HOMOTOPY PROBABILITY THEORY I

GABRIEL C. DRUMMOND-COLE, JAE-SUK PARK, AND JOHN TERILLA

ABSTRACT. This is the first of two papers that introduce a deformation theoretic framework to explain and broaden a link between homotopy algebra and probability theory. In this paper, cumulants are proved to coincide with morphisms of homotopy algebras. The sequel paper outlines how the framework presented here can assist in the development of homotopy probability theory, allowing the principles of derived mathematics to participate in classical and noncommutative probability theory.

1. INTRODUCTION

The second author found a link between homotopy algebra and probability [4]. This paper introduces a deformation theoretic framework to explain and broaden the link between homotopy algebra and probability theory. This framework is then used in a sequel paper [1] to develop a homotopy theory of probability.

In probability theory one considers an algebra of random variables and a linear map e from this algebra to the complex numbers $\mathbb{C}$. Typically, the map e does not respect the product structure; that is for random variables X and Y,

$$e(XY) \neq e(X)e(Y).$$

In fact, the failure of e to be an algebra map measures important correlations between random variables. For example, the bilinear map defined on the space of random variables by

$$\kappa_2(X, Y) := e(XY) - e(X)e(Y)$$

defines the covariance. The map κ_2 fits into an infinite hierarchy of multilinear maps $\{\kappa_n\}$ called *cumulants*. The important notion that a set of random variables $\{X_1, \dots, X_n\}$ be *independent* is defined using the vanishing of the cumulants.

This paper presents a mathematical framework for studying linear maps between algebras that do not respect the products. The framework is a manifestation of the following idea:

> *the failure of a map to respect structure has structure, if you know where to look.*

Specifically, Section 2 contains a construction, which in a simple case has the following input and output:

Input:

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- chain complexes $C = (V, d)$ and $C' = (V', d')$
- degree zero bilinear maps $V \times V \to V$ and $V' \times V' \to V'$
- a chain map $f : C \to C'$

Output:

- a sequence of degree zero multlinear maps $\{f_n : V^n \to V'\}_{n=1}^{\infty}$.

For the input, the differentials and the chain map are not assumed to have any compatibility with the bilinear maps. The sequence of maps $\{f_n\}$ in the output constitute an A_∞ algebra morphism between two A_∞ algebras which arise during the construction.

A probability space provides an example of the input data for the construction. The chain complexes C and C' are, respectively, $(V, 0)$ and $(\mathbb{C}, 0)$, the space of random variables and the complex numbers both with zero differential. The bilinear maps are given by the products of random variables and complex numbers. The chain map $C \to C'$ is the expectation $e : V \to \mathbb{C}$. Thus, the construction can be applied producing an A_∞ morphism

$$\{e_n : V^n \to \mathbb{C}\}_{n=1}^{\infty}.$$

The main result of this paper is that this A_∞ morphism coincides with the cumulants: $\kappa_n = e_n$ for all n.

The paper proceeds as follows. Section 2 contains the details of the construction of the output A_∞ morphism and includes a brief introduction to A_∞ algebras and morphisms. Section 3 contains the definition of the cumulants and the main proposition.

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2. A_∞ ALGEBRAS AND MORPHISMS

The book [3] is good reference for A_∞ algebras.

2.1. **Definitions.** Let V be a graded vector space. Let $V^{\otimes n}$ denote the nth tensor power of V and $TV = \oplus_{n=1}^{\infty} V^{\otimes n}$. As a direct sum, linear maps from TV to a vector space W correspond to collections of linear maps $\{V^{\otimes n} \to W\}_{n=1}^{\infty}$. Also, TV is a coalgebra, free in a certain sense, so that coalgebra maps from a coalgebra $\mathcal{C}$ to TV correspond to linear maps $\mathcal{C} \to V$. This freeness also implies that coderivations from a coalgebra $\mathcal{C}$ to TV correspond to linear maps $\mathcal{C} \to V$. All these correspondences are one-to-one: any linear map $C \to V$ can be lifted uniquely to a coderivation $\mathcal{C} \to TV$, or lifted uniquely to a coalgebra map $\mathcal{C} \to TV$.

