TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9947(XX)0000-0

EXPLICIT DETERMINATION OF ROOT NUMBERS OF ABELIAN VARIETIES

ARMAND BRUMER, KENNETH KRAMER, AND MARIA SABITOVA

ABSTRACT. Let A be an abelian variety over a non-archimedean local field of definition K and let $W(A)$ be the root number of A. Let F be a Galois extension of K over which A has semistable reduction, allowing $F = K$. We analyze $W(A)$ in terms of contributions from the toric and abelian variety components of the closed fibers of the Néron models of A over the ring of integers of K and of F. In particular, our results can be used to calculate $W(A)$ in two main instances: first, for abelian varieties with additive reduction over K and totally toroidal reduction over F, provided that the residue characteristic of K is odd; second, for the Jacobian $A = J(C)$ of a stable curve C over K.

1. Introduction

The root number $W(A)$ associated to an abelian variety A over a number field L is the sign in the (conjectural) functional equation of its L-function, and so the conjectures of Birch and Swinnerton-Dyer imply that the rank of the Mordell–Weil group $A(L)$ satisfies

$$W(A) = (-1)^{\operatorname{rank} A(L)}.$$

For each place v of L, let L_v denote the completion of L with respect to v. Let $W(A_v)$ be the local root number associated to $A_v = A \times_L L_v$. By definition,

$$W(A) = \prod_v W(A_v),$$

where v runs through all the places of L. If $v|\infty$, then $W(A_v) = (-1)^{\dim A}$ (see [Ro93, Prop. 1] or [S07, Lemma 2.1]). If v is finite, then $W(A_v)$ is defined as follows. Let $\overline{L}_v$ be a fixed algebraic closure of L_v. For a rational prime l different from the residue characteristic of L_v let $\mathbb{T}_l(A_v)$ be the l-adic Tate module of A_v and let $\mathbb{V}_l(A_v) = \mathbb{T}_l(A_v) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$. Let $\sigma'_v = \sigma'_{v,A}$ denote a (complex) representation of the Weil–Deligne group $\mathcal{W}'(\overline{L}_v/L_v)$ of L_v associated to $\mathbb{V}_l(A_v)$ via the Deligne–Grothendieck construction (see for example [Ro94]). Then $W(A_v) = W(\sigma'_v)$.

We assume from now on a non-archimedean local field of definition K of A and we let F/K be a (possibly trivial) Galois extension such that A has semistable reduction over F. Denote by $\mathcal{A}_{\mathcal{O}_K}$ and $\mathcal{A}_{\mathcal{O}_F}$ the Néron models over the corresponding rings of integers and write $I(F/K)$ for the inertia group inside $\mathrm{Gal}(F/K)$. We analyze $W(A)$ in terms of contributions from the toric and abelian variety components of the closed fibers of these Néron models. In particular, our results can be used to calculate local root numbers of abelian varieties in two main instances: first,

2010 *Mathematics Subject Classification.* Primary 11G10; Secondary 11G25, 11S20.
Key words and phrases. Abelian variety, root number, Weil-Deligne group.

for abelian varieties with additive reduction over K and totally toroidal reduction over F, provided that the residue characteristic of K is odd, and second, for the Jacobian $J(C)$ of a stable curve C over K. More precisely, in the first case

$$W(A) = \chi(-1)^n,$$

where χ is any ramified quadratic character of $K^\times$ and n is the number of irreducible components of the complex representation of $I(F/K)$ associated to (the toric part of) the closed fiber of $\mathcal{A}_{\mathcal{O}_F}$ (see (2.2.10), (2.2.11), Prop. 2.2.17, and Prop. 3.4 below). In the second case,

$$W(J(C)) = (-1)^{n_k+s_k+1} \prod_{[z]\in\Sigma_k} \tau_{[z]},$$

where k is the residue field of K, Σ_k is the set of the singular k-points of the reduction $\mathcal{C}_{v_K}$ of C over k, $n_k = |\Sigma_k|$, s_k is the number of irreducible k-components of $\mathcal{C}_{v_K}$ and $\tau_{[z]} = \pm 1$ is defined in Prop. 5.4 below.

For a somewhat different viewpoint on epsilon factors associated to abelian varieties which acquire semistable reduction over a tamely ramified extension of a local field, see [Sa93, Lemma 1]. Additional information about root numbers, including their behavior under twisting can be found in [DD09, §3.5].

The paper is organized as follows. Section 2 contains general facts and notation concerning root numbers and abelian varieties. In Section 3, we find the contribution to the local root number $W(A)$ from the toric parts of the closed fibers of $\mathcal{A}_{\mathcal{O}_K}$ and $\mathcal{A}_{\mathcal{O}_F}$ when the residue characteristic of K is odd. In Section 4, we analyze the *delta factor* of $W(A)$, cf. (2.2.10), in terms of the action of the Frobenius automorphism of an algebraic closure of k on the character group of the toric part of the closed fiber of $\mathcal{A}_{\mathcal{O}_K}$. We calculate the root number of the Jacobian of a stable curve C over K in terms of the reduction of C over K in Section 5. Finally, Section 6 provides several examples of hyperelliptic curves C of genus 2 over $\mathbb{Q}$ for which the global root number of $J(C)$ can be determined by our results.

