimes_R P_R^2 \to S \otimes_R P_R^2$$

and the composition

$$A \otimes_R P_R^2 \xrightarrow{\epsilon_A^2} P_R^2 \otimes_R A \xrightarrow{\mathrm{id}_{P_R^2} \otimes f} P_R^2 \otimes_R S \xrightarrow{(\epsilon_S^2)^{-1}} S \otimes_R P_R^2.$$

The latter is equivalent to f being differential. □

Example 4.26. Let $D_k = k \cdot \partial$, where ∂ is a formal symbol that denotes the trivial derivation from k to itself, K be a differential field over (k, D_k) such that $k = K^{\partial}$, let $C = \mathbf{DMod}(K, D_K)$ with $D_K = K \cdot \partial$, $\omega_0 : C \to \mathbf{Vect}(k)$ be a fiber functor, and let $\omega : C \to \mathbf{Vect}(K)$ be the forgetful functor. Since the left and the right k-module structures on P_k^2 coincide, C has the trivial D_k-structure with

$$\phi_C^2(M) := P_k^2 \otimes_k M \cong M \oplus M$$

for a ∂-module M over K. Since

$$\omega_0\left(P_k^2 \otimes_k M\right) \cong P_k^2 \otimes_k \omega_0(M) \cong \omega_0(M) \otimes_k P_k^2,$$

we see that ω_0 is a differential functor. By Proposition 3.42, for any ∂-module M over K, there is a canonical isomorphism of $(K \otimes K)$-modules

$$M \otimes_K P_K^2 \cong P_K^2 \otimes_K M.$$

Since

$$\left(P_K^2 \otimes_K M\right)_K \cong \left(P_k^2 \otimes_k M\right)_K,$$

we obtain that ω is a differential functor. Let A be the K-algebra that corepresents the functor

$$\underline{\mathrm{Isom}}^{\otimes}\left((K \otimes_k -) \circ \omega_0, \omega\right).$$

Proposition 4.25 provides a ∂-structure on A. This ∂-structure coincides with the one defined in [13, 9.2] (note that the definition of a ∂-structure from [13, 9.2] works well for the whole category $\mathbf{DMod}(K, D_K)$, not just a subcategory tensor generated by one object).

Theorem 4.27. *Let C be a D_k-Tannakian category over a differential field (k, D_k), R be a D_k-algebra over k, and let $\omega : C \to \mathbf{Mod}(R)$ be a differential functor. Then there exists a D_k-Hopf algebroid (R, A) over (k, D_k) such that A is faithfully flat over $R \otimes_k R$ and ω lifts up to an equivalence of D_k-categories over k*

$$C \xrightarrow{\sim} \mathbf{Comod}^{fg}(R, A).$$

Proof. Apply Proposition 4.25 to the differential functors ${}_{R\otimes R}\,\omega$ and $\omega_{R\otimes R}$ from C to $\mathbf{Mod}(R\otimes_k R)$, where, as above, for X in C, we put

$$({}_{R\otimes R}\,\omega)(X) := (R\otimes_k R)\otimes_R \omega(X) \cong R\otimes_k \omega(X), \quad (\omega_{R\otimes R})(X) := \omega(X)\otimes_R (R\otimes_k R) \cong \omega(X)\otimes_k R.$$

This gives a differential algebra A over $R\otimes_k R$, where the differential structure on $R\otimes_k R$ is defined as on the tensor product of D_k-algebras (Remark 3.22). From the properties of the functor from $\mathbf{DAlg}(R\otimes_k R)$ to $\mathbf{Sets}$ corepresented by A, it follows that (R, A) is a D_k-Hopf algebroid over k and ω lifts to a differential functor between D_k-categories

$$C \to \mathbf{Comod}^{fg}(R, A)$$

(Example 4.8). Finally, by [13, 1.12] (Theorem A.14), the latter functor is an equivalence of categories and A is faithfully flat over $R\otimes_k R$. □

In particular, when $R = k$, Theorem 4.27 recovers [64, Theorem 2].

Now let us discuss finiteness properties of the algebra A from Proposition 4.25.

Proposition 4.28. *In the notation of Proposition 4.25, suppose that C is D_k-tensor generated by an object X (Definition 4.19). Then A is D_k-generated over R by the matrix entries of the canonical isomorphism*

$$\omega(X)_A \xrightarrow{\sim} \eta(X)_A$$

and the matrix entries of its inverse with respect to any choice of systems of generators of $\omega(X)_A$ and $\eta(X)_A$ over A.

Proof. This follows from Proposition A.13, Remark 4.20, and the calculation of $\mathrm{At}^1_R(\phi)$ at the end of Section 4.3. □

Corollary 4.29. *Suppose that (k, D_k) is differentially closed,* $\operatorname{char} k = 0$*, and the category C is D_k-tensor generated by one object. Then all differential functors from C to* $\mathbf{Vect}(k)$ *are isomorphic.*

Proof. Let $\omega, \eta : C \to \mathbf{Vect}(k)$ be differential functors. By Proposition 4.25, isomorphisms between ω and η as differential functors are in bijection with morphisms of D_k-algebras $A \to k$. By Proposition 4.28, A is D_k-finitely generated over (k, D_k). By [13, 1.12], A is non-zero, being faithfully flat over k. Since $\operatorname{char} k = 0$, there is a morphism from A to k (for example, see [76, Definition 4] and the references given there), which finishes the proof. □

Finally, let us describe the differential structure on the ring A from Proposition 4.25 explicitly. We use its notation. First, recall an explicit construction of A. Consider the R-module

$$F := \bigoplus_{X\in \mathrm{Ob}(C)} \mathrm{Hom}_R(\omega(X), \eta(X))$$

and the R-submodule T of F generated by all elements of type

$$(\psi\circ\omega(\phi)) \oplus (-\eta(\phi)\circ\psi) \in \mathrm{Hom}_R(\omega(X), \eta(X)) \oplus \mathrm{Hom}_R(\omega(Y), \eta(Y)),$$

where $\phi \in \mathrm{Hom}_C(X, Y)$, $\psi \in \mathrm{Hom}_R(\omega(Y), \eta(X))$, and X, Y are objects in C. Then we have $A = F/T$ ([15]). For each object X in C, choose an R-linear section

$$s_X : \eta(X) \to \mathrm{At}^1_R(\eta(X))$$

of the morphism $\mathrm{At}^1_R(\eta(X)) \to \eta(X)$. By Remark 4.16(1) and Proposition 3.42, s_X corresponds to a, possibly, non-integrable D_R-structure on $\eta(X)$. This uniquely defines an R-linear morphism

$$t_X : \mathrm{At}^1_R(\eta(X)) \to \Omega_R \otimes_R \eta(X)$$

such that the canonical morphism

$$\Omega_R \otimes_R \eta(X) \to \mathrm{At}^1_R(\eta(X))$$

is a section of t_X and $t_X \circ s_X = 0$. Next, for any $\partial \in D_R$, consider the additive map

$$\partial : \mathrm{Hom}_R(\omega(X), \eta(X)) \to \mathrm{Hom}_R\left(\omega\left(\mathrm{At}^1_C(X)\right), \eta\left(\mathrm{At}^1_C(X)\right)\right), \quad \partial(\psi) := s_X \circ \left(\partial \otimes \mathrm{id}_{\eta(X)}\right) \circ t_X \circ \mathrm{At}^1_R(\psi),$$

where $\psi \in \mathrm{Hom}_R(\omega(X), \eta(X))$ and we use the functorial isomorphism

$$\omega\left(\mathrm{At}^1_C(X)\right) \xrightarrow{\sim} \mathrm{At}^1_R(\omega(X)).$$

Taking the direct sum over all objects X in C, we get the additive map $\partial : F \to F$. One can show that ∂ preserves the submodule T and defines a derivation on the R-algebra A. All together, this defines a D_R-structure on A.

5. Parameterized Atiyah extensions

5.1. Construction

Throughout this section, we fix a differential field (k, D_k) and a parameterized differential algebra (R, D_R) over (k, D_k) (Definition 3.14). Recall that we have a differential ring $(R, D_{R/k})$, where $D_{R/k}$ is the kernel of the structure map $D_R \to R \otimes_k D_k$ associated with the morphism of differential rings $(k, D_k) \to (R, D_R)$. Put $\Omega_{R/k} := D^\vee_{R/k}$.

Theorem 5.1. *There is a canonical D_k-structure on the category* $\mathbf{DMod}(R, D_{R/k})$ *such that the forgetful functor from the D_k-category* $\mathbf{DMod}(R, D_{R/k})$ *over (k, D_k) to the D_R-category* $\mathbf{Mod}(R)$ *over (R, D_R) is a differential functor.*

Proof. We follow the explicit approach from Section 4.3. First, we need to construct a right-exact k-linear tensor functor

$$\phi^1 : \mathbf{DMod}(R, D_{R/k}) \to {}_k\!\left(P^1_k \otimes_k \mathbf{DMod}(R, D_{R/k})\right)$$

together with certain isomorphisms between tensor functors. Then we need to functorially construct a P^2_k-submodule $\mathrm{At}^2(M)$ in $\mathrm{At}^1(\mathrm{At}^1(M))$ satisfying several properties.

Recall that we distinguish between a P^1_k-module in $\mathbf{DMod}(R, D_{R/k})$ and the corresponding object in $\mathbf{DMod}(R, D_{R/k})$, which makes the difference between ϕ^1 and At^1 (Remark 4.16(3)). In particular, $\phi^1(\phi^1(M))$ is not well-defined, while $\mathrm{At}^1(\mathrm{At}^1(M))$ is well-defined. We call $\mathrm{At}^2(M)$ a *parameterized Atiyah extension*. The proof is divided into several steps.

Step 1. Construction of $\phi^1(M)$

Let M be a $D_{R/k}$-module. Put

$$\mathrm{At}^1(M) := \left\{ m\otimes 1 + \sum_i m_i \otimes \omega_i \,\middle|\, \forall \xi \in D_{R/k},\ \xi(m) = \sum_i \omega_i(\xi) m_i \right\} \subset M \otimes_R P^1_R, \tag{17}$$

where $m, m_i \in M$, $\omega_i \in \Omega_R$. Here we use that $D_{R/k}$ is an R-submodule in D_R, whence, $\omega_i(\xi)$ is well-defined. Equivalently, $\mathrm{At}^1(M)$ is the kernel of the map

$$\lambda : M\otimes_R P^1_R \to \Omega_{R/k}\otimes_R M, \quad m\otimes a + \sum_i m_i\otimes\omega_i \mapsto a\nabla_M(m) + \mathbf{d}a\otimes m - \sum_i [\omega_i]\otimes m_i, \tag{18}$$

where the brackets mean the application of the natural quotient map $\Omega_R \to \Omega_{R/k}$. The Leibniz rule for ∇_M implies that λ is well-defined. Also, λ is R-linear with respect to the right R-module structure on $M\otimes_R P^1_R$ defined by the homomorphism $r : R \to P^1_R$. Hence, $\mathrm{At}^1(M)$ is an R-submodule in $M\otimes_R P^1_R$ with respect to r. Explicitly, we have

$$a\cdot\left(m\otimes 1 + \sum_i m_i\otimes\omega_i\right) = am\otimes 1 + m\otimes\mathbf{d}a + \sum_i m_i\otimes a\omega_i. \tag{19}$$

Let us define a weak $D_{R/k}$-module structure on $\mathrm{At}^1(M)$ (Section 3.10). Recall that we have a weak $D_{R/k}$-module structure on P^1_R. Hence, we obtain a weak $D_{R/k}$-module structure on the tensor product $M\otimes_R P^1_R$. We claim that the corresponding action of an arbitrary element $\partial\in D_{R/k}$ on $M\otimes_R P^1_R$ preserves $\mathrm{At}^1(M)$. Indeed, for any

$$m\otimes 1 + \sum_i m_i\otimes\omega_i \in \mathrm{At}^1(M),$$

we have

$$\partial\left(m\otimes 1 + \sum_i m_i\otimes\omega_i\right) = \partial(m)\otimes 1 + \sum_i (\partial(m_i)\otimes\omega_i + m_i\otimes L_\partial(\omega_i))$$

(see Definition 3.48 for L_∂). Hence, we need to show that, for any $\xi\in D_{R/k}$, we have

$$\xi(\partial(m)) = \sum_i (\omega_i(\xi)\cdot\partial(m_i) + L_\partial(\omega_i)(\xi)\cdot m_i).$$

By (8), the right-hand side is equal to

$$\sum_i (\omega_i(\xi)\cdot\partial(m_i) + \partial(\omega_i(\xi))\cdot m_i - \omega_i([\partial,\xi])\cdot m_i).$$

Further, by (17), the latter equals

$$\partial(\xi(m)) - [\partial,\xi](m).$$

Thus, we conclude by the integrability condition for the $D_{R/k}$-module structure on M. Let us check that the above weak $D_{R/k}$-module structure actually defines a $D_{R/k}$-module structure. For all

$$a\in R, \quad \partial\in D_{R/k}, \quad m\otimes 1 + \sum_i m_i\otimes\omega_i \in \mathrm{At}^1(M),$$

we have

$$a\cdot\partial\left(m\otimes 1+\sum_i m_i\otimes\omega_i\right)=a\cdot\left(\partial(m)\otimes 1+\sum_i(\partial(m_i)\otimes\omega_i+m_i\otimes L_\partial(\omega_i))\right)=$$
$$=a\partial(m)\otimes 1+\partial(m)\otimes\mathbf{d}a+\sum_i(\partial(m_i)\otimes a\omega_i+m_i\otimes aL_\partial(\omega_i))=$$
$$=a\partial(m)\otimes 1+\sum_i(\omega_i(\partial)m_i\otimes\mathbf{d}a+\partial(m_i)\otimes a\omega_i+m_i\otimes aL_\partial(\omega_i))=$$
$$=a\partial(m)\otimes 1+\sum_i(a\partial(m_i)\otimes\omega_i+m_i\otimes L_{a\partial}(\omega_i))=(a\partial)\left(m\otimes 1+\sum_i m_i\otimes\omega_i\right),$$

where we have used (9) and (19). Thus, we have shown that $\mathrm{At}^1(M)$ is an object in $\mathbf{DMod}\,(R,D_{R/k})$.

Now let us extend $\mathrm{At}^1(M)$ to an object $\phi^1(M)$ in $P^1_k\otimes_k\mathbf{DMod}\,(R,D_{R/k})$, that is, let us define a P^1_k-module structure on $\mathrm{At}^1(M)$ with respect to the right homomorphism $r:k\to P^1_k$. For this, note that $M\otimes_R P^1_R$ is a P^1_R-module. In addition, the multiplication by $P^1_k\subset P^1_R$ preserves $\mathrm{At}^1(M)$: for $k\subset P^1_k$ this follows from the existence of the R-linear structure on $\mathrm{At}^1(M)$, while, for any

$$\eta\in\Omega_k\quad\text{and}\quad m\otimes 1+\sum_i m_i\otimes\omega_i\in\mathrm{At}^1(M),$$

we have

$$\left(m\otimes 1+\sum_i m_i\otimes\omega_i\right)\cdot\eta=m\otimes\eta$$

and $\eta(\xi)=0$ for any $\xi\in D_{R/k}$. Moreover, the multiplication by P^1_k commutes with the $D_{R/k}$-structure on $\mathrm{At}^1(M)$, because the product on P^1_R respects the weak $D_{R/k}$-structure via the Leibniz rule (Section 3.10) and

$$\xi(a+\eta)=0$$

in the above notation. All together, this defines an object $\phi^1(M)$ in $P^1_k\otimes_k\mathbf{DMod}\,(R,D_{R/k})$.

Step 2. The functor $M\mapsto\phi^1(M)$

It follows that $\phi^1(M)$ depends functorially on M. Moreover, the explicit description of $\phi^1(M)$ from (17) implies a functorial exact sequence in $\mathbf{DMod}\,(R,D_{R/k})$:

$$0\to\Omega_k\otimes_k M\to\phi^1(M)\xrightarrow{\pi}M\to 0,\quad \pi\left(m\otimes 1+\sum_i m_i\otimes\omega_i\right)=m.$$

It follows that the functor ϕ^1 is exact. By construction, it is also k-linear with respect to the left homomorphism $l:k\to P^1_k$, because the left R-linear structure on P^1_R is involved in the tensor product $M\otimes_R P^1_R$.

