Computational complexity of solving polynomial differential equations over unbounded domains

Amaury Pouly
Joint work with Daniel Graça

10 May 2018

Ordinary Differential Equations (ODEs)

System of ODEs:

$$\begin{cases} y_1(0) = y_{0,1} \\ \qquad \vdots \\ y_n(0) = y_{0,n} \end{cases} \qquad \begin{cases} y_1'(t) = f_1(y_1(t), \ldots, y_n(t), t) \\ \qquad \vdots \\ y_n'(t) = f_n(y_1(t), \ldots, y_n(t), t) \end{cases}$$

More compactly:

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In this talk: autonomous first order explicit system of ODEs

$$y(0) = y_0 \qquad y' = f(y) \qquad y : (a, b) \to \mathbb{R}^n$$

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In this talk (unless specified)

We use Computable Analysis.

Computability of solutions: the theory

Let $I = (a, b)$ and $f \in C^0(\mathbb{R}^n)$. Assume $y \in C^1(I, \mathbb{R}^n)$ satisfies $\forall t \in I$:

$$y(0) = 0, \qquad y'(t) = f(y(t)). \tag{1}$$

Given $t \in I$ and $n \in \mathbb{N}$, can we compute $q \in \mathbb{Q}^n$ s.t. $\|q - y(t)\| \leqslant 2^{-n}$?

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There exists a computable (hence continuous) f such that **none of the solutions** to (1) is computable.

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Theorem (Collins and Graça)

The map $f \mapsto y(\cdot)$ for those f where y is unique, is computable.

Complexity of solutions: typical textbook result

Assume f **Lipschitz** and computable, and $y : I \to \mathbb{R}^n$ satisfies $\forall t \in I$:

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Problems with this approach:

- Accuracy of the result? $\mathcal{O}\left(h^4\right) \leqslant Ah^4$ but A is unknown
- Same problem with complexity
- f is Lipschitz: typically only holds over compact domains

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$$\frac{hM}{2K}\left(e^{Kt} - 1\right) \qquad \text{where } M = \sup_{u \in I} \left\| y''(u) \right\|.$$

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- you know K: f must be Lipschitz on "$\{y(u) : u \in I\}$" or globally
- you know M: but it depends on y !!

Chicken-and-egg problem: the constant in the accuracy bound depends on computing the solution.

Complexity of solutions: the rescaling "myth"

Assume f computable and K-Lipschitz, and $y : I \to \mathbb{R}^n$ satisfies $\forall t \in I$:

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To compute $y(T)$ we could:

1. Define $z(u) = y(Tu)$, then
$$y(T) = z(1)$$
2. Observe that
$$z'(u) = Tf(z) =: f_T(z)$$
3. Solve $z(0) = y_0, z' = f_T(z)$
$[0, 1]$ is a compact!

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Accuracy: $O(h)$ (compact)

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Better analysis:
Accuracy: $A_{K_T,M_z} h$ where

$K_T =$ Lipschitz constant of f_T

$$M_z = \max_{u\in[0,1]} \left\|z''(u)\right\| = \max_{t\in[0,T]} \left\|y''(t)\right\|$$

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Note: now f really needs to be globally Lipschitz.

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Conclusion

This tells us **nothing** about the complexity of the problem.

Side note on practical methods

Assume $y : [0,1] \to \mathbb{R}^n$ satisfies $\forall t \in [0,1]$:

$$y(0) = 0, \qquad y'(t) = f(y(t)).$$

There exists methods of the form:
given h and t, compute $q \in \mathbb{Q}^n$ and $\varepsilon > 0$ such that $\|y(t) - q\| \leqslant \varepsilon$ with the guarantee that $\varepsilon \to 0$ as $h \to 0$.

These methods have **no upper bound** on complexity.

They usually rely on interval arithmetic.

Nonuniform complexity-theoretic approach

Assume $y : [0,1] \to \mathbb{R}^n$ satisfies $\forall t \in [0,1]$:

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Assumption on f	**Lower bound on y**	**Upper bound on y**
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PTIME + analytic	—	PTIME

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PTIME + analytic	—	PTIME

But those results can be deceiving...

$$\begin{cases} y_1(0) = 1 \\ y_2(0) = 1 \\ \vdots \\ y_n(0) = 1 \end{cases} \qquad \begin{cases} y_1' = y_1 \\ y_2' = y_1 y_2 \\ \vdots \\ y_n' = y_{n-1} y_n \end{cases} \quad \to \quad y(t) = \mathcal{O}\left(e^{e^{\cdot^{\cdot^{\cdot^{e^t}}}}}\right)$$

y is PTIME over $[0,1]$

Nonuniform complexity: limitation

Example:

$$f \text{ PTIME analytic } \Rightarrow y \text{ PTIME } \Rightarrow y(t) \pm 2^{-n} \text{ in time } An^k$$

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Conclusion

This only **slightly** better than the previous approach.