The authors are grateful to the anonymous referee for valuable comments and suggestions.

2. General facts and notation

2.1. **Representations of the Weil–Deligne group.** Let K be a local non-archimedean field with ring of integers $\mathcal{O}_K$ and residue field k of characteristic p. Fix a separable closure $\overline{K}$ of K and the coresponding algebraic closure $\overline{k}$ of k. Let φ be the inverse of the Frobenius automorphism of $G_k = \mathrm{Gal}(\overline{k}/k)$ and let Φ denote a preimage of φ in $G_K = \mathrm{Gal}(\overline{K}/K)$. Write $I_K = \mathrm{Gal}(\overline{K}/K^{nr})$ for the inertia group of K, where $K^{nr} \subset \overline{K}$ denotes the maximal unramified extension of K contained in $\overline{K}$. By definition, the Weil group $\mathcal{W}_K = \mathcal{W}(\overline{K}/K)$ of K is a subgroup of G_K equal to $I_K \rtimes \langle\Phi\rangle$, where $\langle\Phi\rangle$ denotes the infinite cyclic group generated by Φ. The group $\mathcal{W}(\overline{K}/K)$ is a topological group (see e.g., [Ro94]). A (*multiplicative*) *character of* $K^\times$ is a continuous homomorphism $\mu : K^\times \to \mathbb{C}^\times$. Throughout the paper we will identify one-dimensional complex continuous representations of $\mathcal{W}_K$ with characters of $K^\times$ via local class field theory, assuming that a uniformizer of K corresponds to a geometric Frobenius $\Phi \in G_K$ as above. Let $\omega\colon \mathcal{W}_K \to \mathbb{C}^\times$ be the one-dimensional representation of $\mathcal{W}_K$ given by

$$\omega|_{I_K} = 1, \quad \omega(\Phi) = q^{-1}, \quad q = |k|.$$

Let $\mathcal{W}'_K = \mathcal{W}'(\overline{K}/K)$ denote the Weil–Deligne group of K. A representation ρ' of $\mathcal{W}'_K$ is a continuous homomorphism $\rho' \colon \mathcal{W}'_K \to \mathrm{GL}(U)$, where U is a finite-dimensional complex vector space and the restriction of ρ' to the subgroup $\mathbb{C}$ of $\mathcal{W}'_K$ is complex analytic. There is a bijection between representations of $\mathcal{W}'_K$ and pairs (ρ, N), where $\rho\colon \mathcal{W}_K \to \mathrm{GL}(U)$ is a continuous complex representation of $\mathcal{W}_K$ and N is a nilpotent endomorphism on U such that $\rho(g)N\rho(g)^{-1} = \omega(g)N$ for all $g \in \mathcal{W}_K$. We identify ρ' with the corresponding pair (ρ, N) and write $\rho' = (\rho, N)$. Furthermore, a representation ρ of $\mathcal{W}_K$ is identified with the representation $(\rho, 0)$ of $\mathcal{W}'_K$ (see e.g., [Ro94, §§1–3]).

2.2. **Abelian varieties.** Let A be an abelian variety over K. For a rational prime l different from $p = \mathrm{char}(k)$, let $\mathbb{T}_l(A)$ be the l-adic Tate module of A, a free $\mathbb{Z}_l$-module of rank $2g$, where $g = \dim A$. Put $V = \mathbb{V}_l(A) = \mathbb{T}_l(A) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ and let

$$\tilde{\rho}_l : G_K \longrightarrow \mathrm{GL}(V)$$

denote the natural l-adic representation of G_K on V. The Weil pairing together with an isogeny from A to the dual abelian variety of A induces a nondegenerate, skew-symmetric, G_K-equivariant pairing

$$\langle -, - \rangle \colon V \times V \longrightarrow \mathbb{Q}_l \otimes \omega_l, \tag{2.2.1}$$

where ω_l is the l-adic cyclotomic character of G_K (see e.g., [S07] for more details).

Let $F/K \subset \overline{K}/K$ be a finite Galois extension over which A acquires semistable reduction ([G67, Thm. 6.1]) and let k_F be the residue field of F. A crucial role is played by the *fixed* and *essentially fixed* G_K-submodules of V, namely

$$V_1 = V^{I_K} \quad \text{and} \quad V_2 = V^{I_F}.$$

Clearly $V_1 \subseteq V_2$.