Definition 1. *An A_∞ algebra* is a pair (V, D) where V is a graded vector space and $D : TV \to TV$ is a degree one[1] coderivation satisfying $D^2 = 0$. *An A_∞ morphism* between two A_∞ algebras (V, D) and (V', D') is a differential coalgebra map $F : (TV, D) \to (TV', D')$. In other words, an A_∞ map from (V, D) to (V', D') is a degree zero coalgebra map $F : TV \to TV'$ satisfying $FD = D'F$.

The identification

$$\operatorname{Coder}(TV, TV) \simeq \prod_{n=1}^{\infty} \hom(V^{\otimes n}, V) \tag{1}$$

can be used to give the data of an A_∞ algebra. That is, a coderivation $D : TV \to TV$ can be given by a sequence $\{d_n\}$ of degree one linear maps $d_n : V^{\otimes n} \to V$ on the graded

[1] Readers familiar with A_∞ algebras will be aware that the definition of an A_∞ algebra sometimes involves a shift of degree, but no degree shifts are used in this paper.

vector space V. The condition that $D^2 = 0$ implies an infinite number of relations satisfied by various compositions among the $\{d_n\}$. Likewise, the identification

$$\mathrm{Coalg}(TV, TV') \simeq \prod_{n=1}^{\infty} \hom(V^{\otimes n}, V') \tag{2}$$

can be used to give the data of an A_∞ morphism between (V, D) and (V', D'). That is, a coalgebra map $F : TV \to TV'$ can be given by a sequence $\{f_n\}$ of degree zero maps $f_n : V^{\otimes} \to V'$. The condition that $FD = D'F$ encodes an infinite number of relations among the $\{f_n\}$, the $\{d_n\}$ and the $\{d'_n\}$. So, the identifications in Equations (1) and (2) provide two equivalent ways of describing A_∞ algebras and morphisms and it is convenient to move between the ways since certain notions or computations are easier to describe in one or the other descriptions of the equivalent data. For example, it is straightforward to compose two differential coalgebra maps $F : (TV, D) \to (TV', D')$ and $G : (TV', D') \to (TV'', D'')$ as $GF : (TV, D) \to (TV'', D'')$ and thus define the composition of A_∞ morphisms, but it is more involved to express the components $(gf)_n : V^{\otimes n} \to V''$ in terms of the $f_k : V^{\otimes k} \to V'$ and $g_m : V'^{\otimes m} \to V''$.

Note that A_∞ structures can be transported via isomorphisms. In particular, if (V, D) is an A_∞ algebra and $G : TV \to TW$ is any degree zero coalgebra isomorphism, then for

$$D^G := G^{-1} D G,$$

the pair (W, D^G) is again an A_∞ algebra.

Morphisms can be transported as well. If F is an A_∞ morphism between (V, D) and (V', D') and $G : TV \to TW$ and $H : TV' \to TW'$ are coalgebra isomorphisms, then

$$F^{G,H} := H^{-1} F G$$

is an A_∞ morphism between the A_∞ algebras (W, D^G) and $(W', (D')^H)$.

Definition 2. Two A_∞ algebras (V, D) and (V', D') are *equivalent* if there exists a coalgebra isomorphism $G : TV \to TV'$ so that $D' = D^G$.

2.2. **Spaces of A_∞ algebras.** Consider an A_∞ algebra (V, D). The condition that D has degree one and that $D^2 = 0$ imply that the first component $d_1 : V \to V$ of D has degree one and satisfies $d_1 \circ d_1 = 0$. So, the pair (V, d_1) is a chain complex. Often it is appropriate to view the chain complex (V, d_1) as a fundamental object and to consider the remaining components $d_2, d_3, \ldots$ of D as structure on the chain complex (V, d_1).

Definition 3. Let $C = (V, d)$ be a chain complex. An A_∞ *structure on* C is an A_∞ algebra (V, D) with $d_1 = d$. Let $\mathcal{M}_C$ denote the set of A_∞ structures on C.