Let us show that the functor ϕ^1 is tensor. Let M and N be $D_{R/k}$-modules. We have a natural isomorphism

$$\left(M\otimes P^1_R\right)\otimes_{P^1_R}\left(N\otimes P^1_R\right)\xrightarrow{\sim}(M\otimes_R N)\otimes_R P^1_R\,.$$

This induces a map

$$\phi^1(M)\otimes_{P^1_k}\phi^1(N)\to(M\otimes_R N)\otimes_R P^1_R.$$

The Leibniz rule for the action of $D_{R/k}$ on $M \otimes_R N$ implies that the image of this map lies in the subset

$$\phi^1(M \otimes_R N) \subset (M \otimes_R N) \otimes_R P^1_R,$$

which defines a morphism of P^1_k-modules

$$m : \phi^1(M) \otimes_{P^1_k} \phi^1(N) \to \phi^1(M \otimes_R N).$$

Our aim is to show that m is an isomorphism. Note that the morphism π from above coincides with taking modulo the ideal $\Omega_k \subset P^1_k$. Also denote taking modulo the ideal in any P^1_k-module by π. Then the morphism m commutes with the identity map from $M \otimes_R N$ to itself via the corresponding morphisms π. By Example 4.5, the kernel of π on

$$\phi^1(M) \otimes_{P^1_k} \phi^1(N)$$

is equal to

$$\Omega_k \otimes_k (M \otimes_R N).$$

It follows that the morphism m induces the identity map from $\Omega_k \otimes_k (M \otimes_R N)$ to itself on the kernels of π. Therefore, m is an isomorphism, which fixes a tensor structure for the functor ϕ^1. Also, we obtain an isomorphism of tensor functors

$$(e \otimes \mathrm{id}) \circ \phi^1 \cong \mathrm{id},$$

where, as above, $e : P^1_k \to k$ is taking modulo Ω_k.

Step 3. Construction of $\mathrm{At}^2(M)$

Put

$$\mathrm{At}^2(M) := \mathrm{At}^1\left(\mathrm{At}^1(M)\right) \cap M \otimes_R P^2_R \subset M \otimes_R P^1_R \otimes_R P^1_R.$$

By Remark 3.49, the subring

$$P^2_R \subset P^1_R \otimes_R P^1_R$$

is preserved under the action of $D_{R/k}$, whence $\mathrm{At}^2(M)$ is a weak $D_{R/k}$-module. Besides, as shown above, $\mathrm{At}^1(\mathrm{At}^1(M))$ is a $D_{R/k}$-module, whence $\mathrm{At}^2(M)$ is also a $D_{R/k}$-module. Since $\mathrm{At}^1(\mathrm{At}^1(M))$ is preserved under the right multiplication by $P^1_k \otimes_k P^1_k$, we obtain that $\mathrm{At}^2(M)$ is preserved under the right multiplication by

$$P^2_k \subset \left(P^1_k \otimes_k P^1_k\right) \cap P^2_R.$$

Since multiplication by $P^1_k \otimes_k P^1_k$ on $\mathrm{At}^1(\mathrm{At}^1(M))$ commutes with the $D_{R/k}$-structure, multiplication by P^2_k commutes with the $D_{R/k}$-structure on $\mathrm{At}^2(M)$. Thus, we see that $\mathrm{At}^2(M)$ is a P^2_k-submodule in $\mathrm{At}^1(\mathrm{At}^1(M))$ in the category $\mathbf{DMod}(R, D_{R/k})$.

It follows that $\mathrm{At}^2(M)$ depends functorially on M. By Step 2, the tensor structure on $\mathrm{At}^1 \circ \mathrm{At}^1$ is induced by the isomorphism

$$\left(M \otimes_R P^1_R \otimes_R P^1_R\right) \otimes_{(P^1_R \otimes_R P^1_R)} \left(N \otimes_R P^1_R \otimes_R P^1_R\right) \xrightarrow{\sim} (M \otimes_R N) \otimes_R \left(P^1_R \otimes_R P^1_R\right).$$

Since P^2_R is a subring in $P^1_R \otimes_R P^1_R$, we see that the product map

$$\mathrm{At}^1\left(\mathrm{At}^1(M)\right) \otimes \mathrm{At}^1\left(\mathrm{At}^1(N)\right) \to \mathrm{At}^1\left(\mathrm{At}^1(M \otimes_R N)\right)$$

preserves At^2.

Consider the filtration by ideals:

$$P_k^1 \otimes_k P_k^1 \supset \left(\Omega_k \otimes_k P_k^1 + P_k^1 \otimes_k \Omega_k\right) \supset \Omega_k \otimes_k \Omega_k \supset 0.$$

This defines a decreasing filtration on $\mathrm{At}^1\left(\mathrm{At}^1(M)\right)$ with the following adjoint quotients (see Section 4.3 for more computational details):

$$M, \quad (\Omega_k \otimes_k M) \oplus (\Omega_k \otimes_k M), \quad \Omega_k \otimes_k \Omega_k \otimes_k M.$$

Consider the intersection of this filtration with $\mathrm{At}^2(M)$. Since $\mathrm{At}^2(M)$ is contained in $M \otimes_R P_R^2$, the corresponding adjoint quotients are contained in

$$M, \quad \Omega_k \otimes_k M, \quad \mathrm{Sym}_k^2 \Omega_k \otimes_k M.$$

Hence, by Proposition 4.18, $\mathbf{DMod}(R, D_{R/k})$ with the functor At^2 is a D_k-category, provided that the induced map

$$\mathrm{At}^2(M) \to M = \mathrm{gr}^0 \mathrm{At}^1\left(\mathrm{At}^1(M)\right)$$

is surjective.

Step 4. Surjectivity of $\mathrm{At}^2(M) \to M$

Take any

$$m \in M \quad \text{and} \quad m \otimes 1 + \sum_i m_i \otimes \omega_i \in \mathrm{At}^1(M).$$

First, let us prove that there exists $x \in M \otimes_R \Omega_R \otimes_R \Omega_R$ such that the image of x under the map

$$M \otimes_R \Omega_R \otimes_R \Omega_R \to M \otimes_R \wedge_R^2 \Omega_R$$

is equal to

$$y := \sum_i m_i \otimes \mathbf{d}\omega_i$$

and the image of x under the map

$$M \otimes_R \Omega_R \otimes_R \Omega_R \to M \otimes_R \Omega_{R/k} \otimes_R \Omega_R$$

is equal to

$$z := -\sum_i \nabla(m_i) \otimes \omega_i,$$

where we apply the isomorphism

$$\Omega_{R/k} \otimes_R M \cong M \otimes_R \Omega_{R/k}.$$

For short,

$$A := \Omega_R \otimes_R \Omega_R, \quad B := \mathrm{Ker}\left(\Omega_R \otimes_R \Omega_R \to \wedge_R^2 \Omega_R\right), \quad C := \mathrm{Ker}\left(\Omega_R \otimes_R \Omega_R \to \Omega_{R/k} \otimes_R \Omega_R\right).$$

We have the following exact sequence

$$A \to (A/B) \oplus (A/C) \to A/(B + C) \to 0,$$

where the first map is given by the diagonal embedding and the second arrow is induced by taking the difference. Since the R-modules

$$A = \Omega_R \otimes_R \Omega_R, \quad A/B \cong \wedge^2_R \Omega_R, \quad A/C \cong \Omega_{R/k} \otimes_R \Omega_R, \quad \text{and} \quad A/(B+C) \cong \wedge^2_R \Omega_{R/k}$$

are projective and, henceforth, flat, we obtain the exact sequence

$$M \otimes_R \Omega_R \otimes \Omega_R \to \left(M \otimes_R \wedge^2_R \Omega_R\right) \oplus (M \otimes_R \Omega_{R/k} \otimes_R \Omega_R) \to M \otimes_R \wedge^2_R \Omega_{R/k} \to 0.$$

The integrability condition on M implies that $y \oplus z$ is in the kernel of the rightmost non-zero map (note that we have switched the tensor factors M and $\Omega_{R/k}$ unlike in Definition 3.19, whence there is a sign change). Hence, by the exactness in the middle, there exists x with the required properties.

Now let us show that the element

$$n := m \otimes 1 \otimes 1 + \sum_i m_i \otimes \omega_i \otimes 1 + \sum_i m_i \otimes 1 \otimes \omega_i - x \in M \otimes_R P^1_R \otimes_R P^1_R$$

belongs to $\mathrm{At}^2(M)$. Since x is sent to y, we see that n belongs to $M \otimes_R P^2_R$. By the hypotheses,

$$m \otimes 1 \otimes 1 + \sum_i m_i \otimes \omega_i \otimes 1 \in \mathrm{At}^1(M) \otimes 1 \subset \mathrm{At}^1(M) \otimes_R P^1_R.$$

Since x is sent to z, we see that the map

$$\lambda \otimes \mathrm{id}_{P^1_R} : M \otimes_R P^1_R \otimes_R P^1_R \to \Omega_{R/k} \otimes_R M \otimes_R P^1_R$$

sends $\sum_i m_i \otimes 1 \otimes \omega_i - x$ to zero (recall that λ is defined in (18)). Since P^1_R is a projective and, therefore, flat R-module, we conclude that

$$\sum_i m_i \otimes 1 \otimes \omega_i - x \in \mathrm{At}^1(M) \otimes_R \Omega_R.$$

Therefore,

$$n \in \mathrm{At}^1(M) \otimes_R P^1_R.$$

It remains to check that

$$n \in \mathrm{At}^1\left(\mathrm{At}^1(M)\right).$$

For this, we need to show that, for any $\xi \in D_{R/k}$, we have

$$\xi\left(m \otimes 1 + \sum_i m_i \otimes \omega_i\right) = \sum_i \omega_i(\xi) \cdot (m_i \otimes 1) - x(- \otimes \xi) \in \mathrm{At}^1(M),$$

where

$$x(- \otimes \xi) \in M \otimes_R \Omega_R \cong \mathrm{Hom}_R(D_R, M)$$

sends any $\partial \in D_R$ to $x(\partial \otimes \xi) \in M$. By the explicit formula for the $D_{R/k}$-module structure on $\mathrm{At}^1(M)$ given in Step 1, the left-hand side is equal to

$$\xi(m) \otimes 1 + \sum_i \xi(m_i) \otimes \omega_i + \sum_i m_i \otimes L_\xi(\omega_i).$$

By the explicit formula (19) for the R-module structure on $\mathrm{At}^1(M)$ also given in Step 1, the right-hand side is equal to

$$\sum_i \omega_i(\xi)m_i \otimes 1 + \sum_i m_i \otimes \mathbf{d}(\omega_i(\xi)) - x(- \otimes \xi).$$

Since

$$m \otimes 1 + \sum_i m_i \otimes \omega_i \in \mathrm{At}^1(M),$$

we have that

$$\xi(m) \otimes 1 = \sum_i \omega_i(\xi)m_i \otimes 1.$$

Further, by the definition of the Lie derivative, we have

$$\sum_i m_i \otimes L_\xi(\omega_i) = \sum_i m_i \otimes \mathbf{d}(\omega_i(\xi)) + \sum_i m_i \otimes (\mathbf{d}\omega_i)(\xi \wedge -).$$

Since x is sent to y, we have that

$$\sum_i m_i \otimes (\mathbf{d}\omega_i)(\xi \wedge -) = x(\xi \otimes -) - x(- \otimes \xi).$$

Finally, since x is sent to z, we have that

$$x(\xi \otimes -) = -\sum_i \xi(m_i) \otimes \omega_i,$$

which shows the required equality.

Step 5. The forgetful functor $\mathbf{DMod}\,(R, D_{R/k}) \to \mathbf{Mod}(R)$

It remains to show that the forgetful functor $\mathbf{DMod}\,(R, D_{R/k}) \to \mathbf{Mod}(R)$ is differential. By Definition 4.9 and (10) from Section 4.1, it is enough to show that the canonical morphism of P^2_R-modules

$$\mathrm{At}^2(M) \otimes_{(P^2_k \otimes_k R)} P^2_R \to M \otimes_R P^2_R$$

is an isomorphism. This follows directly from Lemma 4.17 applied to the filtered ring P^2_R. □

Remark 5.2.

1. If $(R, D_R) = (k, D_k)$, then we have $\mathbf{DMod}\,(R, D_{R/k}) = \mathbf{Vect}(k)$. It follows from the construction in Step 1 in the proof of Theorem 5.1 that the D_k-structure on $\mathbf{DMod}\,(R, D_{R/k})$ given by Theorem 5.1 coincides with the usual D_k-structure on $\mathbf{Vect}(k)$.
2. There is a motivating example for the construction of a $D_{R/k}$-structure on $\mathbf{DMod}(R, D_{R/k})$. Let M be a D_R-module over R and put $N := M^{D_{R/k}}$ (Definition 3.19). Note that N is a k-vector subspace in M. Moreover, there is a D_k-module structure on N over k defined as follows. For $\partial \in D_k$, consider any lift $\tilde{\partial} \in D_R$ of $1 \otimes \partial$ with respect to the structure map $D_R \to R \otimes_k D_k$. Then, for any $n \in N$, put

$$\partial(n) := \tilde{\partial}(n).$$

   In Theorem 5.1, M is replaced by the category $\mathbf{Mod}(R)$ and, correspondingly, N is replaced by $\mathbf{DMod}(R, D_{R/k})$. It seems that both constructions can be generalized for a wider class of D_R-objects or categories instead of M or $\mathbf{Mod}(R)$.

3. In [5, 1.6.3], one finds an alternative definition of the $D_{R/k}$-module structure on $\mathrm{At}^1(M)$ in terms of lifts of the $D_{R/k}$-structure on M to, possibly, non-integrable D_R-structures on M. The construction from op.cit. is given for families of varieties but it applies in the setting of parameterized differential algebras as well. However, the approach to $\mathrm{At}^1(M)$ from Step 1 of the proof of Theorem 5.1 seems to be more convenient to show that one, thus, obtains a D_k-structure on $\mathbf{DMod}(R, D_{R/k})$.

In Section 5.3, we use the following result.

Lemma 5.3. *Given a morphism $(R, D_R) \to (S, D_S)$ of parameterized differential algebras over (k, D_k), the extension of scalars functor (Definition 3.23)*

$$S \otimes_R - : \mathbf{DMod}\,(R, D_{R/k}) \to \mathbf{DMod}\,(S, D_{S/k})$$

is canonically a differential functor between D_k-categories over (k, D_k).

Proof. For a $D_{R/k}$-module M, consider the morphism

$$M \otimes_R P^1_R \to M_S \otimes_S P^1_S = M \otimes_R P^1_S.$$

It follows that this morphism sends $\mathrm{At}^1(M)$ to $\mathrm{At}^1(M_S)$. Hence, the morphism

$$M \otimes_R P^2_R \to M_S \otimes_S P^2_S = M \otimes_R P^2_S$$

sends $\mathrm{At}^2(M)$ to $\mathrm{At}^2(M_S)$. Thus, we obtain a morphism of P^2_S-modules

$$\mathrm{At}^2(M) \otimes_{(P^2_R \otimes_R S)} P^2_S \to \mathrm{At}^2(M_S).$$

By Lemma 4.17 applied to the filtered ring P^2_S, this is an isomorphism. □

5.2. *Matrix description*

Let us describe the differential structure on $\mathrm{At}^1(M)$ in the case of a parameterized field explicitly. In the particular case when D_k is one-dimensional, this will coincide with the prolongation functor from [64, Section 5]. Let (K, D_K) be a parameterized differential field over (k, D_k). Let $\partial_{t,1}, \ldots, \partial_{t,q}$ be a basis of D_k over k, and let

$$\partial_{x,1}, \ldots, \partial_{x,p}, \tilde{\partial}_{t,1}, \ldots, \tilde{\partial}_{t,q}$$

be a basis of D_K over K such that $\tilde{\partial}_{t,i}$ are sent to $1 \otimes \partial_{t,i}$ under the structure map $D_K \to K \otimes_k D_k$. Let $\omega_{t,1}, \ldots, \omega_{t,q}$ be the dual basis in Ω_k to $\partial_{t,1}, \ldots, \partial_{t,q}$, and let

$$\widetilde{\omega}_{x,1}, \ldots, \widetilde{\omega}_{x,p}, \omega_{t,1}, \ldots, \omega_{t,q}$$

be the dual basis in Ω_K to $\partial_{x,1}, \ldots, \partial_{x,p}, \tilde{\partial}_{t,1}, \ldots, \tilde{\partial}_{t,q}$. Thus, we have $\widetilde{\omega}_{x,i}(\tilde{\partial}_{t,j}) = 0$.