Let $\mathcal{A}$ denote the Néron model of A. Let $\mathcal{A}_{v_K} = \mathcal{A} \times_{\mathcal{O}_K} k$ and $\mathcal{A}_{v_F} = \mathcal{A} \times_{\mathcal{O}_K} k_F$ be the closed fibers of $\mathcal{A}$ over k and k_F, respectively, with corresponding connected components $\mathcal{A}^0_{v_K}$ and $\mathcal{A}^0_{v_F}$. Let $\mathcal{T}_{v_K}$ and $\mathcal{T}_{v_F}$ be the maximal tori of $\mathcal{A}^0_{v_K}$ and $\mathcal{A}^0_{v_F}$, respectively. As an algebraic group, $\mathcal{A}^0_{v_K}$ has a natural decomposition

$$0 \to \mathcal{U} \to \mathcal{A}^0_{v_K}/\mathcal{T}_{v_K} \to \mathcal{B}_{v_K} \to 0, \tag{2.2.2}$$

in which $\mathcal{U}$ is a unipotent algebraic group over k and $\mathcal{B}_{v_K}$ is an abelian variety over k. Since A acquires semistable reduction over F,

$$\mathcal{A}^0_{v_F}/\mathcal{T}_{v_F} \cong \mathcal{B}_{v_F}, \tag{2.2.3}$$

where $\mathcal{B}_{v_F}$ is an abelian variety over k_F.

The reduction map $\pi : \mathcal{O} \twoheadrightarrow k$ induces isomorphisms:

$$V_1 \cong \mathbb{T}_l(\mathcal{A}^0_{v_K}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l \quad \text{and} \quad V_2 \cong \mathbb{T}_l(\mathcal{A}^0_{v_F}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l, \tag{2.2.4}$$

as follows from [ST68, Lemma 2]. We endow $\mathbb{T}_l(\mathcal{A}^0_{v_K})$ and $\mathbb{T}_l(\mathcal{A}^0_{v_F})$ with the action of G_K via the isomorphisms (2.2.4). The Igusa-Grothendieck orthogonality theorem [G67, Thm. 2.4] implies that π induces an isomorphism of G_K-modules

$$V_1 \cap V_1^{\perp} \cong \mathbb{T}_l(\mathcal{T}_{v_K}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l, \tag{2.2.5}$$

where $V_1^{\perp}$ is the orthogonal complement to V_1 in V under the pairing (2.2.1).

Proposition 2.2.6. *We have $V_1 \cap V_1^{\perp} \subseteq V_2^{\perp} \subseteq V_2$.*

Proof. This follows from [G67, Prop. 3.5] and [E95, Equation (2.7)]. □

Note that (2.2.2) and (2.2.3) induce exact sequences

$$0 \longrightarrow \mathbb{T}_l(\mathcal{T}_{v_K}) \longrightarrow \mathbb{T}_l(\mathcal{A}^0_{v_K}) \longrightarrow \mathbb{T}_l(\mathcal{B}_{v_K}) \longrightarrow 0$$

and

$$0 \longrightarrow \mathbb{T}_l(\mathcal{T}_{v_F}) \longrightarrow \mathbb{T}_l(\mathcal{A}^0_{v_F}) \longrightarrow \mathbb{T}_l(\mathcal{B}_{v_F}) \longrightarrow 0.$$

Indeed, $\mathcal{T}(\overline{k})$ is a divisible group for a torus $\mathcal{T}$ over k, so taking inverse limit is an exact functor. Moreover, $\mathbb{T}_l(\mathcal{U})$ is trivial. To simplify notation, we use $\mathbb{V}_l$ for the Tate vector space obtained by tensoring the Tate module with $\mathbb{Q}_l$. By (2.2.5) and Prop. 2.2.6, the following diagrams, in which the arrows are inclusions, correspond to each other via the reduction map:

(2.2.7)
$$\begin{array}{ccc} V_1 \cap V_1^{\perp} & \longrightarrow & V_2^{\perp} \\ \downarrow & & \downarrow \\ V_1 & \longrightarrow & V_2 \end{array} \qquad \begin{array}{ccc} \mathbb{V}_l(\mathcal{T}_{v_K}) & \longrightarrow & \mathbb{V}_l(\mathcal{T}_{v_F}) \\ \downarrow & & \downarrow \\ \mathbb{V}_l(\mathcal{A}^0_{v_K}) & \longrightarrow & \mathbb{V}_l(\mathcal{A}^0_{v_F}) \end{array}$$

Since $V_1 \cap V_2^{\perp} = V_1 \cap V_1^{\perp}$ by Prop. 2.2.6, we have the G_K-module isomorphisms:

$$V_1/(V_1 \cap V_1^{\perp}) \cong \mathbb{V}_l(\mathcal{B}_{v_K}) \quad \text{and} \quad V_2/V_2^{\perp} \cong \mathbb{V}_l(\mathcal{B}_{v_F}),$$

as well as an injection of G_K-modules

$$\mathbb{V}_l(\mathcal{A}^0_{v_K})/\mathbb{V}_l(\mathcal{T}_{v_K}) \cong \mathbb{V}_l(\mathcal{B}_{v_K}) \xrightarrow{\text{injection}} \mathbb{V}_l(\mathcal{B}_{v_F}) \cong \mathbb{V}_l(\mathcal{A}^0_{v_F})/\mathbb{V}_l(\mathcal{T}_{v_F}).$$