Definition 4. Let $C = (V, d)$ be a chain complex. The *gauge group* $\mathcal{G}_C$ is the subgroup of $\mathrm{GL}(TV)$ consisting of degree zero coalgebra automorphisms with first component $\mathrm{id} : V \to V$. The gauge group $\mathcal{G}_C$ acts, on the right, by conjugation on $\mathcal{M}_C$:

$$D \cdot G = D^G.$$

2.3. **The gauge group $\mathcal{G}_C$.** The *gauge group* $\mathcal{G}_C$ is a Lie subgroup of $\mathrm{GL}(TV)$. The Lie algebra of $\mathcal{G}_C$ is the Lie subalgebra of $\mathrm{gl}(TV)$ consisting of all degree zero coderivations $TV \to TV$ with first component $0 : V \to V$. This Lie subalgebra can be identified via Equation (1) with the vector space of degree zero maps $\prod_{k=2}^{\infty} \hom(V^{\otimes k}, V)$. Any map $a : V^{\otimes k} \to V$ with $k > 1$ can be lifted to a coderivation $A : TV \to TV$ and then exponentiated to obtain a gauge group element $\mathbf{a} \in \mathcal{M}_C$ where

$$\mathbf{a} = \exp(A) := \mathrm{id} + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

The orbits of the one parameter subgroup $t \mapsto \exp(tA)$ of the gauge group $\mathcal{G}_C$ are curves in the space $\mathcal{M}_C$. Each of these curves connects an A_∞ structure D at $t = 0$ to an equivalent A_∞ structure $D^{\mathbf{a}}$ at $t = 1$.

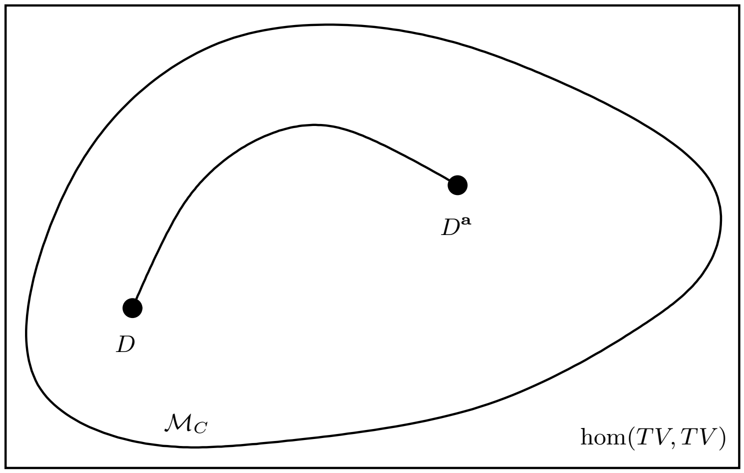

FIGURE 1. A picture of the family of A_∞ structures $\mathcal{M}_C$ on a chain complex $C = (V, d)$ as a subset of $\hom(TV, TV)$. There is a basepoint $D : TV \to TV$ which is the lift of $d : V \to V$ as a coderivation—the basepoint corresponds to C itself with zero additional structure. The path is part of the orbit of the one parameter subgroup of the gauge group passing through the basepoint D at $t = 0$ and an equivalent A_∞ structure $D^{\mathbf{a}} : TV \to TV$ at $t = 1$.

Remark 1. In classical deformation theory, equivalent structures are identified to form a quotient moduli set. Rather than identifying equivalent structures, a simplicial moduli space can be constructed. The points of the simplicial moduli space consist of all structures. The paths in the simplicial moduli space consist of equivalences between structures. Higher dimensional parts correspond to equivalences between equivalences. This paper involves A_∞ structures which are equivalent to trivial A_∞ structures. In order to see the application to probability theory, gauge equivalent structures should not be identified, so the relevant moduli space is the simplicial moduli space.

2.4. **Spaces of A_∞ morphisms.** Consider an A_∞ morphism $F : TV \to TV'$ between A_∞ algebras (V, D) and (V', D'). The conditions that F has degree zero and that $FD = D'F$ imply that the first component $f_1 : V \to V'$ has degree zero and satisfies $f_1 d_1 = d'_1 f_1$. Thus $f_1 : (V, d_1) \to (V', d'_1)$ is a chain map. Here, the chain map $f_1 : (V, d_1) \to (V', d'_1)$ is viewed as a fundamental object and the remaining components $f_2, f_3, \ldots$ of F are viewed as a structure on the chain map f_1.