Let M be a finite-dimensional $D_{K/k}$-module over K and $\{e_1, \ldots, e_m\}$ be a basis of M over K. For $\partial \in D_{K/k}$, let $A_\partial \in \mathrm{Mat}_{m\times m}(K)$ be the connection matrix on M [67, Section 1.2], that is, we have

$$\partial(\overline{e}) = -\overline{e} \cdot A_\partial,$$

where $\overline{e} := (e_1, \ldots, e_m)$. Put $A_i := A_{\partial_{x,i}}$, $1 \leqslant i \leqslant p$. Then we obtain the following basis for $\mathrm{At}^1(M)$:

$$\{f_1, \ldots, f_m, e_i \otimes \omega_{t,j}\}, \quad 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant q, \quad (f_1, \ldots, f_m) = \overline{f}, \quad \overline{f} := \overline{e} \otimes 1 - \sum_{i=1}^{p} \overline{e} \cdot A_i \otimes \widetilde{\omega}_{x,i}.$$

Proposition 5.4. *In the above basis for* $\mathrm{At}^1(M)$*, the connection matrix for* $\partial \in D_{K/k}$ *is equal to*

$$\begin{pmatrix} A_\partial & 0 & \dots & 0 & 0 \\ B_1 & A_\partial & \dots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ B_{q-1} & 0 & \dots & A_\partial & 0 \\ B_q & 0 & \dots & 0 & A_\partial \end{pmatrix}, \quad B_i := -\tilde{\partial}_{t,i}(A_\partial) - A_{[\partial,\tilde{\partial}_{t,i}]}, \quad 1 \leqslant i \leqslant q.$$

Proof. We use the construction of the differential structure on $\mathrm{At}^1(M)$ as given in Step 1 of the proof of Theorem 5.1. By definition, we have

$$\partial(e_i \otimes \omega_{t,j}) = \partial(e_i) \otimes \omega_{t,j}$$

and

$$\partial\big(\overline{f}\big) = -\overline{e} \cdot A_\partial \otimes 1 - \sum_{i=1}^{p} \overline{e} \cdot \partial(A_i) \otimes \widetilde{\omega}_{x,i} + \sum_{i=1}^{p} \overline{e} \cdot A_i A_\partial \otimes \widetilde{\omega}_{x,i} - \sum_{i=1}^{p} \overline{e} \cdot A_i \otimes L_\partial(\widetilde{\omega}_{x,i}).$$

On the other hand, by the definition of K-linear structure (19) on $\mathrm{At}^1(M)$, we have

$$\overline{f} \cdot A_\partial = \overline{e} \cdot A_\partial \otimes 1 + \overline{e} \otimes \mathbf{d}A_\partial - \sum_{i=1}^{p} \overline{e} \cdot A_i \otimes A_\partial \widetilde{\omega}_{x,i}.$$

Since the action of ∂ is well-defined on $\mathrm{At}^1(M)$, the sum

$$\partial\big(\overline{f}\big) + \overline{f} \cdot A_\partial = -\sum_{i=1}^{p} \overline{e} \cdot \partial(A_i) \otimes \widetilde{\omega}_{x,i} - \sum_{i=1}^{p} \overline{e} \cdot A_i \otimes L_\partial(\widetilde{\omega}_{x,i}) + \overline{e} \otimes \mathbf{d}A_\partial$$

belongs to $M \otimes_k \Omega_k$ and, hence, it is uniquely determined by its values at all $\tilde{\partial}_{t,j}$. Evaluating this explicitly and using that $\widetilde{\omega}_{x,i}(\tilde{\partial}_{t,j}) = 0$, we obtain the needed result. □

5.3. *PPV extensions and differential functors*

The following statement is a parameterized version of [13, 9.6] (see also Proposition 2.4).

Theorem 5.5. *Let* (K, D_K) *be a parameterized differential field over a differential field* (k, D_k)*,* $\operatorname{char} k = 0$*,* M *be a finite-dimensional* $D_{K/k}$*-module over* K*. Then there is an equivalence of categories*

$$\Phi : \mathbf{PPV}(M) \xrightarrow{\sim} \mathbf{Fun}_k^D(C, \mathbf{Vect}(k)),$$

where $C := \langle M \rangle_{\otimes,D}$ *is the full subcategory in* $\mathbf{DMod}(K, D_{K/k})$ D_k*-tensor generated by* M *(Definition 4.19), where the* D_k*-structure on* $\mathbf{DMod}(K, D_{K/k})$ *is as in Theorem 5.1.*

Proof. First, let us construct the functor Φ. Let L be a PPV extension for M. By construction, the solution space functor

$$\omega_0 : C \to \mathbf{Vect}(k), \quad X \mapsto X_L^{D_{L/k}}$$

is k-linear. By definition of a PPV extension, there is a canonical isomorphism

$$L \otimes_k \omega_0(X) \xrightarrow{\sim} X_L \tag{20}$$

in $\mathbf{DMod}(L, D_{L/k})$. Therefore, the functor ω_0 is exact and tensor. Let us show that ω_0 is a differential functor between D_k-categories over k. By Remark 5.2(1) and Lemma 5.3 applied to the morphism $(k, D_k) \to (L, D_L)$,

$$L \otimes_k - : \mathbf{Vect}(k) \to \mathbf{DMod}\,(L, D_{L/k})$$

is a differential functor. Since the functor $L \otimes_k -$ is also fully faithful, by Lemma 4.11, it is enough to prove that the composition

$$(L \otimes_k -) \circ \omega_0 : C \to \mathbf{DMod}\,(L, D_{L/k})$$

is a differential functor. By isomorphism (20), this composition is isomorphic to the extension of scalars functor

$$L \otimes_K - : C \to \mathbf{DMod}\,(L, D_{L/k}).$$

By Lemma 5.3, the $L \otimes_K -$ is a differential functor, which implies that ω_0 is a differential functor. We put $\Phi(L) := \omega_0$. One checks that Φ extends to a functor.

Now let us construct a quasi-inverse functor Ψ to Φ. Let $\omega_0 : C \to \mathbf{Vect}(k)$ be a differential functor. Consider the forgetful functor $\omega : C \to \mathbf{Vect}(K)$. By Theorem 5.1, ω is a differential functor. By Theorem 4.2, there exists the extension of scalars $K \otimes_k C$. By Proposition 4.12(1), $K \otimes_k C$ has a canonical D_K-structure and, by Proposition 4.12(2), the functor ω corresponds to a differential functor

$$\eta : K \otimes_k C \to \mathbf{Vect}(K)$$

between D_K-categories over K. By Remark 4.13(2), we also have a differential functor

$$K \otimes_k \omega_0 : K \otimes_k C \to K \otimes_k \mathbf{Vect}(k) = \mathbf{Vect}(K)$$

between D_K-categories over K. By Proposition 4.25, the functor

$$\underline{\mathrm{Isom}}^{\otimes, D}(K \otimes_k \omega_0, \eta) : \mathbf{DAlg}(K, D_K) \to \mathbf{Sets}$$

is corepresented by a D_K-algebra A over K. We will show that A is a domain and $L := \mathrm{Frac}(A)$ is a PPV extension for M. For this, we use analogous results from [13, 9]. By Proposition 4.25, A as a K-algebra corepresents the functor

$$\underline{\mathrm{Isom}}^{\otimes}(K \otimes_k \omega_0, \eta) : \mathbf{Alg}(K) \to \mathbf{Sets}.$$

By Definition 4.1, there is an equivalence of categories

$$\mathbf{Fun}^{r,\otimes}_K(K \otimes_k C, \mathbf{Vect}(K)) \xrightarrow{\sim} \mathbf{Fun}^{r,\otimes}_k(C, \mathbf{Vect}(K)),$$

which sends $K \otimes_k \omega_0$ to $(K \otimes_k -) \circ \omega_0$ and sends η to ω by the construction of η. Therefore, A corepresents the functor

$$\underline{\mathrm{Isom}}^{\otimes}((K \otimes_k -) \circ \omega_0, \omega) : \mathbf{Alg}(K) \to \mathbf{Sets}. \tag{21}$$

Let C_i be the full subcategory in C tensor generated by $(\,\mathrm{At}^1\,)^{\circ i}(M)$ and let A_i be the K-algebra that corepresents the functor

$$\underline{\mathrm{Isom}}^{\otimes}\,((K \otimes_k -) \circ \omega_0|_{C_i}, \omega|_{C_i}).$$

By Remark 4.20, the category C is a union of all C_i's, whence we have that

$$A = \varinjlim_i A_i.$$

Each ring A_i is a particular example of a ring considered in [13, 9.2], where it is denoted by $\Gamma(P, O)$. A $D_{K/k}$-structure on A_i is defined in [13, 9.2]. Moreover, all morphisms

$$A_i \to A_j, \quad i \leqslant j,$$

are morphisms of $D_{K/k}$-algebras over k, which defines a $D_{K/k}$-structure on A. By Example 4.26, this $D_{K/k}$-structure coincides with the one obtained from the D_K-structure on A. Thus, it follows from [13, 9.3] that A is a domain and the field $L := \mathrm{Frac}(A)$, being a D_K-field over K, has no new $D_{K/k}$-constants. Since A corepresents functor (21), the embedding $A \hookrightarrow L$ induces an isomorphism

$$L \otimes_k \omega_0(M) \cong M_L.$$

It follows from [13, 9.6] that this isomorphism identifies $1 \otimes \omega_0(M)$ with $M_L^{D_{L/k}}$. Thus, we have an isomorphism

$$L \otimes_k M_L^{D_{L/k}} \to M_L.$$

Hence, by Proposition 4.28, L is D_K-generated by the coordinates of horizontal vectors in a basis of M over K, whence L is a PPV extension. We put $\Psi(\omega_0) := L$. One checks that Ψ extends to a functor. The proof of the fact that Φ and Ψ are quasi-inverses of each other is the same as the proof of [13, Proposition 9.5]. $\square$

Remark 5.6. It follows from the proof of Theorem 5.5 and Proposition 4.28 that A as above is equal to the PPV ring associated with L (Definition 3.28). Moreover, by the construction of A, for any D_k-algebra R, there is a canonical isomorphism

$$\mathrm{Aut}^{D_K}(R \otimes_k A / R \otimes_k K) \cong \mathrm{Isom}^{\otimes, D}(\omega_R, \omega_R).$$

6. Definability of differential Hopf algebroids

6.1. Reduction to faithful flatness

The goal of this section is to prove Theorem 6.1. This technical result is needed for the proof of Theorem 2.5.

Theorem 6.1. *Let (K, H) be a D_k-Hopf algebroid (Example 3.43(3)) over a differential field (k, D_k) with K being a field and* $\mathrm{char}\, k = 0$*. Suppose that H is a D_k-finitely generated (Definition 3.12) faithfully flat algebra over $K \otimes_k K$. Then there exist a D_k-finitely generated subalgebra R in K over k and a D_k-Hopf algebroid (R, A) over k such that A is a D_k-finitely generated faithfully flat algebra over $R \otimes_k R$ and there is an isomorphism of D_k-Hopf algebroids over k*

$$(K, {}_K A_K) \cong (K, H).$$

The following statement is not used in the paper, but we include it for its own interest.

Corollary 6.2. *Let C be a D_k-Tannakian category over a differentially closed field (k, D_k) with* $\mathrm{char}\, k = 0$*. Suppose that C is D_k-tensor generated by one object. Then there exists a differential (fiber) functor $C \to$* **Vect**(k)*.*

Proof. There is a D_k-morphism from any D_k-algebra over k to a D_k-field over k. Thus, it follows from Definition 4.22 that there is a differential functor $C \to \mathbf{Vect}(K)$ for a D_k-field K over k. Combining Theorem 4.27, Proposition 4.28, Theorem 6.1, Section A.1, and Example 4.10(3), we obtain a differential functor $C \to \mathbf{Mod}(R)$, where R is a D_k-finitely generated D_k-algebra over k. Since $\operatorname{char} k = 0$, there is a morphism from R to k (for example, see [76, Definition 4] and the references given there), which finishes the proof. □

The proof of Theorem 6.1 uses the following statements.

Lemma 6.3. *Let B be a D_k-finitely generated D_k-Hopf algebra over a differential field (k, D_k) with $\operatorname{char} k = 0$. Then B is of D_k-finite presentation over k (Definition 3.13).*

Proof. By [8, Proposition 12], B is a quotient of the Hopf algebra of the differential algebraic group GL_n, that is, we have a surjective morphism of D_k-Hopf algebras

$$C := k\{T_{ij}\}[1/\det] \xrightarrow{\varphi} B.$$

Since C is of D_k-finite presentation, it is enough to prove that the kernel I of φ is D_k-finitely generated. Let $C_n \subset C$ be the subring generated over k by all derivatives of T_{ij} of order at most n with respect to D_k. Put $J_n := I \cap C_n$. Then J_n is a finitely generated Hopf ideal [79, Section 2.1] in the finitely generated Hopf algebra C_n over k, because the comultiplication $\Delta : C \to C \otimes_k C$ is a D_k-morphism.

Let I_n be the D_k-ideal in C generated by J_n. Again, since Δ is a D_k-morphism, I_n is a D_k-finitely generated Hopf ideal in the Hopf algebra C over k. Therefore, I_n is radical [79, Theorem 11.4]. Since $I = \bigcup_n I_n$, by [42, Theorem 7.1], $I = I_n$ for some n, whence I is D_k-finitely generated. □

Lemma 6.4. *Let (K, H) be a D_k-Hopf algebroid over a differential field (k, D_k) with K being a field and $\operatorname{char} k = 0$. Suppose that H is a D_k-finitely generated faithfully flat algebra over $K \otimes_k K$. Then H is of D_k-finite presentation over $K \otimes_k K$.*

Proof. Since H is D_k-finitely generated over $K \otimes_k K$, we have that

$$B := K \otimes_{K \otimes K} H$$

is a D_k-finitely generated D_k-Hopf algebra over K. Therefore, B is of D_k-finite presentation over K by Lemma 6.3. Since $\operatorname{Spec}(H)$ is a D_k-pseudo-torsor under the group scheme $\operatorname{Spec}(B \otimes_k K)$ over $K \otimes_k K$ (Section A.2), we have an isomorphism of D_k-algebras over H:

$$B \otimes_K H \cong H \otimes_{K \otimes K} H.$$

Hence, $H \otimes_{K \otimes K} H$ is of D_k-finite presentation over H. By the hypotheses of the lemma, H is faithfully flat over $K \otimes_k K$. The same argument as in the non-differential case (for example, see [28, Proposition 2.7.1(vi)]) implies that H is of D_k-finite presentation over $K \otimes_k K$. □

Proposition 6.5. *Let (R, A) be a D_k-Hopf algebroid over a differential field (k, D_k) with R being a domain and $\operatorname{char} k = 0$. Suppose that R and A are D_k-finitely generated over k and $A_F \neq 0$, where F is the total fraction ring of $R \otimes_k R$. Then there exists a non-zero element $f \in R$ such that the localization ${}_fA_f$ is faithfully flat over the localization $R_f \otimes_k R_f$.*

Proof of Theorem 6.1. By Lemma 6.4, H is of D_k-finite presentation over $K \otimes_k K$. A standard argument implies that there is a D_k-finitely generated subalgebra R in K over k and a D_k-Hopf algebroid (R, A) over k such that A is of D_k-finite presentation over $R \otimes_k R$ and there is an isomorphism of D_k-Hopf algebroids

$$(K, {}_K A_K) \cong (K, H)$$

over k. Since H is faithfully flat over $K \otimes_k K$, we have $A_F \neq 0$. Hence, by Proposition 6.5, localizing R by a non-zero element, we obtain that A is faithfully flat over $R \otimes_k R$. □

Remark 6.6. Proposition 6.5 is implied by the following hypothetical statement: given a morphism $S \to A$ between D_k-finitely generated algebras over k, suppose that there is a multiplicative set $\Sigma \subset S$ such that the localization $\Sigma^{-1}A$ is faithfully flat over $\Sigma^{-1}S$; then there is $g \in \Sigma$ such that A_g is faithfully flat over S_g. The validity of this statement seems to be not clear, while its non-differential version is well-known (for example, see [29, 8.10.5(vi), 11.2.6.1(ii)]). Proposition 6.5 provides the partial case of the above statement in the remark in which A comes from the differential Hopf algebroid (R, A) and $S = R \otimes_k R$.