We are interested in the complex representation $\rho' = (\rho, N)$ of $\mathcal{W}'_K$ associated to $\tilde{\rho}_l$. Fix an embedding $\iota\colon \mathbb{Q}_l \hookrightarrow \mathbb{C}$ and recall the standard procedure ([Ro94, §4] or [T79, Thm. 4.2.1]) for transforming an l-adic representation $\tilde{\lambda}_l$ of G_K on a finite-dimensional vector space U over $\mathbb{Q}_l$ into a complex representation λ' of $\mathcal{W}'_K$ on $U \otimes_\iota \mathbb{C}$. If $t_l\colon I_K \longrightarrow \mathbb{Q}_l$ is a non-trivial continuous homomorphism, then there exists a unique nilpotent endomorphism $M_l \in \mathrm{End}_{\mathbb{Q}_l}(U)$ such that

$$\tilde{\lambda}_l(h) = \exp(t_l(h)M_l),$$

for all h in an open subgroup of I_K. Fix a Frobenius Φ as in §2.1 and define $\lambda_l : \mathcal{W}_K \longrightarrow \mathrm{GL}(U)$ by

$$\lambda_l(g) = \tilde{\lambda}_l(g)\exp(-t_l(h)M_l), \quad g = \Phi^m h \in \mathcal{W}_K,\ m \in \mathbb{Z},\ h \in I_K.$$

(Note that λ_l generally is not the restriction of $\tilde{\lambda}_l$ to $\mathcal{W}_K$.) Let $\lambda = \lambda_l \otimes_\iota \mathbb{C}$ be the (continuous) representation of $\mathcal{W}_K$ on $U \otimes_\iota \mathbb{C}$ obtained from λ_l via the extension-of-scalars inclusion $\mathrm{GL}(U) \hookrightarrow \mathrm{GL}(U \otimes_\iota \mathbb{C})$ and let $M = M_l \otimes_\iota \mathbb{C}$ be the endomorphism of $U \otimes_\iota \mathbb{C}$ obtained from M_l by extending scalars via ι. Then by definition $\lambda' = (\lambda, M)$. It is known that λ' does not depend on the choice of t_l.

Remark 2.2.8. Given $\mathbb{Q}_l[G_K]$-submodules $0 \subseteq U_1 \subseteq U_2 \subseteq U$, consider the l-adic representation $\tilde{\gamma}_l\colon G_K \to \mathrm{GL}(U_2/U_1)$ obtained from $\tilde{\lambda}_l$. Let $\gamma' = (\gamma_l \otimes_\iota \mathbb{C}, \overline{M}_l \otimes_\iota \mathbb{C})$ be the complex representation of $\mathcal{W}'_K$ associated to $\tilde{\gamma}_l$, where $\gamma_l\colon \mathcal{W}_K \to \mathrm{GL}(U_2/U_1)$ is a homomorphism and $\overline{M}_l \in \mathrm{End}_{\mathbb{Q}_l}(U_2/U_1)$ is nilpotent. If $M_l(U_2) \subseteq U_1$, then $\overline{M}_l = 0$ and γ_l is the restriction of $\tilde{\gamma}_l$ to $\mathcal{W}_K$. These observations follow from uniqueness of the nilpotent endomorphism in the above procedure.

In the case of $\tilde{\rho}_l$ the procedure can be made more explicit. The inertia group I_F acts on V through its maximal pro-l quotient $I_F(l) = \mathrm{Gal}(F^l/F^{nr}) \cong \mathbb{Z}_l$, where $F^l \subset \overline{K}$ denotes the maximal pro-l extension of F^{nr}. Fix an element $\sigma_F \in I_F$

whose restriction to $I_F(l)$ is a topological generator and define an endomorphism $N_l : V \to V$ by

$$N_l = \tilde{\rho}_l(\sigma_F) - 1.$$

Then $\ker N_l = V_2$ and the image of N_l is contained in $V_2^{\perp}$. By Prop. 2.2.6, $V_2^{\perp} \subseteq V_2$, so $N_l^2 = 0$ and $\tilde{\rho}_l(\sigma_F) = 1 + N_l = \exp(N_l)$. Let e be the power of l in the ramification degree of F/K. Then the image of $I_F(l)$ in $I_K(l)$ is a subgroup of index e. In view of the diagram

$$\begin{array}{ccccc} I_F & \xrightarrow{\text{res}} & I_F(l) & \xrightarrow{\approx} & \mathbb{Z}_l \\ {\scriptstyle\text{inclusion}}\downarrow & & \downarrow & & \downarrow{\scriptstyle\text{inclusion}} \\ I_K & \xrightarrow{\text{res}} & I_K(l) & \xrightarrow{\approx} & \frac{1}{e}\mathbb{Z}_l \end{array}$$

in which $\sigma_F \mapsto 1$ in the top row, we may choose an element $\sigma_K \in I_K$ whose restriction to $I_K(l)$ is a topological generator and such that $\sigma_F = \sigma_K^e$ on K^l. Let $t_l \colon I_K \to \mathbb{Q}_l$ be the composition of arrows in the bottom row, so that $t_l(\sigma_K) = 1/e$.