Definition 5. Let $C = (V, d)$ and $C' = (V', d')$ be chain complexes and let $f : C \to C'$ be a chain map. An A_∞ morphism on f is an A_∞ morphism $F : (TV, D) \to (TV, D')$ between two A_∞ structures on C and C' with $f_1 = f$. Let

$$\mathcal{M}_f \subset \hom(TV, TV) \times \hom(TV, TV') \times \hom(TV', TV')$$

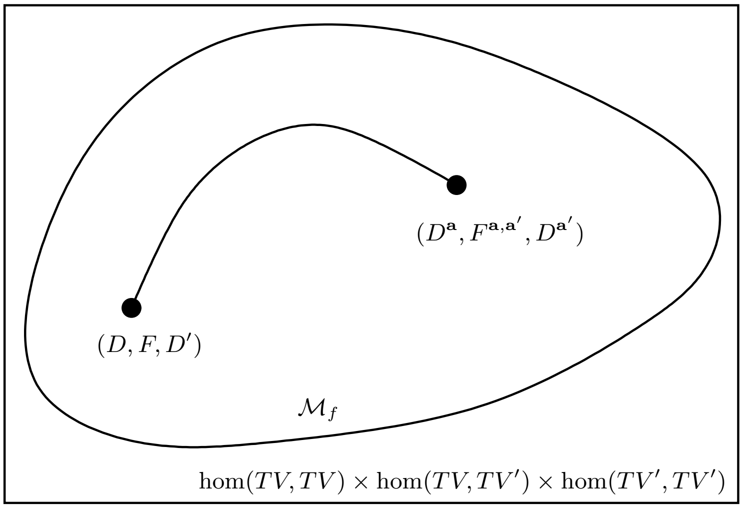

FIGURE 2. The picture for morphisms equivalent to chain maps is similar to that of structures equivalent to chain complexes.

be the set of A_∞ morphisms on f between A_∞ structures on C and C'. That is, a triple $(D, F, D') \in \mathcal{M}_f$ consists of degree one, square zero coderivations $D : TV \to TV$ and $D' : TV' \to TV'$ with $d_1 = d$ and $d_1' = d'$ and a degree zero coalgebra map $F : TV \to TV'$ satisfying $FD = D'F$ with $f_1 = f$.

The product of gauge groups $\mathcal{G}_C \times \mathcal{G}_{C'}$ acts, on the right, on $\mathcal{M}_f$ by

$$(D, F, D') \cdot (G, H) := (D^G, F^{H,G}, (D')^H).$$

Remark 2. A seemingly trivial situation will be important in the next section. An ungraded vector space V can be considered a chain complex $C = (V, 0)$ by setting the degree of every element of V to be zero and setting the differential $d = 0$. The gauge group $\mathcal{G}_C$ acts trivially on $\mathcal{M}_C$ when $d = 0$ since $G^{-1}0G = 0$ for any $G \in \mathcal{M}_C$. Any linear map $f : V \to V'$ between vector spaces V and V' is a chain map between $C = (V, 0)$ and $C' = (V', 0)$ when V and V' are regarded as chain complexes with zero differentials. If $a : V^{\otimes k} \to V$ and $a' : V'^{\otimes m} \to V'$, then a, a' can be lifted to coderivations A, A' and exponentiated to obtain coalgebra automorphisms $\mathbf{a} \in \mathcal{G}_C$ and $\mathbf{a}' \in \mathcal{G}_{C'}$. The A_∞ structures $D^{\mathbf{a}}$ and $D'^{\mathbf{a}'}$ on C and C' are identically zero, but the morphism $F^{\mathbf{a},\mathbf{a}'} : TV \to TV'$ is typically nonzero. That is, $F^{\mathbf{a},\mathbf{a}'}$ is a nonzero A_∞ morphism between two zero A_∞ structures.