The rest of the section is on the proof of Proposition 6.5, which we actually prove in Section 6.3.

6.2. *Auxiliary results*

The following is a modification of [75, Proposition 5]. The authors are grateful to D. Trushin for his suggestion to use this result.

Lemma 6.7. *Let A be a D_S-finitely generated algebra over a differential ring (S, D_S) and let $a_1, \ldots, a_p$ be its generators. Suppose that A is a domain. Consider the following (non-differential) S-subalgebras in A:*

$$A_n := S\left[(\partial_1 \cdot \ldots \cdot \partial_m)(a_i) \mid \partial_j \in D_S,\ m \le n,\ 1 \leqslant i \leqslant p\right],\ n \in \mathbb{N}.$$

Then there exist a natural number N and a non-zero element $g \in A_N$ such that, for any $n \geq N$, there is an isomorphism

$$(A_{n+1})_g \cong (A_n)_g[T_1, \ldots, T_{l_n}]$$

of algebras over the localization $(A_n)_g$, where the T_l's are formal variables.

Proof. Replacing S by its image under the homomorphism $S \to A$, we may assume that this homomorphism is injective and S is a domain. Let $\mathfrak{p}$ be the kernel of the surjective morphism of D_S-algebras over S

$$\varphi : B \to A, \quad y_i \mapsto a_i,$$

where $B := S\{y_1, \ldots, y_p\}$ (Definition 3.12). Then $\mathfrak{p}$ is a prime D_S-ideal. For a natural n, put

$$B_n := S\left[(\partial_1 \cdot \ldots \cdot \partial_m)(y_i) \mid \partial_j \in D_S,\ m \le n,\ 1 \leqslant i \leqslant p\right] \subset B.$$

Then we have

$$A_n \cong B_n/(\mathfrak{p} \cap B_n).$$

Since D_S is a finitely generated projective S-module, localizing S by a non-zero element, assume that D_S is now a finitely generated free S-module. Let

$$D_S = S \cdot \delta_1 \oplus \ldots \oplus S \cdot \delta_d.$$

Then we have

$$[\delta_i, \delta_j] = \sum_{q=1}^{d} c_{ij}^q \delta_q, \quad c_{ij}^q \in S,\ 1 \leqslant i, j \leqslant d,$$

which is exactly the situation considered in [34].

For every D_S-polynomial $f \in B \setminus S$, we define its leader, separant, and initial as in [34, Section 3.2]. More precisely, put

$$\Theta := \{\mathrm{id}\} \cup \left\{\delta_{i_1} \cdot \ldots \cdot \delta_{i_m} \mid 1 \leqslant i_j \leqslant d,\ m \geqslant 1\right\}, \quad M := \{\,\theta y_j \mid \theta \in \Theta,\ 1 \leqslant j \leqslant p\,\} \subset B,$$

and let the order of

$$\delta_{i_1} \cdot \ldots \cdot \delta_{i_m} y_j \in M$$

be m. Thus, B is the ring of polynomials in elements of M. Consider an orderly differential ranking on M [34, Definition 3.3], for example, the ranking that first compares the orders of two elements in M and then compares lexicographically the y_j's and δ_i's. If $u_f \in M$ is the leader of f with respect to this ranking on M and

$$f = I_r u_f^r + \ldots + I_0,$$

where I_i, $0 \leqslant i \leqslant r$, do not contain u_f, then the separant is $S_f := \partial f/\partial u_f$ and the initial I_f is I_r. Let $\Sigma \subset \mathfrak{p}$ be a characteristic set of $\mathfrak{p}$ with respect to our ranking, [34, Section 6.3], and put

$$W := \{u_f \mid f \in \Sigma\},\ \ Z := \left\{\theta u_f \mid f \in \Sigma,\ \theta \in \Theta,\ \theta \neq \mathrm{id}\right\},\ \ X := M \setminus (Z \cup W),\ \text{ and }\ \tilde{g} := \prod_{f \in \Sigma} I_f S_f.$$

Note that $\tilde{g} \notin \mathfrak{p}$, because the differential ideal $\mathfrak{p}$ is prime. By [34, Section 6.1], for every $f \in \mathfrak{p}$, there exists $q \geqslant 0$ such that

$$\tilde{g}^q \cdot f = \sum_i h_i \cdot (\theta_i f_i)^{n_i}, \tag{22}$$

for some $h_i \in B$, $\theta_i \in \Theta$, $f_i \in \Sigma$, and $n_i > 0$, where the polynomials h_i's are free of the elements of Z. Let $N \in \mathbb{N}$ be such that $B_N \supset W$, that is, N is the maximal order of the elements of W. Since the ranking is orderly, this implies that $\Sigma \subset B_N$. Further, I_f and S_f belong to B_N for any $f \in \Sigma$, because they are differential polynomials of order not exceeding N. Hence, $\tilde{g} \in B_N$. Put

$$g := \varphi(\tilde{g}).$$

We have that $g \neq 0$, because $\tilde{g} \notin \mathfrak{p}$ as shown above. Since, again, the ranking is orderly, the localization A_g is generated by $\varphi(W)$, $\varphi(X)$, and $1/g$ over S. Moreover, if

$$f \in S[W \cup X] \subset B$$

is such that $\varphi(f) = 0$ in A_g, then (22) implies that there exists $q \geqslant 0$ such that

$$\tilde{g}^q f \in (\Sigma),$$

the (non-differential) ideal generated by Σ. Therefore,

$$A_g \cong S[W \cup X]_g/(\Sigma)$$

as S-algebras. Thus, for every $n \geqslant N$, we have that $(A_{n+1})_g$ is a polynomial ring over $(A_n)_g$. Precisely, we have

$$(A_{n+1})_g = (A_n)_g[T],$$

where $T := (\varphi(X) \cap A_{n+1}) \setminus A_n$. $\square$

We use the following notation and conventions in our geometric constructions. Given morphisms of schemes $\varphi : Y \to X$ and $\pi : Z \to X$, denote the fibred product $Y \times_X Z$ by $\varphi^* Z$ and the projection to Y by $\varphi^*\pi : \varphi^* Z \to Y$. Thus, there is a Cartesian square of schemes

$$\begin{array}{ccc} \varphi^* Z & \longrightarrow & Z \\ {\scriptstyle \varphi^*\pi}\big\downarrow & & {\scriptstyle \pi}\big\downarrow \\ Y & \xrightarrow{\varphi} & X. \end{array}$$

The morphism $\varphi^*\pi$ is usually called a base change of π by the morphism φ. The notation $\varphi^* Z$ is correct, provided that π is the only considered morphism from Z to X.

Given a morphism of schemes $\varphi : Y \to X$ and an open or closed subscheme $U \subset Y$, denote the restriction of the morphism φ to U by $\varphi|_U$. Given an open or closed subscheme $W \subset X$, denote the restriction of Y to W, that is, the preimage $\varphi^{-1}(W)$, by Y_W, and denote the morphism $\varphi|_{Y_W}$ by

$$\varphi_W : Y_W \to W.$$

In particular, if x is a point in X, then Y_x denotes the fiber of φ over x considered as a scheme over the residue field $k(x)$ at x.

Given a scheme X, denote the projection to the i-th factor by

$$p_i : X \times X \to X, \quad i = 1,\ 2.$$

Denote the projection to the product of the i-th and j-th factors by

$$p_{ij} : X \times X \times X \to X \times X, \quad 1 \leqslant i < j \leqslant 3.$$

Given a scheme X and a field F, denote the set of F-points of X by $X(F)$. That is, an element in $X(F)$ is a morphism of schemes $\mathrm{Spec}(F) \to X$.

Recall that a morphism is faithfully flat if and only if it is both flat and surjective. A base change $\varphi^*\pi$ of a (faithfully) flat morphism π by any morphism φ is (faithfully) flat. Further, if a composition of morphisms of schemes

$$W \xrightarrow{\lambda} Z \xrightarrow{\pi} X,$$

is (faithfully) flat with λ being faithfully flat, then π is (faithfully) flat. Also, we will use the following fact.

Lemma 6.8. *Consider a Cartesian square of schemes*

$$\begin{array}{ccc} \varphi^* Z & \xrightarrow{\pi^*\varphi} & Z \\ {\scriptstyle \varphi^*\pi}\big\downarrow & & {\scriptstyle \pi}\big\downarrow \\ Y & \xrightarrow{\varphi} & X. \end{array}$$

Suppose that φ is faithfully flat and there is an open subset $W \subset \varphi^* Z$ such that the morphism $(\varphi^*\pi)|_W : W \to Y$ is (faithfully) flat and the morphism $(\pi^*\varphi)|_W : W \to Z$ is surjective. Then π is (faithfully) flat. In particular, if $\varphi^*\pi$ is (faithfully) flat, then π is (faithfully) flat.

Proof. The morphism $\pi^*\varphi : \varphi^*Z \to Z$ is faithfully flat, being the base change of the faithfully flat morphism φ by the morphism π. Therefore, the morphism

$$(\pi^*\varphi)|_W : W \to Z$$

is also faithfully flat, being both flat and surjective. On the other hand, the composition $\pi\circ(\pi^*\varphi)|_W$ is (faithfully) flat, because it is equal to the composition of (faithfully) flat morphisms $\varphi\circ(\varphi^*\pi)|_W$. Therefore, the morphism π is (faithfully) flat. $\square$

Definition 6.9. Let $G \to X$ be a group scheme over a scheme X. Suppose that we are given an action of G on a scheme $T \to X$ over X, that is, a morphism $a : G \times_X T \to T$ that satisfies the group action condition. We say that T is a *pseudo-torsor under* G if the morphism

$$(a, \mathrm{pr}_T) : G \times_X T \to T \times_X T$$

is an isomorphism, where pr_T is the projection to T.

Lemma 6.10. *Let $\rho : G \to X$ be a group scheme over a scheme X, and $\pi : T \to X$ be a pseudo-torsor under G over X. Suppose that there exists an open subset $V \subset T$ such that the restriction $\pi|_V : V \to X$ is faithfully flat and the fibers of the morphism $\pi|_V : V \to X$ are dense in the fibers of the morphism $\pi : T \to X$. Then the morphisms ρ and π are faithfully flat.*

Proof. The morphism ρ is surjective because of the existence of the unit section and the morphism π is surjective, because the morphism $\pi|_V$ is faithfully flat and, in particular, surjective. Hence, one needs to show the flatness of ρ and π.

With this aim, we construct a faithfully flat morphism $\varphi : Y \to X$ that satisfies the following two conditions. The first condition is that there is an open subset $W \subset \varphi^*G$ such that the morphism $(\varphi^*\rho)|_W : W \to Y$ is flat and the morphism $(\rho^*\varphi)|_W : W \to G$ is surjective. By Lemma 6.8, this implies that ρ is flat. In particular, $\varphi^*\rho$ is flat. The second condition on $\varphi : Y \to X$ is that there is an isomorphism

$$\varphi^*G \cong \varphi^*T$$

of schemes over Y, thus, $\varphi^*\pi$ is also flat. Again by Lemma 6.8, this implies that π is flat, which gives the needed result.

Now let us construct the required morphism $\varphi : Y \to X$. We claim that

$$Y := V, \quad \varphi := \pi|_V,$$

satisfies all conditions above. Indeed, by the hypotheses of the lemma, φ is faithfully flat. Further, since T is a pseudo-torsor under G, there is an isomorphism

$$G \times_X T \xrightarrow{\sim} T \times_X T$$

that commutes with the right projection to T. After the restriction to the open subset $V \subset T$, we obtain an isomorphism

$$\psi : G \times_X V \xrightarrow{\sim} T \times_X V$$

of schemes over V. In the other notation, ψ is an isomorphism $\varphi^*G \cong \varphi^*T$ of schemes over Y. Further, consider the open subset

$$V \times_X V \subset T \times_X V$$

and put
$$W := \psi^{-1}(V \times_X V) \subset G \times_X V = \varphi^* G.$$
The (right) projection $V \times_X V \to V$ is flat, being the base change of the flat morphism $\pi|_V : V \to X$ by itself. Since ψ is an isomorphism, the projection $W \to V$ is also flat, that is, we obtain the flatness of the morphism
$$(\varphi^*\rho)|_W : W \to Y.$$
It remains to prove that the morphism
$$(\rho^*\varphi)|_W : W \to G$$
is surjective. Take a point $g \in G$. We need to show that the fiber W_g is non-empty. Let F denote the residue field at g and put $x := \rho(g)$ to be the corresponding F-point of X. The point $g \in G_x(F)$ defines an automorphism of the scheme T_x over F, which we denote by the same letter g. By the construction of W, we have the equality
$$W_g = V_x \cap g^{-1} V_x \subset T_x.$$
By the hypotheses of the lemma, V_x is a dense open subset in T_x, whence the latter intersection is non-empty. $\square$

Recall that an affine groupoid Γ acting on an affine scheme X over κ is a pair (X, Γ), where $\Gamma = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(R)$, and the pair (R, A) is a Hopf algebroid. It follows from the definition of a Hopf algebroid that one has a morphism $\pi : \Gamma \to X \times X$ and a morphism of schemes over $X \times X \times X$
$$m : p_{12}^*\Gamma \times_{(X^{\times 3})} p_{23}^*\Gamma \to p_{13}^*\Gamma.$$
Moreover, the morphism
$$(m, \mathrm{pr}) : p_{12}^*\Gamma \times_{(X^{\times 3})} p_{23}^*\Gamma \to p_{13}^*\Gamma \times_{(X^{\times 3})} p_{23}^*\Gamma$$
is an isomorphism, where
$$\mathrm{pr} : p_{12}^*\Gamma \times_{(X^{\times 3})} p_{23}^*\Gamma \to p_{23}^*\Gamma$$
is the projection. Consider the restriction Γ_Δ of Γ to the diagonal $\Delta \subset X \times X$, that is, we have $\Gamma_\Delta = \pi^{-1}(\Delta)$. The morphism
$$\pi_\Delta : \Gamma_\Delta \to \Delta \cong X$$
defines a group scheme over X. Take the base change of the latter morphism by the projection $p_1 : X \times X \to X$ and obtain the group scheme over $X \times X$
$$\rho : G \to X \times X, \quad \rho := p_1^*(\pi_\Delta), \quad G := p_1^*(\Gamma_\Delta).$$
It follows that $\pi : \Gamma \to X \times X$ is a pseudo-torsor under G over $X \times X$. We will need only affine groupoids acting on affine schemes, so, one may suppose this in the following.

