Problem set 1

Math 625 Spring 2024

Due Thursday, March 14

1 Hermite Interpolation

Choose n distinct numbers $x_1 < \cdots < x_{n+1}$ and let f be a differentiable function f. There is a unique polynomial p of degree $2n+1$ satisfying $f(x_i) = p(x_i)$ and $f'(x_i) = p'(x_i)$ for $i = 1, \ldots, n+1$. Let's call p the 1*st order Hermite interpolate* of f.

Problem 1. Explain why p exists and why it is unique.

Problem 2. Find the 11th order Hermite interpolate of $\frac{1}{1+10t^2}$ using the interpolation points $-1, -0.6, -0.2, 0.2, 0.6, 1$.

Problem 3. For $i = 1, \ldots, n+1$, let $L_i(t)$ be the associated i-th Lagrange polynomial. For $i = 1, \ldots, n+1$, define

$$a_i(t) = (1 - 2(t - x_i)L_i'(x_i))(L_i(t))^2. \text{ and } b_i(t) = (t - x_i)(L_i(t))^2.$$

Show that the $2n+2$ polynomials $\{a_1, \ldots, a_{n+1}, b_1, \ldots, b_{n+1}\}$ form an orthonormal basis of the space of polynomials of degree $\leq 2n+1$ with the inner product given by

$$\langle f, g \rangle = \sum_{i=0}^{n} f(x_i)g(x_i) + f'(x_i)g'(x_i).$$

Problem 4. Write a program that inputs a set of points $\{x_1, \ldots, x_{n+1}\}$ and outputs the polynomials $\{a_i, b_i\}$.

Problem 5. Find a *simple* formula for the approximation p of an arbitrary differentiable function f as a linear combination of the $\{a_i, b_i\}$.

Problem 6. Prove:

Theorem. *If f is differentiable at least $2n+2$ times, then for any number x, the difference $f(x) - p(x)$ between the value of f and the Hermite interpolate is given by*

$$f(x) - p(x) = \frac{f^{(2n+2)}(\theta)}{(2n+2)!} w(x)$$

where w is the polynomial $w(t) = \prod_{i=1}^{n+1}(t - x_i)^2$ and θ is some number between x and x_1 and x_{n+1}.

Problem 7. Analyze the error terms for the 11th degree Hermite interpolate for $\dfrac{1}{1+10t^2}$ using

(a) the interpolation points $-1, -0.5, 0, 0.5, 1$.

(b) Analyze the error terms for the 11th degree Hermite interpolate for $\dfrac{1}{1+10t^2}$ using the 5 Chebyshev interpolation points on the interval $[-1, 1]$.

(c) If you want to guarantee that the error in the degree $2n + 1$ Hermite interpolate for $\dfrac{1}{1+10t^2}$ is less than 10^{-100}, how many interpolate points do you need if you use n Chebyshev points?

2 Fourier Series

Problem 8. Let V be the vector space of integrable functions with the inner product defined by

$$\langle f, g\rangle := \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)g(t)dt.$$

Let W be the eleven dimensional subspace of functions spanned by

$$\{1, \sin(t), \sin(2t), \sin(3t), \sin(4t), \sin(5t), \cos(t), \cos(2t), \cos(3t), \cos(4t), \cos(5t)\}.$$

(a) Project the absolute value function onto W.

(b) Let h be the periodic function (with period 2π) defined by

$$h(t) = \begin{cases} -1 & \text{if } -\pi \leq t < 0 \\ 1 & \text{if } 0 \leq t < \pi \end{cases}$$

Project h onto W.

Problem 9. In general, orthogonally projecting a function f onto the space spanned by

$$\{1, \sin(t), \cos(t), \sin(2t), \cos(2t), \ldots, \sin(nt), \cos(nt)\}$$

yields a function of the form

$$A_0 + A_1\cos(t) + B_1\sin(t) + A_2\cos(2t) + B_2\sin(2t) + \cdots + A_n\cos(nt) + B_n\sin(nt).$$

Explain why for $n = 1, 2, \ldots$, the numbers A_n and B_n satisfy

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)dt \text{ and } \frac{1}{\pi}B_n = \int_{-\pi}^{\pi} f(t)\sin(nt)dt.$$