For example, a straightforward computation in the simple case that $a : V^{\otimes 2} \to V$ and $a' : V'^{\otimes 2} \to V$ reveals the first two structure morphisms $f_1^{\mathbf{a},\mathbf{a}'} : V \to V'$ and $f_2^{\mathbf{a},\mathbf{a}'} : V^{\otimes 2} \to V'$ to be the maps defined by

$$f_1^{\mathbf{a},\mathbf{a}'}(X) = f(X)$$
$$f_2^{\mathbf{a},\mathbf{a}'}(X_1, X_2) = f(a(X_1, X_2)) - a'(f(X_1), f(X_2)).$$

3. PROBABILITY SPACES AND CUMULANTS

3.1. **Probability spaces.** One modern approach to probability theory (see, for example, [7]) begins with the following definition:

Definition 6. A *probability space* is a triple (V, e, a) where V is a complex vector space, $e : V \to \mathbb{C}$ is a linear function, and $a : V \times V \to V$ is an associative bilinear product. Elements of V are called *random variables* and the number $e(X)$ is called *the expected value* of the random variable $X \in V$. The notation $X_1 X_2$ is used for $a(X_1, X_2)$. Complex multiplication is denoted by a'. The notation α_n (and α'_n) is used for the multilinear maps $V^{\otimes n} \to V$ (and $\mathbb{C}^{\otimes n} \to \mathbb{C}$) obtained by repeated multiplication. The expectation values of products $e(X_1 \cdots X_n)$ are called *joint moments*.

The product on V is not assumed to be commutative. Elements of V are sometimes called *observables* when the product is not commutative, but here no special terminology is used to distinguish between commutative and noncommutative probability spaces.

3.2. **Cumulants.**

Definition 7. Let (V, e, a) be a probability space. The *nth cumulant* of (V, e, a) is the linear map $\kappa_n : V^{\otimes n} \to \mathbb{C}$ defined recursively by the following equation:

$$e \circ \alpha_n = \sum_{k=1}^{n} \alpha'_k \circ \left(\sum_P \bigotimes_{j=1}^{k} \kappa_{n_j} \right) \tag{3}$$

where P ranges over all ordered partitions $P = (n_1, \ldots, n_k)$ with $\displaystyle\sum_{j=1}^{k} n_j = n$.

Remark 3. The cumulants defined above have been called variously *partial cumulants*, *Waldenfels cumulants*, and *Boolean cumulants* [6]. The adjectives can help distinguish these cumulants from the *classical cumulants*, which are defined in the special case when the product on V is commutative, and the *free cumulants* which are important in free probability theory [5]. For a survey of various kinds of cumulants and their combinatorics, see [2].

Equation (3) expresses the joint moment $e(X_1 \cdots X_n)$ in terms of products of cumulants. For the first few values of n, this equation is given by:

$$e(X_1) = \kappa_1(X_1)$$
$$e(X_1X_2) = \kappa_2(X_1 \otimes X_2) + \kappa_1(X_1)\kappa_1(X_2)$$
$$\begin{aligned} e(X_1X_2X_3) = \kappa_3(X_1 \otimes X_2 \otimes X_3) &+ \kappa_1(X_1)\kappa_2(X_2 \otimes X_3) \\ &+ \kappa_2(X_1 \otimes X_2)\kappa_1(X_3) + \kappa_1(X_1)\kappa_1(X_2)\kappa_1(X_3) \end{aligned}$$

One easily solves these equations for the cumulants expressed in terms of products of joint moments:

$$\kappa_1(X_1) = e(X_1)$$
$$\kappa_2(X_1 \otimes X_2) = e(X_1X_2) - e(X_1)e(X_2)$$
$$\begin{aligned} \kappa_3(X_1 \otimes X_2 \otimes X_3) = e(X_1X_2X_3) &- e(X_1X_2)e(X_3) \\ &- e(X_1)e(X_2X_3) + e(X_1)e(X_2)e(X_3). \end{aligned}$$

and in general finds

$$\kappa_n = \sum_{k=1}^{n} (-1)^{k-1} \alpha'_k \circ \left(\sum_P \bigotimes_{j=1}^{k} e \circ \alpha_{n_j} \right). \tag{4}$$

The sum in Equation (4) above is over the same set: P ranges over all ordered partitions $P = (n_1, \dots, n_k)$ with $\displaystyle\sum_{j=1}^{k} n_j = n$.