Lemma 6.11. *Let $\pi : \Gamma \to X \times X$ be a groupoid acting on a scheme X. Suppose that there are open subsets $U \subset X \times X$ and $V \subset \Gamma$ such that for any $i = 1, 2$, the fibers of the projection $p_i|_U : U \to X$ are dense in the fibers of the projection $p_i : X \times X \to X$, the image $\pi(V)$ is contained in U, the morphism $\pi|_V : V \to U$ is faithfully flat, and the fibers of the morphism $\pi|_V : V \to U$ are dense in the fibers of the morphism $\pi_U : \Gamma_U \to U$. Then the morphism $\pi : \Gamma \to X \times X$ is faithfully flat.*

Proof. The idea of the proof is to construct a faithfully flat morphism $\varphi : Y \to X \times X$ such that the base change

$$\varphi^*\pi : \varphi^*\Gamma \to Y$$

is faithfully flat and to conclude by Lemma 6.8. We are going to define the morphism φ as a composition of two faithfully flat morphisms. First, consider the open subset

$$W := U \times_X U = (U \times X) \cap (X \times U) \subset X \times X \times X.$$

Since the open embedding $W \hookrightarrow X \times X \times X$ and the projection $p_{13} : X^{\times 3} \to X^{\times 2}$ are both flat, their composition

$$p_{13}|_W : W \to X \times X$$

is flat as well. Let us show that the morphism $p_{13}|_W$ is surjective. Take a point z on $X \times X$. We need to show that the fiber W_z is non-empty. Let F denote the residue field at z and put $x_i := p_i(z)$ to be the corresponding F-points in X. By the construction of W, we have the equality

$$W_z = {}_{x_1}U \cap U_{x_2} \subset X_k,$$

where

$${}_{x_1}U := p_1^{-1}(x) \cap U, \quad U_{x_2} := p_2^{-1}(x) \cap U, \quad \text{and} \quad X_F := X \times \mathrm{Spec}(F).$$

By the hypotheses of the lemma, the open subsets ${}_{x_1}U$ and U_{x_2} are dense in X_F, whence their intersection is non-empty. We conclude that the morphism

$$p_{13}|_W : W \to X \times X$$

is surjective, whence it is faithfully flat. Secondly, consider the morphism

$$p_{23}^*\pi : p_{23}^*\Gamma \to X \times X \times X$$

and put

$$Y := \left(p_{23}^*\Gamma\right)_W.$$

Let us show that the morphism $\left(p_{23}^*\pi\right)_W : Y \to W$ is faithfully flat. There is a Cartesian square

$$\begin{tikzcd} p_{23}^*\Gamma \arrow[r] \arrow[d, "p_{23}^*\pi"'] & \Gamma \arrow[d, "\pi"'] \\ X \times X \times X \arrow[r, "p_{23}"] & X \times X. \end{tikzcd}$$

Changing $X \times X$ by the open subset U, we obtain a Cartesian square

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "(p_{23}^*\pi)_W"'] & \Gamma_U \arrow[d, "\pi_U"'] \\ W \arrow[r, "p_{23}|_W"] & U. \end{tikzcd}$$

Hence, it is enough to show that the morphism $\pi_U : \Gamma_U \to U$ is faithfully flat. With this aim, consider the group scheme

$$\rho : G = p_1^*(\Gamma_\Delta) \to X \times X$$

as in the discussion before the lemma. Take the restrictions $\Gamma_U = \pi^{-1}(U)$ and $G_U = \rho^{-1}(U)$. Note that the group scheme $\rho_U : G_U \to U$ over U, the pseudo-torsor $\pi_U : \Gamma_U \to U$ under G_U, and the open subset $V \subset \Gamma_U$ satisfy the hypotheses of Lemma 6.10. Therefore, the morphism $\pi_U : \Gamma_U \to U$ is faithfully flat, whence the morphism

$$\left(p_{23}^*\pi\right)_W : Y \to W$$

is faithfully flat as explained above. Put

$$\varphi := p_{13}|_W \circ \left(p_{23}^*\pi\right)_W : Y \to X \times X.$$

The morphism φ is faithfully flat, being a composition of faithfully flat morphisms.

Now let us prove that the morphism $\varphi^*\pi : \varphi^*\Gamma \to Y$ is faithfully flat. For this, we use another equivalent constructions of the morphism $\varphi^*\pi$. Consider the diagram of Cartesian squares

$$\begin{tikzcd}
p_{23}^*\Gamma \times_{(X^{\times 3})} p_{13}^*\Gamma \arrow[r] \arrow[d] & p_{13}^*\Gamma \arrow[r] \arrow[d, "p_{13}^*\pi"'] & \Gamma \arrow[d, "\pi"'] \\
p_{23}^*\Gamma \arrow[r, "p_{23}^*\pi"] & X \times X \times X \arrow[r, "p_{13}"] & X \times X.
\end{tikzcd}$$

This gives the diagram of Cartesian squares

$$\begin{tikzcd}
\left(p_{23}^*\Gamma\right)_W \times_W \left(p_{13}^*\Gamma\right)_W \arrow[r] \arrow[d] & \left(p_{13}^*\Gamma\right)_W \arrow[r] \arrow[d, "(p_{13}^*\pi)_W"'] & \Gamma \arrow[d, "\pi"'] \\
Y = \left(p_{23}^*\Gamma\right)_W \arrow[r, "(p_{23}^*\pi)_W"] & W \arrow[r, "p_{13}|_W"] & X \times X.
\end{tikzcd}$$

Since

$$\varphi^*\Gamma = \left(p_{23}^*\pi\right)_W^*(p_{13}|_W)^*\Gamma,$$

we obtain that

$$\varphi^*\Gamma = \left(p_{13}^*\Gamma\right)_W \times_W \left(p_{23}^*\Gamma\right)_W$$

and the morphism in question $\varphi^*\pi : \varphi^*\Gamma \to Y$ coincides with the projection

$$\mathrm{pr} : \left(p_{13}^*\Gamma\right)_W \times_W \left(p_{23}^*\Gamma\right)_W \to \left(p_{23}^*\Gamma\right)_W.$$

So, we are reduced to show the faithful flatness of the morphism pr.

Since (X, Γ) is a groupoid, there is an isomorphism

$$p_{12}^*\Gamma \times_{(X^{\times 3})} p_{23}^*\Gamma \xrightarrow{\sim} p_{13}^*\Gamma \times_{(X^{\times 3})} p_{23}^*\Gamma$$

of schemes over $p_{23}^*\Gamma$ (see the discussion before the lemma). Thus, there is an isomorphism

$$(p_{12}^*\Gamma)_W \times_W \left(p_{23}^*\Gamma\right)_W \xrightarrow{\sim} \left(p_{13}^*\Gamma\right)_W \times_W \left(p_{23}^*\Gamma\right)_W$$

of schemes over $(p_{23}^*\Gamma)_W$. This shows that faithful flatness of the morphism pr is equivalent to the faithful flatness of the projection

$$\mathrm{pr}' : (p_{12}^*\Gamma)_W \times_W \left(p_{23}^*\Gamma\right)_W \to \left(p_{23}^*\Gamma\right)_W.$$

Finally, the morphism pr' is the base change of the faithfully flat morphism $\pi_U : \Gamma_U \to U$ by the composition

$$\left(p_{23}^*\Gamma\right)_W \xrightarrow{(p_{23}^*\pi)_W} W \xrightarrow{p_{12}|_W} U.$$

Therefore, the morphism pr' is faithfully flat, which finishes the proof. $\square$

Lemma 6.12. *Let $\psi : G' \to G$ be a morphism between group schemes of finite type over a scheme X, let $\pi : T \to X$ be a pseudo-torsor under G, $\pi' : T' \to X$ be a pseudo-torsor under G', and let $\varphi : T' \to T$ be a morphism compatible with ψ in the following sense: the diagram*

$$\begin{array}{ccc} G' \times_X T' & \longrightarrow & T' \\ {\scriptstyle \psi\times\varphi}\big\downarrow & & {\scriptstyle \varphi}\big\downarrow \\ G \times_X T & \longrightarrow & T \end{array}$$

commutes. Let $V \subset T$ be an open subset and put $V' := \varphi^{-1}(V)$. Suppose that the fibers of the morphism $\pi|_V : V \to X$ are dense in the fibers of the morphism $\pi : T \to X$ and the morphism $\varphi|_{V'} : V' \to V$ is surjective. Then the fibers of the morphism

$$\pi'|_{V'} : V' \to X$$

are dense in the fibers of the morphism $\pi' : T' \to X$.

Proof. First, we reduce the lemma to a question about algebraic groups. Since the needed result is fiber-wise and all data in the lemma are stable under a base change, we may assume that $X = \mathrm{Spec}(F)$, where F is a field. Further, it is enough to show the density after the extension of scalars to the algebraic closure of F. Thus, we assume that F is algebraically closed. Taking an F-point t' on T' and the point $t := \varphi(t')$ on T, we obtain isomorphisms $G' \xrightarrow{\sim} T'$ and $G \xrightarrow{\sim} T$ that send ψ to φ.

Therefore, we may assume that $T' = G'$ and $T = G$. Finally, we may assume that the schemes G and G' are reduced. Summarizing, we have a morphism of algebraic groups $\psi : G' \to G$ and an open dense subset $V \subset G$ such that the morphism $\psi|_{V'} : V' \to V$ is surjective, where $V' = \psi^{-1}(V)$. We need to show that V' is dense in G'.

The image of the morphism ψ is a closed subgroup in G (for example, see [73, Proposition 2.2.5]). On the other hand, this image contains the dense subset V, because the morphism

$$\psi|_{V'} : V' \to V$$

is surjective. Consequently, the morphism ψ is surjective. It follows that all irreducible components of the fibers of ψ have the same dimension $d := \dim(G') - \dim(G)$.

Since $V \subset G$ is a dense open subset and all irreducible components of G have the same dimension $\dim(G)$, we see that all irreducible components of the closed subset $Z := G\backslash V \subset G$ have dimension strictly less than $\dim(G)$. Therefore, all irreducible components of the closed subset $\psi^{-1}(Z) \subset G'$ have dimension strictly less than $d + \dim(G) = \dim(G')$. Since

$$V' = G'\backslash\psi^{-1}(Z)$$

and all irreducible components of G' have the same dimension $\dim(G')$, we conclude that V' is dense in G', which finishes the proof. $\square$

6.3. Proof of Proposition 6.5

We are now ready to give a proof of Proposition 6.5. We use the geometric notation from Section 6.2.

Proof of Proposition 6.5. We will localize the ring R over a finite set of non-zero elements and then prove that the corresponding localization of A is faithfully flat over the obtained localization of $R \otimes_k R$.

Let $\{a_i\}$ be a finite set of D_k-generators of A over $R \otimes_k R$ and put A_0 to be the $(R \otimes_k R)$-subalgebra in A generated by the set $\{a_i\}$. Since $L := \mathrm{Frac}(R)$ is a field, by [13, 3.7, 3.8] (see also [79, §3.3]), the images of a_i's in ${}_LA_L$ are contained in a Hopf subalgebroid of $(L, {}_LA_L)$ finitely generated over $L \otimes_k L$. Therefore, localizing R by a non-zero element and enlarging the finite subset $\{a_i\} \subset A$, we obtain that (R, A_0) is a Hopf subalgebroid in (R, A).

For each natural n, put A_n to be the $(R \otimes_k R)$-subalgebra in A generated by all elements of the form

$$(\partial_1 \cdot \ldots \cdot \partial_m)(a_i), \quad \partial_j \in D_k,\ m \leqslant n.$$

Since (R, A) is a differential Hopf algebroid, it follows that (R, A_n) is a Hopf subalgebroid in (R, A) for all n. Put

$$X := \mathrm{Spec}(R), \quad \Gamma := \mathrm{Spec}(A), \quad \Gamma_n := \mathrm{Spec}(A_n).$$

Denote the groupoid morphisms by $\pi_n : \Gamma_n \to X \times X$.

Since A is D_k-finitely generated over k, we see that Γ has finitely many irreducible components [42, Theorem 7.5]. Applying Lemma 6.7 to each irreducible component of Γ, we see that there exist a natural number N and an affine dense open subset $W_N \subset \Gamma_N$ such that for any $n \geq N$, the morphisms

$$\varphi_n|_{W_n} : W_n \to W_N$$

are faithfully flat, where $W_n := \varphi_n^{-1}(W_N)$ and $\varphi_n : \Gamma_n \to \Gamma_N$ are the morphisms that arise in the projective system formed by Γ_n.

Since $\mathrm{char}\, k = 0$ and R is a domain, the ring $R \otimes_k R$ is reduced. Since the morphism

$$\pi_N|_{W_N} : W_N \to X \times_k X$$

is of finite type, by the generic flatness (for example, see [39, Proposition 7.91.7]), there is a dense open subset $U \subset X \times_k X$ such that the morphism $\pi_N|_{V_N} : V_N \to U$ is flat and of finite presentation, where

$$V_N := W_N \cap \pi_N^{-1}(U).$$

As $A_F \neq 0$, we may also assume that $\pi_N|_{V_N}$ is faithfully flat. It follows that the morphisms

$$\pi_n|_{V_n} : V_n \to U$$

are faithfully flat, where

$$V_n := \varphi_n^{-1}(V_N), \quad n \geqslant N.$$

By [29, 9.5.3], replacing U with a dense open subset, we obtain that the fibers of the morphism

$$\pi_N|_{V_N} : V_N \to U$$

are dense in the fibers of the morphism

$$(\pi_N)_U : (\Gamma_N)_U \to U,$$

because W_N is dense in Γ_n. Since $R \otimes_k R$ has finitely many irreducible components [42, Theorem 7.5], we may assume that U is an affine dense open subset in $X \times_k X$. Localizing R by a non-zero element, we obtain that, for any $i = 1, 2$, the fibers of the projections $p_i|_U : U \to X$ are dense in the fibers of the projection

$$p_i : X \times_k X \to X$$

(by the extension of scalars, this follows from the analogous statement about irreducible varieties over fields). For each n, put

$$G_n := p_1^*((\Gamma_n)_\Delta),$$

where $\Delta \subset X \times_k X$ is the diagonal. Then Γ_n is a pseudo-torsor under the group scheme G_n over $X \times_k X$. The morphism of group schemes $\psi_n : G_n \to G_N$ induced by φ_n is compatible with the morphism of pseudo-torsors $\varphi_n : \Gamma_n \to \Gamma_N$ in the sense of Lemma 6.12. Since the fibers of the morphism

$$\pi_N|_{V_N} : V_N \to U$$

are dense in the fibers of the morphism $(\pi_N)_U : (\Gamma_N)_U \to U$, we see that, by Lemma 6.12, the fibers of the morphism $\pi_n|_{V_N} : V_n \to U$ are dense in the fibers of the morphism

$$(\pi_n)_U : (\Gamma_n)_U \to U.$$

We obtain that, for every $n \geq N$, the groupoid $\Gamma_n \to X \times_k X$ and the open subsets $V_n \subset \Gamma$, $U \subset X \times_k X$ satisfy all hypotheses of Lemma 6.11 (which is also true for schemes over a field k with the product of schemes taken over k). Therefore, the morphism π_n is faithfully flat. In other terms, the ring A_n is faithfully flat over $R \otimes_k R$. Since $A = \bigcup_n A_n$, where $A_n \subset A_{n+1}$, we conclude that A is faithfully flat over $R \otimes_k R$, which finishes the proof. $\square$

7. Proofs of the main results

7.1. *Proof of Theorem 2.5*

We use the notation from Theorem 2.5. Let M be a finite-dimensional $D_{K/k}$-module over K. Consider a D_k-structure on $\mathbf{DMod}(K, D_{K/k})$ as in Theorem 5.1. Let C be the subcategory $\langle M \rangle_{\otimes, D}$ in $\mathbf{DMod}\,(K, D_{K/k})$ D_k-tensor generated by M (Definition 4.19). By Theorem 5.5, to prove the theorem, it is enough to construct a differential functor from C to $\mathbf{Vect}(k)$, which is our goal in what follows.

First, we would like to apply Theorem 4.27 to the forgetful functor $C \to \mathbf{Vect}(K)$ and, thus, obtain a D_k-Hopf algebroid. The problem here is that, a priori, there is no D_k-structure on K. To overcome this, the splitting $\widetilde{D}_k$ is introduced in the hypotheses of the theorem (see also Remark 2.7). This allows to switch between the D_k-structure on C and the D_K-structure on K as follows.

The morphism of differential fields $(k, D_k) \to (k, \widetilde{D}_k)$ defines a $\widetilde{D}_k$-structure on C by Proposition 4.12 (1). Denote the category C with this $\widetilde{D}_k$-structure by $\widetilde{C}$. Thus, the identity functor $C \to \widetilde{C}$ is a differential functor from a D_k-category C to a $\widetilde{D}_k$-category $\widetilde{C}$. By Theorem 5.1, the forgetful functor is a differential functor from the D_k-category C to the D_K-category $\mathbf{Vect}(K)$. Apply the extension of scalars along the vertical morphisms of the diagram from Remark 3.16 to the forgetful functor $C \to \mathbf{Vect}(K)$.

By Proposition 4.12(2), we obtain a differential functor ω from the $\widetilde{D}_k$-category $\widetilde{C}$ to the $\widetilde{D}_K$-category $\mathbf{Vect}(K)$ (the latter category is with the usual $\widetilde{D}_K$-structure as in Example 4.7). Recall that

$$\widetilde{D}_K = K \otimes_k \widetilde{D}_k$$

and K is a $\widetilde{D}_k$-field over k. Thus, we have a differential functor $\omega : \widetilde{C} \to \mathbf{Vect}(K)$ between $\widetilde{D}_k$-categories.