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Assume $y : I \to \mathbb{R}^d$ satisfies $\forall t \in I$:

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where $f : \mathbb{R}^n \to \mathbb{R}^n$ is .... Then $y(t) \pm 2^{-n}$ can be computed in time

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where

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Problem: we cannot predict the behaviour of y based on f only.

Are you confused?

You should be!

- practical methods: “no complexity”
- nonuniform complexity: misleading
- uniform worst-case complexity: everything looks hard

Are you confused?

You should be!

- practical methods: “no complexity”
- nonuniform complexity: misleading
- uniform worst-case complexity: everything looks hard

Question: are we looking at the problem the wrong way?

Parametrized complexity approach

Goal: Assume $y : I \to \mathbb{R}^d$ satisfies $\forall t \in I$:

$$y(0) = 0, \qquad y'(t) = f(y(t)),$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is nice. Then $y(t) \pm 2^{-n}$ can be computed in time

$$\text{poly}(t, n, K_d, K_f, K_y(t))$$

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- K_d: depends on the dimension d
- K_f: depends on f and its representation
- K_y: is a reasonable parameter of y that must be unknown to the algorithm (i.e. not part of the input)

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Important differences with “textbook” approach:

- Result is always correct
- K_y not assumed to be known (e.g. K and M of previous slides)

Parametrized complexity result

Assume $y : I \to \mathbb{R}^d$ satisfies $\forall t \in I$:

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where $p : \mathbb{R}^n \to \mathbb{R}^n$ is vector of multivariate polynomials.

Theorem (TCS 2016)

Assuming $t \in I$, computing $y(t) \pm 2^{-n}$ takes time:

$$\operatorname{poly}(\deg p, \log \Sigma p, n, \ell_y(t))^d$$

where:

- Σp: sum of absolute value of coefficients of p

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where:

- Σp: sum of absolute value of coefficients of p
- $\ell_y(t)$: "length" of y over $[0, t]$

$$\ell_y(t) = \int_0^t \max(1, \|y'(u)\|) du$$

Note: the algorithm find $\ell(t)$ automatically, **it is not part of the input**

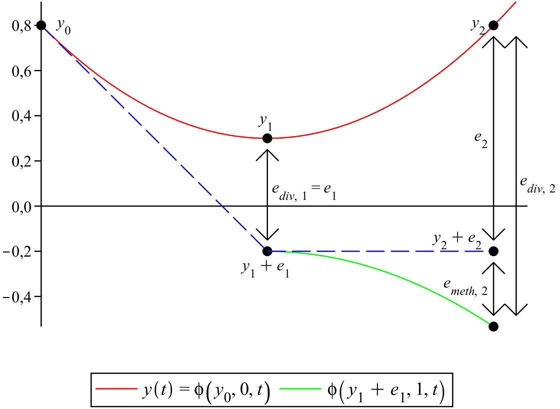
Euler method

$$y(0) = 0 \qquad y'(t) = p(y(t))$$

Time step h, discretize and compute $\tilde{y}^i \approx y(ih)$:

$$y(t+h) \approx y(t) + hy'(t) \quad \rightsquigarrow \quad \tilde{y}^{i+1} = \tilde{y}^i + hp(\tilde{y}^i)$$

Linear approximation at each step.



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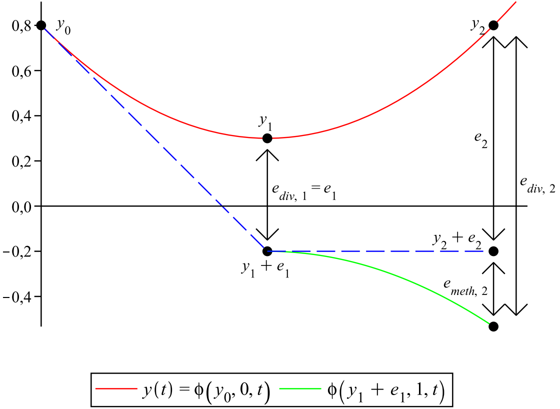
Linear approximation at each step. Does not work well in practice.