3.3. **Main Proposition.**

Hypotheses for the Main Proposition. Let (V, e, a) be a probability space. Consider both V and the complex numbers $\mathbb{C}$ as graded vector spaces concentrated in degree zero. Then $(V, 0)$ and $(\mathbb{C}, 0)$ are A_∞ algebras and the map $e : V \to \mathbb{C}$ defines an A_∞ morphism between these two A_∞ algebras. Then, following the notation of Section 2.3, $E^{\mathbf{a},\mathbf{a}'} : TV \to T\mathbb{C}$ is an A_∞ morphism between $(V, 0^{\mathbf{a}}) = (V, 0)$ and $(\mathbb{C}, 0^{\mathbf{a}'}) = (\mathbb{C}, 0)$. Let $e_n : V^{\otimes n} \to \mathbb{C}$ denote the components of the A_∞ morphism $E^{\mathbf{a},\mathbf{a}'}$.

Main Proposition. *For all n, $\kappa_n = e_n$.*

Proof. Both the collection $\{\kappa_n : V^{\otimes n} \to \mathbb{C}, n \geq 1\}$ and the single map $e : V \to \mathbb{C}$ can be extended as coalgebra maps $TV \to T\mathbb{C}$. Let K and E, respectively, denote these extensions. The statement of the proposition is that these two coalgebra maps are related by means of the coalgebra isomorphisms $\mathbf{a}$ and $\mathbf{a}'$ as in the following diagram.

$$\begin{array}{ccc} TV & \xrightarrow{K} & T\mathbb{C} \\ {\scriptstyle\mathbf{a}}\big\downarrow & & \big\downarrow{\scriptstyle\mathbf{a}'} \\ TV & \xrightarrow{E} & T\mathbb{C} \end{array}$$

It suffices to check that the components

$$\begin{array}{ccccccc} TV & \xrightarrow{\mathbf{a}} & TV & \xrightarrow{E} & T\mathbb{C} & \longrightarrow & \mathbb{C} \\ TV & \xrightarrow{K} & T\mathbb{C} & \xrightarrow{\mathbf{a}'} & T\mathbb{C} & \longrightarrow & \mathbb{C} \end{array}$$

coincide when evaluated on a vector $X_1 \otimes \cdots \otimes X_n \in V^{\otimes n}$.

The map $TV \xrightarrow{E} T\mathbb{C} \to \mathbb{C}$ is zero except for the one to one component $e : V \to \mathbb{C}$. Recall that A denotes the lift of a to a coderivation $TV \to TV$ and $\mathbf{a} = \exp(A)$. So the only components of $\mathbf{a}$ that contribute to the composition in question are $\frac{1}{(n-1)!}A^{n-1} : V^{\otimes n} \to V$.

The nonzero components of A are of the form $a_n : V^{\otimes n} \to V^{\otimes n-1}$ and are given by

$$a_n(X_1 \otimes \cdots \otimes X_n) = \sum_{j=1}^{n-1} X_1 \otimes \cdots \otimes X_j X_{j+1} \otimes \cdots \otimes X_n.$$

Then the only nonzero component of $\frac{1}{(n-1)!}A^{n-1}$ is the composition $\frac{1}{(n-1)!}a_2 \circ \cdots \circ a_n$. The expression for a_n has $(n-1)$ terms, so this composition has $(n-1)!$ terms. Applied to $X_1 \otimes \cdots \otimes X_n$, this composition then yields $X_1 \cdots X_n$. So the composition along the left and bottom takes $X_1 \otimes \cdots \otimes X_n$ to

$$e(X_1 \cdots X_n).$$

The map $K : TV \to T\mathbb{C}$ evaluated on $(X_1 \otimes \cdots \otimes X_n)$ breaks into the following sum.

$$K(X_1 \otimes \cdots \otimes X_n) = \left(\sum_{k=1}^{n} \sum_{P} \bigotimes_{j=1}^{k} \kappa_{n_j} \right) (X_1 \otimes \cdots \otimes X_n) \tag{5}$$

where P ranges over all ordered partitions $P = (n_1, \ldots, n_k)$ with $\sum_{j=1}^{k} n_j = n$.