By Theorem 4.27, there exists a $\widetilde{D}_k$-Hopf algebroid (K, H) over k such that H is faithfully flat over $K \otimes_k K$ and ω lifts up to an equivalence of $\widetilde{D}_k$-categories

$$\widetilde{C} \xrightarrow{\sim} \mathbf{Comod}^{fg}(K, H).$$

Since C is D_k-tensor generated by one object, the $\widetilde{D}_k$-category C is also $\widetilde{D}_k$-tensor generated by one object. Hence, Proposition 4.28 and the proof of Theorem 4.27 imply that H is $\widetilde{D}_k$-finitely generated over $K \otimes_k K$. We apply Theorem 6.1 to (K, H) and obtain the corresponding Hopf algebroid (R, A). The extension of scalars

$$K \otimes_R - : \mathbf{Comod}^{fg}(R, A) \to \mathbf{Comod}^{fg}(K, H)$$

is a differential functor between $\widetilde{D}_k$-categories (Example 4.10(3)). The forgetful functor

$$\mathbf{Comod}^{fg}(R, A) \to \mathbf{Mod}(R)$$

is a differential functor, where we consider the $\widetilde{D}_R$-category structure on $\mathbf{Mod}(R)$ with

$$\widetilde{D}_R := R \otimes_k \widetilde{D}_k$$

(Example 4.10(2)). We have that R is $\widetilde{D}_k$-finitely generated over k, the morphism $(R, \widetilde{D}_R) \to (K, \widetilde{D}_K)$ is strict, where $\widetilde{D}_K := K \otimes_k \widetilde{D}_k$, and (k, D_k) is relatively differentially closed in $(K, \widetilde{D}_K)$ by the hypotheses of the theorem. Therefore, there is a morphism of differential rings $(R, \widetilde{D}_R) \to (k, D_k)$. This defines a differential functor

$$\mathbf{Mod}(R) \to \mathbf{Vect}(k), \quad N \mapsto k \otimes_R N$$

from the $\widetilde{D}_R$-category $\mathbf{Mod}(R)$ to the D_k-category $\mathbf{Vect}(k)$ (Example 4.10(1)). Summarizing, we obtain a collection of differential functors

$$C \to \widetilde{C} \to \mathbf{Comod}^{fg}(K, H) \xleftarrow{K \otimes_R -} \mathbf{Comod}^{fg}(R, A) \to \mathbf{Mod}(R) \to \mathbf{Vect}(k).$$

Since A is faithfully flat over $R \otimes_k R$, the extension of scalars functor $K \otimes_R -$ is an equivalence of categories (see [13, 1.8,3.5] and also Section A.1). All together, this defines a differential functor from C to $\mathbf{Vect}(k)$, which finishes the proof.

7.2. *Proof of Theorem 2.8*

We need the following simple facts.

Lemma 7.1. *Let Y be an irreducible variety over a field k_0 with $\operatorname{char} k_0 = 0$, k be a field extension of k_0, and let $K_0 := k_0(Y)$. Suppose that k_0 is existentially closed in K_0. Then, for any non-empty open subset $U \subset X := Y \times_{k_0} k$, there exists a k_0-point y on Y such that the k-point $x := y \times_{k_0} k$ of X belongs to U.*

Proof. First note that, if the lemma is proven for an extension k' of k, then this implies the lemma for k. Thus, replacing k by its extension, if needed, we may assume that $k^\Gamma = k_0$, where Γ is the group of all field automorphisms of k over k_0, because $\operatorname{char} k_0 = 0$. For a non-empty open subset $U \subset X$, take its complement $Z := X \backslash U$ and consider the intersection

$$Z' := \bigcap_{\sigma\in\Gamma} \sigma(Z).$$

The closed subvariety $Z' \subset X$ is invariant under Γ, therefore there exists a closed subvariety $W \subset Y$ such that $Z' = W \times_{k_0} k$. Moreover, $W \neq Y$, because $Z' \subset Z \neq X$. Put $V := Y \backslash W$, which is a non-empty open subset in Y. Since k_0 is existentially closed in K_0, there exists $y \in V(k_0)$. This defines a k-point $x := y \times_{k_0} k$ in X. If $x \in Z(k)$, then $x \in \sigma(Z(k))$ for any $\sigma \in \Gamma$. Thus, $x \in Z'(k)$, which contradicts to the fact that y is not in W. Hence, we obtain that x belongs to U. $\square$

Lemma 7.2. *Let $k \subset K$ be a field extension and let $\varphi_1, \ldots, \varphi_n : k \to k$ be maps that are linearly independent over k. Then $\varphi_1, \ldots, \varphi_n$ are linearly independent over K considered as maps from k to K.*

Proof. Since $\varphi_1, \ldots, \varphi_n$ are linearly independent over k, the image of the map

$$\Phi : k \to k^{\oplus n}, \quad f \mapsto (\varphi_1(f), \ldots, \varphi_n(f)),$$

spans all $k^{\oplus n}$ over k. Therefore, the image of the composition of Φ and the natural embedding $k^{\oplus n} \subset K^{\oplus n}$ spans all $K^{\oplus n}$ over K, so, $\varphi_1, \ldots, \varphi_n$ are linearly independent over K. $\square$

Lemma 7.3. *Let K be a D_k-field over a differential field (k, D_k) with $\operatorname{char} k = 0$ such that K is of finite transcendence degree over k. Then any finite subset $\Sigma \subset K$ is contained in a D_k-subalgebra R in K over k that is finitely generated as an algebra over k.*

Proof. Let L be the D_k-subfield generated by Σ in K. It follows from [45, Theorem 5.6.3] that L is a finitely generated field over k. Hence, there exists a finite set $S \subset L$ such that $\Sigma \subset S$ and $L = k(S)$. It now follows from differentiating fractions that $R := k[S \cup 1/T] \subset K$ satisfies the requirement of the lemma, where $T \subset K$ is the set of the denominators of $D_k(S)$. $\square$

Now, we prove Theorem 2.8 using its notation. Suppose that condition 1 of the theorem holds. Then the structure map identifies $\widetilde{D}_k$ and $1 \otimes D_k$, where $\widetilde{D}_k$ is given in condition 1. Hence,

$$(k, D_k) \to (k, \widetilde{D}_k)$$

is an isomorphism and K is a D_k-field. Let R be a D_k-finitely generated subalgebra in K over k. We need to show that there is a morphism of D_k-algebras $R \to k$. By Remark 2.9(1), we have

$$K = \operatorname{Frac}(K_0 \otimes_{k_0} k).$$

Let $\{a_i/b_i\}$ be a finite set of D_k-generators of R over k with $a_i, b_i \in K_0 \otimes_{k_0} k$. Let R_0 be the subalgebra in K_0 generated over k by the K_0-components of summands in a_i's and b_i's and put

$$f := \prod_i b_i.$$

Since K_0 is the field of D_k-constants, $R_0 \otimes_{k_0} k$ is a D_k-differential subalgebra in K over k. Hence, R is contained in the localization $(R_0 \otimes_{k_0} k)_f$. By Lemma 7.1 applied to

$$Y := \mathrm{Spec}(R_0), \quad U := \mathrm{Spec}\left((R_0 \otimes_{k_0} k)_f\right),$$

there exists a k_0-point y on $Y(k_0)$ such that the k-point $x := y \times_{k_0} k$ of X belongs to U. The point x defines a morphism of k-algebras $f : R \to k$. The kernel of f is generated by D_k-constants in R, because $K_0 = K^{D_k}$. Therefore, f is a morphism of D_k-algebras, and (k, D_k) is relatively differentially closed in

$$(K, K \otimes_k D_k) \cong (K, K \otimes_k \widetilde{D}_k).$$

Now suppose that condition 2 of the theorem holds. Our first goal is to construct a splitting $\widetilde{D}_k$ of (K, D_K) over (k, D_k) such that the natural map $K \otimes_k \widetilde{D}_k \to D_K$ is an isomorphism. With this aim, we consider the "effective" quotients

$$D_K^{\mathrm{eff}} := \mathrm{Im}(\theta_K : D_K \to \mathrm{Der}(K, K)), \quad D_k^{\mathrm{eff}} := \mathrm{Im}(\theta_k : D_k \to \mathrm{Der}(k, k))$$

of the differential structures D_K and D_k, respectively. It follows from Lemma 7.2 that the natural map

$$K \otimes_k D_k^{\mathrm{eff}} \to \mathrm{Der}(k, K)$$

is injective. Therefore, the composition

$$D_K \to K \otimes_k D_k \to K \otimes_k D_k^{\mathrm{eff}}$$

factors through D_K^{eff}, that is, we have a commutative diagram

$$\begin{array}{ccc} D_K & \longrightarrow & K \otimes_k D_k \\ \downarrow & & \downarrow \\ D_K^{\mathrm{eff}} & \longrightarrow & K \otimes_k D_k^{\mathrm{eff}}. \end{array}$$

Hence, (K, D_K^{eff}) is a parameterized differential field over (k, D_k^{eff}). By condition 2 of the theorem, we have

$$D_{K/k} \xrightarrow{\sim} \mathrm{Der}_k(K, K).$$

Consequently, the natural K-linear morphism

$$D_{K/k} = \mathrm{Ker}\,(D_K \to K \otimes_k D_k) \to \mathrm{Ker}\left(D_K^{\mathrm{eff}} \to K \otimes_k D_k^{\mathrm{eff}}\right) \subseteq \mathrm{Der}_k(K, K)$$

is an isomorphism. It follows that there is an isomorphism

$$D_K \cong D_K^{\mathrm{eff}} \times_{\left(K \otimes D_k^{\mathrm{eff}}\right)} (K \otimes_k D_k). \tag{23}$$

Take commuting bases in D_K^{eff} and D_k^{eff} from Proposition 3.18. Put $\widetilde{D}_k^{\mathrm{eff}}$ to be the k-linear span of the basis in D_K^{eff}. Then $\widetilde{D}_k^{\mathrm{eff}}$ is a splitting of (K, D_K^{eff}) over (k, D_k^{eff}) such that

$$D_K^{\mathrm{eff}} \cong K \otimes_k \widetilde{D}_k^{\mathrm{eff}}.$$

Put

$$\widetilde{D}_k := \widetilde{D}_k^{\mathrm{eff}} \times_{D_k^{\mathrm{eff}}} D_k \subset D_K. \tag{24}$$

Since taking effective quotients is a morphism of Lie rings and by formula (3) from Section 3.2, we have that $\widetilde{D}_k$ is closed under the Lie bracket on D_K. Thus, $\widetilde{D}_k$ is a splitting of (K, D_K) over (k, D_k). Comparing (23) and (24), we see that

$$K \otimes_k \widetilde{D}_k \cong D_K.$$

Note that, in this case,

$$\dim_k(\widetilde{D}_k) = \dim_K(D_K),$$

while, in the previous case (condition 1 of the theorem), $\dim_k(\widetilde{D}_k)$ could be less than $\dim_K(D_K)$. So, in this case, $\widetilde{D}_k$ could be "much larger". Put

$$D := \mathrm{Ker}\left(\widetilde{D}_k \to D_k\right).$$

Then we have $K \otimes_k D \cong D_{K/k}$.

Let (R, D_R) be a differential subalgebra in (K, D_K) over (k, D_k) such that the morphism $(R, D_R) \to (K, D_K)$ is strict and (R, D_R) is differentially finitely generated over k. Extending R by a finite number of elements from K, we obtain that

$$D_R \cong R \otimes_k \widetilde{D}_k$$

and R is a $\widetilde{D}_k$-finitely generated $\widetilde{D}_k$-subalgebra in K over k. Since

$$\dim_K(D_{K/k}) = \dim_k(\mathrm{Der}_k(K, K))$$

is finite, K is of finite transcendence degree over k. Hence, by Lemma 7.3, we may assume that R is finitely generated as an algebra over k. By the hypotheses of the theorem, we have

$$D_{K/k} \cong \mathrm{Der}_k(K, K).$$

Since $\mathrm{char}\, k = 0$ and R is finitely generated, localizing R by a non-zero element, we may assume that R is smooth over k and

$$R \otimes_k D \cong \mathrm{Der}_k(R, R).$$

Since k is existentially closed in K, there is a homomorphism $f : R \to k$ of k-algebras. We claim that f extends to a morphism of differential algebras $(R, D_R) \to (k, D_k)$ over (k, D_k). By definition, to prove this, we have to construct a morphism of Lie rings

$$s : D_k \to \widetilde{D}_k = k \otimes_R D_R$$

such that, for all $\partial \in D_k$ and $a \in R$, we have

$$\partial(f(a)) = f(s(\partial)(a)). \tag{25}$$

We claim that, for any $\partial \in D_k$, there is a unique $s(\partial) \in \widetilde{D}_k$ that satisfies (25). Indeed, consider the derivation δ from R to itself defined as the composition

$$R \xrightarrow{f} k \xrightarrow{\partial} k \longrightarrow R$$

and consider any $\tilde{\partial} \in \widetilde{D}_k$ such that $\tilde{\partial}$ is sent to ∂ by the surjective map $\widetilde{D}_k \to D_k$. The difference $\delta - \theta_K(\tilde{\partial})$ is a k-linear derivation from R to itself, that is, it belongs to

$$R \otimes_k D \cong \mathrm{Der}_k(R, R).$$

Put

$$s(\partial) := \tilde{\partial} + f\big(\delta - \theta_K(\tilde{\partial})\big),$$

where f denotes also the map

$$R \otimes_k \widetilde{D}_k \xrightarrow{f \cdot \mathrm{id}} \widetilde{D}_k.$$

By construction, $s(\partial)$ satisfies (25). The uniqueness of $s(\partial)$ follows from the fact that if $s(\partial)$ and $s(\partial)'$ in $\widetilde{D}_k$ satisfy (25), then $s(\partial) - s(\partial)'$ belongs to

$$\mathrm{Ker}\Big(R \otimes_k \widetilde{D}_k \xrightarrow{f \cdot \mathrm{id}} \widetilde{D}_k\Big),$$

whose intersection with $1 \otimes \widetilde{D}_k$ is trivial. By construction, the obtained map $s : D_k \to \widetilde{D}_k$ is k-linear. The uniqueness of $s(\partial)$ implies that s is a morphism of Lie rings: given $\partial_1, \partial_2 \in D_k$, the commutator $[s(\partial_1), s(\partial_2)]$ satisfies (25) with $\partial = [\partial_1, \partial_2]$. This shows that (k, D_k) is relatively differentially closed in

$$(K, D_K) = (K, K \otimes_k \widetilde{D}_k).$$

8. PPV extensions with non-closed constants

In this section, we discuss two aspects of PPV extension for parameterized differential fields over an arbitrary differential field (k, D_k) (in contrast to the usual assumption [10] that (k, D_k) be differentially closed).

8.1. Galois correspondence

We establish the Galois correspondence for PPV extensions. Basically, we use the classical differential Galois correspondence for PV extensions. Also, we use the differential Tannakian formalism, in particular, Theorem 5.5.

First, let us recall several notions concerning differential algebraic groups. Let (k, D_k) be a differential field and G be a *linear D_k-group*, that is, G is a group-valued functor on $\mathbf{DAlg}(k)$ corepresented by a D_k-finitely generated D_k-Hopf algebra U over k. A *D_k-subgroup H in G* is a corepresentable group subfunctor H in G on the category $\mathbf{DAlg}(k)$. By [79, Theorem 15.3], this corresponds to a surjective morphism $U \to V$ between D_k-Hopf algebras over k. Hence, H is a linear D_k-group.

Suppose that G acts on a D_k-algebra A, that is, we have a morphism of D_k-algebras $m : A \to A \otimes_k U$ that satisfies the axioms of a comodule over a Hopf algebra. Let A be a domain and $L := \mathrm{Frac}(A)$. We put

$$L^G := \{a/b \in L \mid a, b \in A,\ b \cdot m(a) = a \cdot m(b)\}.$$

It follows that L^G is a D_k-subfield in L.

Let (K, D_K) be a parameterized differential field over (k, D_k) and let M be a finite-dimensional $D_{K/k}$-module. Suppose that there exists a PPV extension L for M.

Definition 8.1. The *parameterized differential Galois group of L over K* is the group functor

$$\mathrm{Gal}^{D_K}(L/K) : \mathbf{DAlg}(k, D_k) \to \mathbf{Sets}, \quad R \mapsto \mathrm{Aut}^{D_K}(R \otimes_k A / R \otimes_k K),$$

where A is a PPV ring associated with L (Definition 3.28) and we consider a D_K-structure on the extension of scalars $R \otimes_k K$ as given by Definition 3.23.