Taylor method

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Time step h, discretize and compute $\tilde{y}^i \approx y(ih)$:

$$y(t+h) \approx y(t) + \sum_{i=1}^{\omega} h^i y^{(i)}(t) \qquad \text{using } y^{(i)}(t) = \mathrm{poly}_i(y(t))$$

Do a ω-th order Taylor approximation at each step.

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Do a ω-th order Taylor approximation at each step.

Works well for $\omega \geqslant 3$ but

- How to choose h and ω ? One more parameter to choose!
- Error analysis is less obvious
- Complexity increases with ω

Adaptive Taylor method

Adapt h and ω at each step.

$$y(0) = 0 \qquad y'(t) = p(y(t))$$

Time step h_i, discretize and compute $\tilde{y}^i \approx y(\sum_{j \leqslant i} h_i)$:

$$y(t + h_i) \approx y(t) + \sum_{i=1}^{\omega_i} h_i^i y^{(i)}(t) \qquad \text{using } y^{(i)}(t) = \text{poly}_i(y(t))$$

Do a ω_i-th order Taylor approximation at each step.

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Do a ω_i-th order Taylor approximation at each step.

Adapt the amount of computation to the hardness of the problem but

- Many more parameters to choose
- Error analysis is challenging
- Complexity analysis usually not done

Adaptive Taylor method: parameter choice

How to choose the time steps h_i and orders ω_i:

- h_i: estimate the radius of convergence
- ω_i: try to guess the accuracy loss

Use voodoo magic and interval arithmetic to ensure correctness.

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Our idea: we need to choose h_i, ω_i based on some high-level geometrical feature.

Our algorithm in one sentence: choose h_i, ω_i so that

at each step, we increase the length of the solution by 1

Interesting (practical ?) consequences

Compute $y(t) \pm \varepsilon$

Method	Max. Order	Number of steps
Fixed ω	$\omega - 1$	$\mathcal{O}\left(L^{\frac{\omega+1}{\omega-1}} \varepsilon^{-\frac{1}{\omega-1}}\right)$

where $$L \approx \int_0^t \max(1, \|y'(u)\|) du$$

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Euler ($\omega = 2$)	1	$\mathcal{O}\left(\frac{L^3}{\varepsilon}\right)$

where $\qquad L \approx \int_0^t \max(1, \|y'(u)\|) du$

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Variable	$\mathcal{O}\left(\log \frac{L}{\varepsilon}\right)$	$\mathcal{O}(L)$

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Conclusion

Solving Ordinary Differential Equations numerically:

- vastly different algorithms/results for vastly different expectations
- practical methods: no complexity
- nonuniform complexity: imprecise/misleading
- uniform worst-case complexity: everything is hard
- uniform parametrized complexity: encouraging

Questions:

- how far can we push parametrized complexity?
- can theory bring insight to practice?
- geometric complexity?

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Lemma: $y^{(k)}(t) = P_k(y(t)) = \text{poly}(y(t))$

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- **Fixed** h: wasteful

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What about h ?

- **Fixed** h: wasteful
- **Adaptive** h: choose h depending on i, p, n and $\tilde{y}^i$

Choice of the parameters

Choice of h based on an effective lower bound on radius of convergence of the Taylor series:

Lemma: If $y' = p(y)$, $\alpha = \max(1, \|y_0\|)$, $k = \deg(p)$, $M = (k-1)\Sigma p \alpha^{k-1}$ then:

$$\left\| y^{(k)}(t) - P_k(y(t)) \right\| \leqslant \frac{\alpha (Mt)^k}{1 - Mt}$$

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I spare you the analysis of the global error !

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This is impossible, right ?!

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Example

$$\begin{cases} x(t) = t^{u(t)} \\ u(t) = e^{-t} - (1 - e^{-t})\frac{1}{v(t)} \\ v(t) = v_0 \end{cases} \rightsquigarrow \begin{cases} x(t) \sim t^{\frac{1}{v_0}} \\ u(t) \to \frac{1}{v_0} \\ v(t) = v_0 \end{cases}$$

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Theorem

There is no universal bound in p, y_0, t_0, t and μ.