As in the calculation above for $\mathbf{a}$, the nonzero component of $\mathbf{a}'$ mapping $\mathbb{C}^{\otimes k} \to \mathbb{C}$ is $\frac{1}{(k-1)!}(A')^{k-1}$ which maps $z_1 \otimes \cdots \otimes z_n \mapsto z_1 \cdots z_n$. Hence, the composition along the top and right yields

$$\sum_{k=1}^{n} \alpha' \circ \left(\sum_{P} \bigotimes \kappa_{n_j} \right) \tag{6}$$

where the sum is over the same partitions as above. The coincidence of the two maps now follows from Definition 7 of the cumulants. □

Remark 4. In classical probability theory, random variables are measurable $\mathbb{C}$-valued functions on a measure space and the expectation value of a random variable is defined by integration. The product of measurable functions is measurable and defines the product of random variables. In this situation, the product is commutative and associative. One can define a *classical probability space* as a probability space (V, e, a) for which $a : V \times V \to V$ is commutative. The entire discussion in Sections 2 and 3 can be symmetrized for a classical probability space. The requisite modifications and results are contained in [1].

4. Homotopy Probability Theory

The starting point of homotopy probability theory is to replace the space V of random variables with a chain complex $C = (V, d)$ of random variables.

Definition 8. The data of a *homotopy probability space* consists of a chain complex $C = (V, d)$, a chain map $e : C \to \mathbb{C}$, and a degree zero associative product $a : V^{\otimes 2} \to V$.

The expectation e and the differential d are not assumed to satisfy any properties with respect to $a : V^{\otimes 2} \to V$.

The coincidence of the cumulants for a probability space and an A_∞ morphism on the expectation provides the guide for how to proceed for a homotopy probability space. Cumulants for a homotopy probability space are defined as the A_∞ morphism $E^{\mathbf{a},\mathbf{a}'}$ on the chain map $e : (V, d) \to (\mathbb{C}, 0)$ associated to the product $a : V^{\otimes 2} \to V$ and the product $a' : \mathbb{C}^{\otimes 2} \to \mathbb{C}$ of complex numbers. These cumulants are an A_∞ morphism between the A_∞ structure $(V, D^{\mathbf{a}})$ and the (zero) A_∞ structure $(\mathbb{C}, D^{\mathbf{a}'})$.

These ideas are elaborated and illustrated in [1].

References

[1] G. C. Drummond-Cole and J. Terilla. Homotopy probability theory II. *preprint*, 2013.

[2] F. Lehner. Cumulants in noncommutative probability theory I. Noncommutative exchangeability systems. *Mathematische Zeitschrift*, 248(1):67–100, 2004.

[3] J.-L. Loday and B. Vallette. *Algebraic Operads*. Springer, 2012.

[4] J.-S. Park. Homotopical probability space, November 2, 2011. Lecture at CUNY Graduate Center.

[5] R. Speicher and A. Nica. *Lectures on the Combinatorics of Free Probability*. Number 335 in LMS Lecture Note Series. Cambridge Univ. Press, 2006.

[6] R. Speicher and R. Woroudi. Boolean convolution. In *Free Probability Theory, Waterloo, ON*, volume 12 of *Fields Inst. Commun.* Amer. Math. Soc., 1995.

[7] T. Tao. Notes on free probability theory. http://terrytao.wordpress.com/2010/02/10/245a-notes-5-free-probability/, 2010.

DEPARTMENT OF MATHEMATICS, NORTHWESTERN UNIVERSITY 2033 SHERIDAN ROAD EVANSTON, IL 60208-2730, USA
E-mail address: gabriel@math.northwestern.edu

CENTER FOR GEOMETRY AND PHYSICS, INSTITUTE FOR BASIC SCIENCE IN KORA & DEPARTMENT OF MATHEMATICS, POSTECH, POHANG, GYEONGBUK 790-784, KOREA
E-mail address: jaesuk@postech.ac.kr

DEPARTMENT OF MATHEMATICS ROOM 211 KISSENA HALL, QUEENS COLLEGE, THE CITY UNIVERSITY OF NEW YORK 65-30 KISSENA BLVD QUEENS, NY 11367-1597 USA
E-mail address: jterilla@qc.cuny.edu