Lemma 8.2. *The functor* $\mathrm{Gal}^{D_K}(L/K)$ *is corepresented by a* D_k*-finitely generated* D_k*-Hopf algebra, that is,* $\mathrm{Gal}^{D_K}(L/K)$ *is a linear* D_k*-group.*

Proof. By Theorem 5.5, the PPV extension L corresponds to a differential functor

$$\omega : \langle M\rangle_{\otimes,D} \to \mathbf{Vect}(k).$$

By Remark 5.6, the functor $\mathrm{Gal}^{D_K}(L/K)$ is canonically isomorphic to the functor

$$\mathbf{DAlg}(k, D_k) \to \mathbf{Sets}, \quad R \mapsto \mathrm{Isom}^{\otimes,D}(\omega_R, \omega_R).$$

By Proposition 4.25, the latter functor is corepresentable by a D_k-Hopf algebra U over k. By Proposition 4.28, U is D_k-finitely generated. □

Note that one can also prove Lemma 8.2 more explicitly without using the Tannakian formalism.

Remark 8.3. It follows from Proposition 4.25, Theorem 5.5, and [13, 9.6] that L is a union of (possibly, infinitely many) PV-extensions defined by the $D_{K/k}$-modules $(\mathrm{At}^1)^{\circ i}(M)$ and $\mathrm{Gal}^{D_K}(L/K)$ with forgotten D_k-structure is the differential Galois group $\mathrm{Gal}^{D_{K/k}}(L/K)$.

Recall the differential Galois correspondence in the case of arbitrary constants from [22, Section 4]. Given a Hopf algebra U, an *algebraic subgroup* $\mathrm{Spec}(V)$ in $\mathrm{Spec}(U)$ corresponds to a surjective homomorphism between Hopf algebras $U \to V$.

Proposition 8.4. *There is a bijective correspondence between algebraic subgroups* $H \subset \mathrm{Gal}^{D_{K/k}}(L/K)$ *and* $D_{K/k}$*-subfields* $K \subset E \subset L$ *given by*

$$H \mapsto E := L^H, \quad E \mapsto H := \mathrm{Gal}^{D_{E/k}}(L/E).$$

The Galois correspondence in the parameterized case is as follows.

Proposition 8.5. *There is a bijective correspondence between* D_k*-subgroups* $H \subset \mathrm{Gal}^{D_K}(L/K)$ *and* D_K*-subfields* $K \subset E \subset L$ *given by*

$$H \mapsto E := L^H, \quad E \mapsto H := \mathrm{Gal}^{D_E}(L/E).$$

Proof. By Proposition 8.4 and Remark 8.3, we only need to show that an algebraic subgroup

$$H \subset \mathrm{Gal}^{D_{K/k}}(L/K)$$

is a D_k-subgroup in $\mathrm{Gal}^{D_K}(L/K)$ if and only if the corresponding $D_{K/k}$-subfield $E \subset L$ is a D_K-subfield. Suppose that E is a D_K-subfield. Then L is a PPV extension for the D_E-module M_E over E, where $D_E := E \otimes_K D_K$. Therefore, the corresponding Galois group H has a canonical D_k-structure and corepresents a group subfunctor in G on $\mathbf{DAlg}(k)$ given by Definition 8.1. Thus, H is a D_k-subgroup in G.

Conversely, suppose that H is a D_k-subgroup in $G := \mathrm{Gal}^{D_K}(L/K)$. Consider the extension of scalars G_K from (k, D_k) to (K, D_K) for G (Definition 3.23). We have a D_K-subgroup H_K in G_K. By the adjunction between restriction and extension of scalars (Definition 3.37) and Definition 8.1, G_K acts on the D_K-field L over K and $L^H = L^{H_K}$. By Proposition 8.4, we have $E = L^H$, whence, E is an D_K-subfield in L. □

The proof of the normal subgroup case uses the differential Tannakian formalism.

Proposition 8.6. *Under the correspondence from Proposition 8.5, a normal D_k-subgroup H corresponds to a PPV extension E over K, and we have an isomorphism of D_k-groups*

$$\mathrm{Gal}^{D_K}(L/K)/H \cong \mathrm{Gal}^{D_K}(E/K).$$

Proof. For short, put $G := \mathrm{Gal}^{D_K}(L/K)$. Let

$$\omega : \langle M \rangle_{\otimes,D} \to \mathbf{Vect}(k)$$

be the differential functor that corresponds to the PPV extension L by Theorem 5.5. It follows from Theorem 4.27 and the proof of Lemma 8.2 that ω lifts up to an equivalence of D_k-categories

$$\langle M \rangle_{\otimes,D} \xrightarrow{\sim} \mathbf{Rep}^{fg}(G). \tag{26}$$

Let H be a normal D_k-subgroup in G. Then $\mathbf{Rep}^{fg}(G/H)$ is a full D_k-subcategory in $\mathbf{Rep}^{fg}(G)$. By [8, Proposition 15], G/H is a linear D_k-group, that is, there is a faithful finite-dimensional representation of G/H over k. Let N be the corresponding $D_{K/k}$-module over K under the equivalence (26). Taking the restriction of ω to the subcategory $\langle N \rangle_{\otimes,D}$ in $\langle M \rangle_{\otimes,D}$, by Theorem 5.5, we obtain a PPV extension E for N, which is embedded into L as a D_K-subfield. Moreover, by construction, we have an isomorphism

$$G/H \cong \mathrm{Gal}^{D_K}(E/K).$$

We need to show that E is the subfield in K associated with H by the Galois correspondence, that is, $E = L^H$. By Proposition 8.5, it is enough to show the equality $\mathrm{Gal}^{D_E}(L/E) = H$. It is implied by the fact that $\mathrm{Gal}^{D_E}(L/E)$ is the kernel of the restriction homomorphism

$$\mathrm{Gal}^{D_K}(L/K) = G \to \mathrm{Gal}^{D_K}(E/K) = G/H. \tag{27}$$

Conversely, if E is a PPV extension of K in L, then $H := \mathrm{Gal}^{D_E}(L/E)$ is the kernel of the group homomorphism (27), whence H is normal. $\square$

8.2. *Extension of constants in parameterized differential fields*

Now let us consider the behavior of PPV extensions and the corresponding differential categories under extensions of the differential field (k, D_k). Let (K, D_K) be a parameterized differential field over (k, D_k) with $\operatorname{char} k = 0$. Let l be a D_k-field over k (Definition 3.12). In particular, we have a differential field (l, D_l) with $D_l := l \otimes_k D_k$. Since $\operatorname{char} k = 0$, the field k is algebraically closed in K and the ring

$$R := l \otimes_k K$$

is a domain (for example, see [36, Corollary 1, p. 203]). Denote the fraction field of R by L. By Definition 3.23, R is a D_K-algebra over K. Therefore, L is a D_K-field over K and we have morphisms of differential rings

$$(l, D_l) \to (R, D_R) \to (L, D_L),$$

where $D_R := R \otimes_K D_K$ and $D_L := L \otimes_K D_K$. Also, we have

$$D_{R/l} := \mathrm{Ker}(D_R \to R \otimes_l D_l) \cong R \otimes_K D_{K/k} \cong l \otimes_k D_{K/k}, \quad D_{L/l} := \mathrm{Ker}(D_L \to L \otimes_l D_l) \cong L \otimes_K D_{K/k},$$

because the functors $R \otimes_K -$ and $L \otimes_K -$ are exact.

Lemma 8.7. *The $D_{K/k}$-algebra R over K has no non-zero $D_{K/k}$-ideals besides R itself.*

Proof. Let I be a non-zero $D_{K/k}$-ideal in R and consider $0 \neq f \in I$ with

$$f = \sum_{i=1}^{n} c_i \otimes f_i, \quad 0 \neq c_i \in l, \ 0 \neq f_i \in K$$

such that $c_1 \ldots, c_n$ are linearly independent over k. Suppose that f has the minimal possible number n among all non-zero elements in I. Take any $\partial \in D_{K/k}$. Since $\partial f \in I$, we have

$$g := (1 \otimes f_1)\partial f - (1 \otimes \partial f_1) f \in I.$$

On the other hand,

$$\partial f = \sum_{i=1}^{n} c_i \otimes \partial f_i,$$

hence,

$$g = \sum_{i=2}^{n} c_i \otimes (f_1 \partial f_i - \partial f_1 \, f_i)$$

has less summands than f. Therefore, $g = 0$. Since $c_1, \ldots, c_n$ are linearly independent over k, we obtain that $\partial(f_i/f_1) = 0$ for all $i = 2, \ldots, n$ and for all $\partial \in D_{K/k}$. Hence,

$$h_i := f_i/f_1 \in k = K^{D_{K/k}}$$

and we have

$$f = \left(\sum_{i=1}^{n} c_i h_i \right) \otimes f_1.$$

Thus, f is invertible in R and $I = R$. $\square$

Lemma 8.8. *Let P and P' be $D_{R/l}$-modules over R such that P is a finitely generated R-module and let $\phi : P_L \to P'_L$ be a morphism between the corresponding differential modules over $(L, D_{L/l})$. Then we have*

$$\phi(P \otimes 1) \subseteq P' \otimes 1.$$

Proof. Consider the subset $I \subset R$ that consists of all $f \in R$ such that, for all $v \in P \otimes 1$, we have

$$f \cdot \phi(v) \in P' \otimes 1.$$

It is readily seen that I is an ideal in R. Moreover, since the module P is finitely generated over R, the ideal I is non-zero. Take any $\partial \in D_{K/k}$. Since the R-submodule $P \otimes 1 \subset P_L$ is stable under ∂ and ϕ is L-linear and commutes with ∂, for all $f \in I$ and $v \in P \otimes 1$, we get

$$\partial f \cdot \phi(v) = \partial(f \cdot \phi(v)) - f \cdot \phi(\partial v) \in P' \otimes 1.$$

Hence, I is a non-zero $D_{K/k}$-ideal in R. By Lemma 8.7, we conclude that $I = R$. $\square$

Corollary 8.9. *For all finite-dimensional $D_{K/k}$-modules M and M' over K, the natural map*

$$l \otimes_k \mathrm{Hom}_{D_{K/k}}(M, M') \cong \mathrm{Hom}_{D_{L/l}}(M_L, M'_L)$$

is an isomorphism, where, for a differential ring (A, D_A), $\mathrm{Hom}_{D_A}(-,-)$ denotes morphisms between differential modules over (A, D_A). In particular, we have

$$l \otimes_k M^{D_{K/k}} = M_L^{D_{L/l}} \quad \text{and} \quad l = L^{D_{L/l}}.$$

Proof. First, by Lemma 8.8 with $P = M_R$ and $P' = M'_R$, the natural morphism

$$\mathrm{Hom}_{D_{R/l}}(M_R, M'_R) \to \mathrm{Hom}_{D_{L/l}}(M_L, M'_L)$$

is an isomorphism. Since $D_{R/l} \cong l \otimes_k D_{K/k}$ acts trivially on l and $M_R \cong l \otimes_k M$, we have canonical isomorphisms:

$$\mathrm{Hom}_{D_{R/l}}(M_R, M'_R) = \mathrm{Hom}_R(M_R, M'_R)^{D_{R/l}} \cong (l \otimes_k \mathrm{Hom}_K(M, M'))^{l \otimes_k D_{K/k}} \cong l \otimes_k \mathrm{Hom}_{D_{K/k}}(M, M'). \square$$

Thus, we see that (L, D_L) is a parameterized differential field over (l, D_l), and that there are no non-trivial new solutions over L of linear $D_{K/k}$-differential equations given over K. The following result is implied directly by Corollary 8.9 (more precisely, by its last assertion $l = L^{D_{L/l}}$).

Corollary 8.10. *Let M be a finite-dimensional $D_{K/k}$-module over K, E be a PPV extension for M, and let A be the D_k-Hopf algebra of the parameterized Galois group of E over K. Then the D_L-field $F := \mathrm{Frac}(l \otimes_k E)$ is a PPV extension for M_L and the D_l-Hopf algebra of the parameterized Galois group of F over L is A_l.*

By Theorem 5.5, Corollary 8.10 also follows from the following categorical statement, which makes sense without the assumption of the existence of a PPV extension (or, equivalently, the existence of a differential functor) and has interest on its own right.

Proposition 8.11. *Let M be a finite-dimensional $D_{K/k}$-module over K. Then the differential functor from a D_k-category over k to a D_l-category over l (Definition 4.19, Proposition 4.12, and Theorem 5.1)*

$$\langle M \rangle_{\otimes, D} \to \langle M_L \rangle_{\otimes, D}, \quad X \mapsto X_L$$

induces an equivalence D_l-categories

$$\Phi : l \otimes_k \langle M \rangle_{\otimes, D} \xrightarrow{\sim} \langle M_L \rangle_{\otimes, D}.$$

Proof. It is known that Hom-spaces in the extension of scalars category $l \otimes_k \langle M \rangle_{\otimes, D}$ are obtained by taking $l \otimes_k -$ from the Hom-spaces in the category $\langle M \rangle_{\otimes, D}$ ([54, p.407], [74]). Thus, it follows from Corollary 8.9 that Φ is fully faithful. Let us show that Φ is essentially surjective. Any object N in $\langle M_L \rangle_{\otimes, D}$ is a subquotient of Q_L for some object Q in $\langle M \rangle_{\otimes, D}$, that is, there are $D_{L/l}$-submodules

$$N_1 \subset N_2 \subset Q_L \quad \text{such that} \quad N \cong N_2/N_1.$$

Indeed, this is true for M_L, and also this property is preserved under taking direct sums, tensor products, duals, subquotients, and the functor At^1. Put

$$P_i := N_i \cap (l \otimes_k Q) \subset Q_L, \quad i = 1, 2, \quad \text{and} \quad P := P_1/P_2.$$

We have

$$P_i = \varinjlim (P_i \cap (V \otimes_k Q)),$$

where the limit is taken over all finite-dimensional over k subspaces V in l. Recall that objects in

$$l \otimes_k \langle M \rangle_{\otimes,D}$$

are l-modules in the category of ind-objects in $\langle M \rangle_{\otimes,D}$ ([54, p. 407], [74]). Therefore, P_i and P are objects in $l \otimes_k \langle M \rangle_{\otimes,D}$. Finally, $\Psi(P) = N$, because $L = \mathrm{Frac}(l \otimes_k K)$, whence $L \otimes_R P \cong N$. $\square$

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Appendix

Here we recollect several known definitions and results and fix some notation we extensively use in the paper.

A.1. Hopf algebroids

There are many references concerning Hopf algebroids, for example, see [13, 1.6, 1.14]. Also, the book [39] is very useful. A *Hopf algebroid* is a pair of rings (R, A) with the following data and properties. First, there are two ring homomorphisms $l : R \to A$ and $r : R \to A$, that is, A is an algebra over $R \otimes R$. In particular, A is an R-bimodule with the left and the right R-module structures given by the homomorphisms l and r, respectively. Further, there are morphisms of algebras over $R \otimes R$:

$$\Delta : A \to A \otimes_R A, \quad e : A \to R, \quad \iota : A \to A^s.$$

According to our notation, the tensor product $A \otimes_R A$ involves both left and right R-module structures on A. The ring R is considered as an algebra over $R \otimes R$ via the multiplication on R. In particular, we have the identities $e \circ l = \mathrm{id}_R$ and $e \circ r = \mathrm{id}_R$. Also, A^s denotes the same ring A with the right and left R-module structures being the initial left and right R-module structures on A, respectively, that is, we have

$$\iota(l(f)a) = \iota(a)\, r(f) \quad \text{and} \quad \iota(ar(f)) = l(f)\iota(a) \quad \text{for all } f \in R,\ a \in A.$$

The morphisms (l, r, Δ, e, ι) are required satisfy the following set of axioms, which are similar to the axioms in the definition of a Hopf algebra. The coassociativity axiom requires the equality of the compositions

$$\begin{tikzcd} A \arrow[r, "\Delta"] & A \otimes_R A \arrow[r, shift left, "\Delta \otimes \mathrm{id}_A"] \arrow[r, shift right, "\mathrm{id}_A \otimes \Delta"'] & A \otimes_R A \otimes_R A \end{tikzcd}.$$

The counit axiom requires that both compositions

$$A \xrightarrow{\Delta} A \otimes_R A \underset{\mathrm{id}_A \otimes e}{\overset{e \otimes \mathrm{id}_A}{\rightrightarrows}} A$$

are equal to the identity. Finally, the antipode axiom requires that the following diagrams commute:

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & A \otimes_R A \\ \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle \iota \cdot \mathrm{id}_A} \\ R & \xrightarrow{r} & A, \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\Delta} & A \otimes_R A \\ \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle \mathrm{id}_A \cdot \iota} \\ R & \xrightarrow{l} & A. \end{array}$$

In particular, it follows that ι is an involution and that $e \circ \iota = e$. Also, ι is uniquely defined by Δ and e. Note that a Hopf algebroid (R, A) with $l = r$ is the same as a Hopf algebra A over R.