Define a representation $\rho_l : \mathcal{W}_K \to \mathrm{GL}(V)$ by

$$\rho_l(g) = \tilde{\rho}_l(g)\exp(-t_l(h)N_l), \quad g = \Phi^m h \in \mathcal{W}_K,\ m \in \mathbb{Z},\ h \in I_K. \tag{2.2.9}$$

Note that $I_F \subseteq \ker \rho_l$. Finally, let $\rho = \rho_l \otimes_\iota \mathbb{C}$ and let $N = N_l \otimes_\iota \mathbb{C}$. Then as above $\rho' = (\rho, N)$ is a representation of $\mathcal{W}_K'$ on $V \otimes_\iota \mathbb{C}$. In our context, ρ' does not depend on the choice of $l \neq p$ nor ι (see e.g., [S07]).

For the definition and properties of the root number $W(\rho)$ associated to ρ, see [T79, §3] or [Ro94, §11, §12]. An additional contribution to $W(A)$ comes from the *delta factor*, given by

$$\delta(\rho') = \det\left(-\Phi|_{(V\otimes_\iota\mathbb{C})^{\rho(I_K)}/(\ker N)^{\rho(I_K)}}\right). \tag{2.2.10}$$

By definition,

$$W(A) = W(\rho') = W(\rho)\cdot\delta(\rho'). \tag{2.2.11}$$

To proceed further, we introduce various representations of $\mathcal{W}_K$, which are restrictions to $\mathcal{W}_K$ of the induced G_K-action (via $\tilde{\rho}_l$) on subquotients of the l-adic representation space $V = \mathbb{V}_l(A)$. The indicated isomorphisms follow from (2.2.7). Thus

$$\rho_l^{tor} \colon \mathcal{W}_K \to \mathrm{GL}(\mathbb{V}_l(\mathcal{T}_{v_F})) \cong \mathrm{GL}(V_2^{\perp}) \tag{2.2.12}$$

and we obtain

$$\bar{\rho}_l^{tor} \colon \mathcal{W}_K \to \mathrm{GL}(\overline{U}_{tor}) \text{ on } \overline{U}_{tor} = \mathbb{V}_l(\mathcal{T}_{v_F})/\mathbb{V}_l(\mathcal{T}_{v_K}) \cong V_2^{\perp}/(V_1 \cap V_1^{\perp}). \tag{2.2.13}$$

Similarly, we have

$$\rho_l^{av} : \mathcal{W}_K \to \mathrm{GL}(\mathbb{V}_l(\mathcal{B}_{v_F})) \cong \mathrm{GL}\left(V_2/V_2^{\perp}\right) \tag{2.2.14}$$

and we obtain

$$\bar{\rho}_l^{av} \colon \mathcal{W}_K \to \mathrm{GL}(\overline{U}_{av}) \text{ on } \overline{U}_{av} = \mathbb{V}_l(\mathcal{B}_{v_F})/\mathbb{V}_l(\mathcal{B}_{v_K}) \cong V_2/(V_1 + V_2^{\perp}). \tag{2.2.15}$$

Since $\mathrm{Image}\, N_l \subseteq V_2^{\perp} \subseteq V_2$ and $\ker N_l = V_2$, Remark 2.2.8 implies that the following are the continuous complex representations of $\mathcal{W}_K$ arising from the standard construction of representations of $\mathcal{W}_K'$ over $\mathbb{C}$ associated to the corresponding l-adic G_K-modules:

$$\rho^{tor} = \rho_l^{tor} \otimes_\iota \mathbb{C}, \quad \bar{\rho}^{tor} = \bar{\rho}_l^{tor} \otimes_\iota \mathbb{C}, \quad \rho^{av} = \rho_l^{av} \otimes_\iota \mathbb{C}, \quad \bar{\rho}^{av} = \bar{\rho}_l^{av} \otimes_\iota \mathbb{C}.$$

Remark 2.2.16. By Prop. 2.2.6, I_F acts trivially on $\overline{U}_{tor}$ and so the action of I_K on $\overline{U}_{tor}$ factors through $H = I_K/I_F = \mathrm{Gal}(F^{nr}/K^{nr})$. Moreover, $\overline{U}_{tor}^H = \{0\}$. Denote the restriction of $\bar{\rho}^{tor}$ to H by $\bar{\rho}^{tor}|_H$. According to [G67, p. 360], the representation $\bar{\rho}^{tor}|_H$ is realizable over $\mathbb{Q}$.

Proposition 2.2.17. *Let $\rho = \rho_l \otimes_\iota \mathbb{C}$ with ρ_l as in* (2.2.9). *Then*

$$W(\rho) = W(\bar{\rho}^{av}) \cdot (\det \bar{\rho}^{tor})(-1).$$

In particular, if A already is semistable over K, then $W(\rho) = 1$.

Proof. By additivity, $W(\rho)$ is the product of the root numbers of the complex representations of $\mathcal{W}_K$ arising from the successive quotients in

$$0 \subseteq V_2^\perp \subseteq V_2 \subseteq V. \tag{2.2.18}$$

Since N_l acts as zero on each of these successive quotients, $\rho_l(\mathcal{W}_K)$ and $\tilde{\rho}_l(\mathcal{W}_K)$ agree on them. The first piece $V_2^\perp$ affords the representation ρ_l^{tor} in (2.2.12) and the second piece $V_2/V_2^\perp$ affords the representation ρ_l^{av} in (2.2.14). The perfect G_K-equivariant pairing

$$V_2^\perp \times (V/V_2) \to \mathbb{Q}_l \otimes \omega_l$$

induced by (2.2.1) shows that the representation ν of $\mathcal{W}_K$ afforded by the third piece $(V/V_2) \otimes_\iota \mathbb{C}$ satisfies $\nu \cong (\rho^{tor})^* \otimes \omega$, where $(\rho^{tor})^*$ is the contragredient representation of ρ^{tor}. Hence, $W(\rho^{tor} \oplus \nu) = (\det \rho^{tor})(-1)$. But any lift $\tau \in \mathcal{W}_K$ of $-1 \in K^\times \cong \mathcal{W}_K^{ab}$ is in I_K, so τ acts trivially on $V_1 = V^{I_K}$. It follows that $(\det \rho^{tor})(-1) = (\det \bar{\rho}^{tor})(-1)$.

Since the subrepresentation $\rho_0\colon \mathcal{W}_K \to \mathrm{GL}(\mathbb{V}_l(\mathcal{B}_{v_K}) \otimes_\iota \mathbb{C})$ of ρ^{av} is unramified, $W(\rho_0) = 1$ by standard properties of root numbers. Hence $W(\rho^{av}) = W(\bar{\rho}^{av})$. □

3. Calculation of $\det \bar{\rho}^{tor}(-1)$ for odd p

Let $\bar{\rho}^{tor} = \bar{\rho}_l^{tor} \otimes_\iota \mathbb{C}$ be the representation associated with the toric component of the special fiber of the abelian variety A, as above. According to Remark 2.2.16, $\bar{\rho}^{tor}$ factors through $H = \mathrm{Gal}(F^{nr}/K^{nr})$ and the restriction $\bar{\rho}^{tor}|_H$ is realizable over $\mathbb{Q}$. When p is odd, ramification of 2-power degree in F^{nr}/K^{nr} is tame, so the 2-Sylow subgroups of H are cyclic. We have the following group-theoretic lemma.

Lemma 3.1. i) *An irreducible complex representation of a group of odd order with real-valued character is trivial.*

ii) *Let σ be a non-trivial irreducible orthogonal representation of a finite group G over $\mathbb{C}$ and assume that a 2-Sylow subgroup $C = \langle c \rangle$ of G is cyclic. Then $\det \sigma(c) = -1$.*

Proof. Item (i) follows from [Se77, Prop. 3.9]. In (ii), a standard induction argument shows that there is a normal complement B of C, i.e., $G = B \rtimes C$ is a semi-direct product in which C acts on the normal subgroup B, with $|B|$ odd. (Note that writing $G = \sqcup Cg_i$ as a union of distinct right cosets, shows that the image of c in the left regular permutation representation of G is a product of $[G\!:\!C]$ cycles of length $|C|$. Hence c is an odd permutation, the sign gives a surjective map $G \to \{\pm 1\}$, and the even permutations in G form an index 2 normal subgroup of G.)

If the restriction $\mathrm{Res}_B^G \sigma$ of σ to B is trivial, then $\sigma(c) = -1$ and (ii) is obvious. Hence we assume from now on that $\mathrm{Res}_B^G \sigma$ is non-trivial.

Using the Mackey method of small subgroups, σ can be constructed from an irreducible representation ψ of B in the following way. For $g \in G$, let ψ_g denote the representation of B given by $\psi_g(b) = \psi(g^{-1}bg)$ for all $b \in B$. Denote the stabilizer of ψ in G by $G_\psi = \{g \in G \,|\, \psi_g \cong \psi\}$. Then $G_\psi = B \rtimes \langle c^x \rangle$ for some positive integer x, which we can choose to divide $e = |C|$.