A Hopf algebroid (R, A) defines a Hopf algebra B over R by the formula

$$B := R \otimes_{(R \otimes R)} A.$$

Further, by the extension of scalars, B defines a Hopf algebra $B \otimes R$ over $R \otimes R$. It follows from the definition of a Hopf algebroid that $\mathrm{Spec}(A)$ is a pseudo-torsor under the group scheme $\mathrm{Spec}(B \otimes R)$ over $\mathrm{Spec}(R \otimes R)$ (for example, see Definition 6.9).

A *Hopf algebroid over a ring* κ is a Hopf algebroid (R, A) such that A and R are κ-algebras, the morphisms l and r are morphisms of κ-algebras and the morphisms (Δ, e, ι) are morphisms of algebras over $R \otimes_\kappa R$. In this case, $\mathrm{Spec}(A)$ is a pseudo-torsor under the group scheme $\mathrm{Spec}(R \otimes_\kappa B)$ over $\mathrm{Spec}(R \otimes_\kappa R)$, where B is defined as above.

A *comodule* over a Hopf algebroid (R, A) is an R-module M together with a morphism of A-modules

$$\epsilon_M : M \otimes_R A \to A \otimes_R M$$

that satisfies two axioms, which are similar to the axioms in the definition of a comodule over a Hopf algebra. The first axiom requires the equality

$$R \otimes_A \epsilon_M = \mathrm{id}_M,$$

where the A-module structure on R is defined by the ring homomorphism $e : A \to R$ and we use that

$$R \otimes_A (M \otimes_R A) \cong R \otimes_A (A \otimes_R M) \cong M.$$

The second axiom requires the equality of the composition

$$M \otimes_R A \otimes_R A \xrightarrow{\epsilon_M \otimes_R A} A \otimes_R M \otimes_R A \xrightarrow{A \otimes_R \epsilon_M} A \otimes_R A \otimes_R M$$

to the extension of scalars

$$(A \otimes_R A) \otimes_A \epsilon_M : M \otimes_R A \otimes_R A \to A \otimes_R A \otimes_R M,$$

where the A-module structure on $A \otimes_R A$ is given by the ring homomorphism Δ and we use that

$$(A \otimes_R A) \otimes_A (M \otimes_R A) \cong M \otimes_R A \otimes_R A \quad \text{and} \quad (A \otimes_R A) \otimes_A (A \otimes_R M) \cong A \otimes_R A \otimes_R M.$$

One proves that ϵ_M is an isomorphism. By adjunction between extension and restriction of scalars, one obtains a left R-linear morphism

$$\phi_M : M \to A \otimes_R M,$$

and one can give an equivalent definition of a comodule in terms of ϕ_M. Denote the category of comodules over a Hopf algebroid (R, A) by $\mathbf{Comod}(R, A)$. Denote the full subcategory of comodules over (R, A) that are finitely generated as R-modules by $\mathbf{Comod}^{fg}(R, A)$.

Remark A.12. Given a Hopf algebroid (R, A) over a ring κ, the pair $(\mathrm{Spec}(A), \mathrm{Spec}(R))$ defines a category $\mathcal{G}$ fibred in groupoids over κ-schemes. A comodule over (R, A) is the same as a quasi-coherent sheaf on $\mathcal{G}$, or, equivalently, a morphism of fibred categories from $\mathcal{G}$ to the fibred category of quasi-coherent sheaves, [13, 3.3].

Given a morphism of rings $R \to S$ and a Hopf algebroid (R, A), there is a canonical structure of a Hopf algebroid on the extension of scalars $(S, {}_SA_S)$, where

$${}_SA_S := (S \otimes S) \otimes_{(R \otimes R)} A.$$

The extension of scalars also induces a functor

$$\mathbf{Comod}(R, A) \to \mathbf{Comod}(S, {}_SA_S), \quad M \mapsto S \otimes_R M.$$

If (R, A) is a Hopf algebroid over a ring κ and κ' is an algebra over κ, then $(R_{\kappa'}, A_{\kappa'})$ is a Hopf algebroid over κ' with

$$R_{\kappa'} := \kappa' \otimes_\kappa R \quad \text{and} \quad A_{\kappa'} := \kappa' \otimes_\kappa A.$$

For a Hopf algebroid (R, A) over a ring κ, suppose that A is a *faithfully flat* module over $R \otimes_\kappa R$, that is,

$$N \mapsto A \otimes_{(R \otimes_\kappa R)} N$$

is a faithful exact functor on the category of modules over $R \otimes_\kappa R$. Then a very important fact is that any R-finitely generated comodule M in $\mathbf{Comod}^{fg}(R, A)$ is a projective R-module, [13, 1.9,3.5]. It follows that $\mathbf{Comod}^{fg}(R, A)$ is a Tannakian category with the forgetful fiber functor

$$\omega : \mathbf{Comod}^{fg}(R, A) \to \mathbf{Mod}(R)$$

(Section A.2). Further, given a morphism of κ-algebras $R \to S$, the extension of scalars ${}_SA_S$ is faithfully flat over $S \otimes_\kappa S$ and the functor

$$S \otimes_R - : \mathbf{Comod}^{fg}(R, A) \to \mathbf{Comod}^{fg}(S, {}_SA_S)$$

is an equivalence of categories, [13, 1.8,3.5].

A.2. *Tannakian categories*

General references for Tannakian categories are [13] and [15]; also, an outline is given in [67, B.3]. By a *tensor category*, we mean a category C together with a functor

$$C \times C \to C, \quad (X, Y) \mapsto X \otimes Y, \tag{A.1}$$

a unit object $\mathbb{1}_C$, and functorial isomorphisms

$$X \otimes \mathbb{1}_C \cong X, \quad X \otimes Y \cong Y \otimes X, \quad X \otimes (Y \otimes Z) \cong (X \otimes Y) \otimes Z$$

that satisfy a set of axioms, which can be found in the references above. A functor $F : \mathcal{C} \to \mathcal{D}$ between tensor categories is *tensor* if there are functorial isomorphisms

$$F(X) \otimes F(Y) \cong F(X \otimes Y), \quad F(\mathbb{1}_{\mathcal{C}}) \cong \mathbb{1}_{\mathcal{D}} \tag{A.2}$$

that are compatible with the commutativity and associativity isomorphisms above. A *morphism between tensor functors*, $F \to G$, is a morphism between functors that commutes with the isomorphisms (A.2) for F and G. Denote the category of tensor functors between tensor categories $\mathcal{C}$ and $\mathcal{D}$ by $\mathbf{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$.

An *internal Hom object* $\mathcal{H}om_{\mathcal{C}}(X, Y)$ in a tensor category $\mathcal{C}$ is an object that represents the functor from $\mathcal{C}$ to the category of sets $U \mapsto \mathrm{Hom}_{\mathcal{C}}(U \otimes X, Y)$, that is, there is a functorial isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(U, \mathcal{H}om_{\mathcal{C}}(X, Y)) \cong \mathrm{Hom}_{\mathcal{C}}(U \otimes X, Y).$$

An internal Hom object $\mathcal{H}om_{\mathcal{C}}(X, Y)$ is unique up to a canonical isomorphism if it exists. Denote the internal Hom object $\mathcal{H}om_{\mathcal{C}}(X, \mathbb{1})$ by $X^{\vee}$. An object X in $\mathcal{C}$ is *dualizable* if, for any object Y, there exists the internal Hom object $\mathcal{H}om_{\mathcal{C}}(X, Y)$ and the natural morphism

$$X^{\vee} \otimes Y \to \mathcal{H}om_{\mathcal{C}}(X, Y)$$

is an isomorphism, [13, 2.3]. A tensor category $\mathcal{C}$ is *rigid* if all objects in $\mathcal{C}$ are dualizable.

Let $\mathcal{C}$ and $\mathcal{D}$ be tensor categories. Then, for any tensor functor $F : \mathcal{C} \to \mathcal{D}$ and any dualizable object X in $\mathcal{C}$, the object $F(X)$ is also dualizable and the natural morphism

$$F(\mathcal{H}om_{\mathcal{C}}(X, Y)) \to \mathcal{H}om_{\mathcal{D}}(F(X), F(Y))$$

is an isomorphism for any object Y in $\mathcal{C}$, [13, 2.7]. If $\mathcal{C}$ is rigid, then any morphism between tensor functors from $\mathcal{C}$ to $\mathcal{D}$ is an isomorphism, [13, 2.7].

Recall that, in an *abelian category*, morphisms between objects form abelian groups, there is a zero object, there are finite direct sums of objects, and there are kernels and cokernels of morphisms, satisfying some conditions. In particular, an analogue of the homomorphism theorem for groups is satisfied. Also, in an abelian category, exact sequences are well-defined. A functor $F : \mathcal{C} \to \mathcal{D}$ between abelian categories is *(left, right)-exact* if it sends (left, right)-exact sequences to (left, right)-exact sequences.

By an *abelian tensor category*, we mean a tensor category such that the tensor product functor is additive and right-exact on both arguments. Let $F : \mathcal{C} \to \mathcal{D}$ be a right-exact tensor functor between abelian tensor categories with $\mathcal{C}$ being rigid. Then, the functor F is exact, [13, 2.10(i)], and *faithful*, that is, injective on morphisms with the same source and target [13, 2.13(ii)].

For an abelian rigid tensor category $\mathcal{C}$ and an object X in $\mathcal{C}$, denote the minimal full rigid tensor subcategory in $\mathcal{C}$ that contains X and is closed under taking subquotients by $\langle X \rangle_{\otimes}$. We say that $\langle X \rangle_{\otimes}$ *is tensor generated by* X. It follows that $\langle X \rangle_{\otimes}$ is an abelian subcategory in $\mathcal{C}$.

Let R be a commutative ring. An *R-linear category* $\mathcal{C}$ is an additive category $\mathcal{C}$ such that, for all objects X, Y in $\mathcal{C}$, the group of morphisms $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is given with an R-module structure and the composition of morphisms is R-bilinear, that is, induces morphisms of R-modules

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \otimes_R \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$$

for all object X, Y, and Z in $\mathcal{C}$. A functor $F : \mathcal{C} \to \mathcal{D}$ between R-linear categories is *R-linear* if it induces R-linear maps

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y)).$$

Denote the category of R-linear functors between R-linear categories $\mathcal{C}$ and $\mathcal{D}$ by $\mathbf{Fun}_R(\mathcal{C}, \mathcal{D})$.

Given a tensor category $\mathcal{C}$, a ring homomorphism $R \to \mathrm{End}_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}})$ induces an R-linear category structure on $\mathcal{C}$. By an *R-linear tensor category* we mean a tensor category with an R-linear structure obtained as above. Equivalently, one requires that the tensor product functor (A.1) is R-linear in both variables. For example, for a finite commutative group G, the tensor category $\mathbf{Rep}(G)$ is $k[G]$-linear, but it is not a $k[G]$-linear tensor category with the tensor structure given by the usual tensor product of representations. On the other hand, $\mathbf{Rep}(G)$ is a k-linear tensor category.

A *Tannakian category over a field k* is an abelian rigid tensor category $\mathcal{C}$ with a fixed isomorphism $\mathrm{End}_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}}) \cong k$ such that there exist a k-algebra R and a right-exact k-linear tensor functor $\omega : \mathcal{C} \to \mathbf{Mod}(R)$. The functor ω is called a *fiber functor*. It follows from the above that ω is exact and faithful. A Tannakian category $\mathcal{C}$ is *neutral* if, in the above notation, one can take $R = k$, that is, there exists a fiber functor $\omega : \mathcal{C} \to \mathbf{Vect}(k)$.

Given a Tannakian category $\mathcal{C}$ and two fiber functors

$$\omega, \eta : \mathcal{C} \to \mathbf{Mod}(R),$$

denote the set of all tensor isomorphisms between ω and η by $\mathrm{Isom}^{\otimes}(\omega, \eta)$. Given an R-algebra S, one has the fiber functor

$$\omega_S : \mathcal{C} \to \mathbf{Mod}(S), \quad X \mapsto S \otimes_R \omega(X).$$

Note that the functor ω_S is denoted by $S \otimes_R \omega$ in [13]. It is more convenient for us to reserve the notation $S \otimes_R \omega$ for the extension of scalars of the functor defined in Section 4.1. The functor

$$\underline{\mathrm{Isom}}^{\otimes}(\omega, \eta) : \mathbf{Alg}(R) \to \mathbf{Sets}, \quad S \mapsto \mathrm{Isom}^{\otimes}(\omega_S, \eta_S),$$

is corepresented by an R-algebra A, [13]. In particular, the identity map from A to itself corresponds to a canonical isomorphism of tensor functors $\omega_A \xrightarrow{\sim} \eta_A$.

Proposition A.13. *In the above notation, suppose that $\mathcal{C}$ is tensor generated by an object X. Then the R-algebra A is generated by the matrix entries of the canonical isomorphism*

$$\omega(X)_A \xrightarrow{\sim} \eta(X)_A$$

and the matrix entries of its inverse with respect to any choice of systems of generators of $\omega(X)_A$ and $\eta(X)_A$ over A.

Proof. Let B be a k-subalgebra in A generated by the matrix entries as in the proposition. We need to show that $B = A$. Given projective B-modules P and Q, the extension of scalars map

$$\mathrm{Hom}_B(P, Q) \to \mathrm{Hom}_A(P_A, Q_A)$$

is injective, because P and Q are direct summands in free B-modules and B is embedded into A. Therefore, by the universal property of A, it is enough to prove that the canonical isomorphisms

$$\omega(Y)_A \xrightarrow{\sim} \eta(Y)_A$$

are defined over B, where Y runs through all objects in $\mathcal{C}$. By the construction of B, this is true for the tensor generator X. Further, this property is preserved under taking direct sums, tensor products, and duals of objects in $\mathcal{C}$.

It remains to show that this property is preserved under taking subquotients, for which it is enough only to consider subobjects. Assume that there is an isomorphism of B-modules $\lambda : \omega(Y)_B \xrightarrow{\sim} \eta(Y)_B$ whose extension of scalars λ_A is equal to the canonical isomorphism

$$\omega(Y)_A \xrightarrow{\sim} \eta(Y)_A.$$

Given a subobject $Z \subset Y$, consider the composition

$$\mu : \omega(Z)_B \to \omega(Y)_B \xrightarrow{\lambda} \eta(Y)_B \to \eta(Y/Z)_B.$$

Since $\mu_A = 0$ and $\omega(Z)_B$, $\eta(Y/Z)_B$ are projective B-modules, we have that $\mu = 0$. Hence, $\lambda(\omega(Z)_B) = \eta(Z)_B$, which implies the needed condition for Z. $\square$

Theorem A.14. *Let C be a Tannakian category over k and let $\omega : C \to \mathbf{Mod}(R)$ be a fiber functor. Then there exists a Hopf algebroid (R, A) over k such that A is faithfully flat over $R \otimes_k R$ and ω lifts up to a tensor k-linear equivalence of tensor categories*

$$C \xrightarrow{\sim} \mathbf{Comod}^{fg}(R, A).$$

That is, for any object X in C, there is a functorial in X structure of a comodule over (R, A) on $\omega(X)$ giving the above equivalence ([13, 1.12]).

In particular, for a neutral Tannakian category (C, ω), there exists a Hopf algebra A over k such that ω lifts up to an equivalence between C and $\mathbf{Comod}^{fg}(A)$ (equivalently, there exists an affine group scheme G over k such that ω induces an equivalence between C and $\mathbf{Rep}^{fg}(G)$). The Hopf algebroid A from Theorem A.14 corepresents the functor

$$\underline{\mathrm{Isom}}^{\otimes}({}_{R\otimes R}\,\omega, \omega_{R\otimes R}) : \mathbf{Alg}(R \otimes_k R) \to \mathbf{Sets},$$

where, as above, we put

$$({}_{R\otimes R}\,\omega)(X) := (R \otimes_k R) \otimes_R \omega(X) \cong R \otimes_k \omega(X), \quad (\omega_{R\otimes R})(X) := \omega(X) \otimes_R (R \otimes_k R) \cong \omega(X) \otimes_k R.$$

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