It turns out that ψ can be extended to a representation of G_ψ. Let m be the degree of ψ. From $\psi_{c^x} \cong \psi$, one can easily show that there exists some $M \in \mathrm{GL}_m(\mathbb{C})$ such that

$$\psi(c^{-x}bc^x) = M^{-1}\psi(b)M, \quad \forall b \in B \tag{3.2}$$

and by Schur's Lemma we can arrange that $M^{e/x} = I_m$, where I_m is $m \times m$ identity. Then ψ can be extended to an (irreducible) representation of G_ψ via $\psi(c^x) = M$. We will denote the extension of ψ to G_ψ also by ψ, so that

$$\psi(bc^{xy}) = \psi(b)\, M^y, \quad b \in B,\ y \in \{0, 1, \ldots, (e/x) - 1\}.$$

Mackey guarantees that there is a choice of ψ and M for which $\sigma \cong \mathrm{Ind}_{G_\psi}^G \psi$. (See [S14] for more details.)

The restriction $\mathrm{Res}_B^G \sigma \cong \psi \oplus \psi_c \oplus \cdots \oplus \psi_{c^{x-1}}$ is orthogonal, because σ is orthogonal. But by assumption, σ and so also ψ, is not trivial on B. Hence ψ is not equivalent to its contragredient representation ψ^* by (i). Since each ψ_{c^i} is irreducible, we have $\psi^* \cong \psi_{c^j}$ for some j. Then one can easily check that x is even and $\psi^* \cong \psi_{c^{x/2}}$.

Since σ is orthogonal, there is a non-degenerate, symmetric G-equivariant pairing $\langle\ ,\ \rangle \colon V \times V \to \mathbb{C}$ on the representation space V of σ. Let W be an m-dimensional subspace of V affording the representation ψ of G_ψ and let $e_1, \ldots, e_m$ be a basis for W. We have $V = W \oplus cW \oplus c^2W \oplus \cdots \oplus c^{x-1}W$ as modules over $R = \mathbb{C}[G_\psi]$. If $0 \le j \le x-1$ and $j \ne x/2$, irreducibility of W implies that

$$\mathrm{Hom}_R(W \otimes_R c^jW, \mathbb{C}) \cong \mathrm{Hom}_R(W, (c^jW)^*) = 0.$$

But $\langle\ ,\ \rangle$ is non-degenerate, so its restriction to $W \times c^{x/2}W$ must be non-degenerate. Thus the matrix $D = (d_{ij})$ with $d_{ij} = \langle e_i, c^{x/2}e_j \rangle$ for $1 \le i, j \le m$ is non-singular. Since multiplication by c^x on W is given by the matrix $M = (a_{ij})$, we have

$$\begin{aligned} d_{ij} &= \langle e_i, c^{\frac{x}{2}}e_j \rangle \overset{\text{sym}}{=} \langle c^{\frac{x}{2}}e_j, e_i \rangle \overset{c^{\frac{x}{2}}\text{-invar}}{=} \langle c^x e_j, c^{\frac{x}{2}}e_i \rangle = \langle Me_j, c^{\frac{x}{2}}e_i \rangle \\ &= \langle \sum_{r=1}^m a_{rj}e_r, c^{\frac{x}{2}}e_i \rangle = \sum_{r=1}^m a_{rj} \langle e_r, c^{\frac{x}{2}}e_i \rangle = \sum_{r=1}^m a_{rj}d_{ri}. \end{aligned}$$

Hence $D = D^tM$ and we find that $\det M = 1$.

To conclude the computation, we use the Frobenius formula for the determinant of an induced representation, which we review for the convenience of the reader. Let T be a complete set of representatives for the right cosets of G_ψ in G and let $\mathrm{sign}(g)$ be the sign of the permutation of these right cosets induced by right multiplication by $g \in G$. Then

$$\det\sigma(g) = \det(\mathrm{Ind}_{G_\psi}^G \psi)(g) = (\mathrm{sign}(g))^{\deg\psi} \prod_{(s,t)\in\Delta(T)} \det(\psi(sgt^{-1})), \tag{3.3}$$

where $\Delta(T) = \{(s,t) \in T \times T \,|\, sgt^{-1} \in G_\psi\}$. Take $T = \{1, c, \ldots, c^{x-1}\}$ and $g = c$. Then g acts as a cycle of length x on the right cosets of G_ψ and $\mathrm{sign}(g) = -1$, since x is even. Moreover, sgt^{-1} is in G_ψ if and only if it is in $\langle c^x \rangle$. But $\psi(c^x) = M$

and $\det M = 1$, so the product term in (3.3) is 1. Finally, $m = \deg\psi$ is odd, so $\det\sigma(c) = -1$. □

Proposition 3.4. *Assume that p is odd. Let χ be any ramified quadratic character of $K^\times$ and let n be the number of irreducible components of $\bar{\rho}^{tor}|_H$. Then*

$$(\det\bar{\rho}^{tor})(-1) = \chi(-1)^n. \tag{3.5}$$

Proof. Since the ratio of the two different ramified quadratic characters of $K^\times$ is the unramified quadratic character of $K^\times$, which vanishes on the units U_K, the right side of (3.5) is well-defined.

Suppose $\alpha \in I_K$ maps to -1 under class field theory homomorphism $\mathcal{W}_K \to K^\times$ and let $h \in H$ be the image of α under the natural projection $I_K \to H$. Then $\det\bar{\rho}^{tor}(-1)$ means $\det\bar{\rho}^{tor}|_H(h)$, independent of the choice of α. By raising to a suitable odd power, we can choose h in a 2-Sylow subgroup of H.

If $|H|$ is odd, the left side of (3.5) therefore is 1. By Remark 2.2.16, $\bar{\rho}^{tor}|_H$ can be realized over $\mathbb{Q}$ and the trivial representation does not appear in $\bar{\rho}^{tor}|_H$. Then Lemma 3.1(i) implies that $\bar{\rho}^{tor}|_H \cong \mu \oplus \mu^*$ for some representation μ of H. Hence n is even and the right side of (3.5) also is 1.

Assume for the rest of the proof that $|H|$ is even and let c generate the 2-Sylow subgroup of H that contains h. We can write

$$\bar{\rho}^{tor}|_H \cong \mu \oplus \mu^* \oplus \lambda_1 \oplus \cdots \oplus \lambda_m \oplus \nu^q,$$

where μ is a representation of H, each λ_i is an irreducible orthogonal representation of H with $\dim\lambda_i \geq 2$ and ν is the one-dimensional representation of H of order 2. Thus $\nu(c) = -1$ and by Lemma 3.1, $\det\lambda_i(c) = -1$ for all i. Furthermore, $n \equiv m + q \pmod 2$. Hence

$$(\det\bar{\rho}^{tor})(-1) = (\det\bar{\rho}^{tor}|_H)(h) = \begin{cases} 1 & \text{if } h \in \langle c^2\rangle, \\ (-1)^n & \text{otherwise.} \end{cases} \tag{3.6}$$

Let L be a ramified quadratic extension of K contained in F and let χ be the corresponding character of $K^\times$. Then LK^{nr} is the unique quadratic extension of K^{nr} in F^{nr}. Thus h is in $\langle c^2\rangle$ if and only if h is trivial on LK^{nr} if and only if $\chi(-1) = 1$, by the choice of h. This completes verification of (3.5). □

4. Calculation of $\delta(\rho')$

We keep the notation of Section 2.2: A is an abelian variety defined over K which acquires semistable reduction over the finite Galois extension F of K. The action of G_K on $V = \mathbb{V}_l(A)$, $V_1 = V^{I_K}$, $V_2 = V^{I_F}$ is given by the l-adic representation $\tilde{\rho}_l$. We also have an action of $\mathcal{W}_K$ on $V \otimes_\iota \mathbb{C}$ via the complex representation $\rho = \rho_l \otimes_\iota \mathbb{C}$ with ρ_l defined in (2.2.9). By construction, $\rho(I_F) = 1$, so $\rho|_{I_K}$ factors through the finite group $H = I_K/I_F = \mathrm{Gal}(F^{nr}/K^{nr})$. If N_l is the nilpotent endomorphism of V described in §2.2, then $\ker N_l = V_2$.

Recall that $N = N_l \otimes_\iota \mathbb{C}$ and that $\Phi \in G_K$ is a pre-image of $\varphi \in G_k$, where φ is the inverse of the Frobenius automorphism.

Proposition 4.1. *There is a perfect $\mathcal{W}_K$-equivariant pairing*

$$\Big((V \otimes_\iota \mathbb{C})^{\rho(I_K)}/\left(\ker N\right)^{\rho(I_K)}\Big) \times (\mathbb{V}_l(\mathcal{T}_{v_K}) \otimes_\iota \mathbb{C}) \to \mathbb{C} \otimes \omega,$$

with the action of Φ *on* $(V \otimes_\iota \mathbb{C})^{\rho(I_K)}/(\ker N)^{\rho(I_K)}$ *corresponding to that of* φ *on* $\mathbb{V}_l(\mathcal{T}_{v_K})$.

Proof. Since $\rho|_H$ is semisimple, we find the $\rho(I_K)$-invariants of $V \otimes_\iota \mathbb{C}$ by considering successive quotients corresponding to the filtration $V \supseteq V_1^\perp + V_2 \supseteq V_2 \supseteq 0$, i.e.,

$$V/(V_1^\perp + V_2) \;\oplus\; (V_1^\perp + V_2)/V_2 \;\oplus\; V_2. \tag{4.2}$$

Then N_l is 0 on each piece, so $\rho_l(I_K)$ and $\tilde{\rho}_l(I_K)$ have the same invariants.

By virtue of the pairing (2.2.1), $(\tilde{\rho}_l(h) - 1)(V) \subseteq V_1^\perp$ for all $h \in I_K$. Hence $\tilde{\rho}_l(I_K)$ acts trivially on the first piece $V/(V_1^\perp + V_2)$.

We say that the action of a group on a vector space is *fixed point free* if the only fixed point is 0. To verify that the action of $\tilde{\rho}_l(I_K)$ on $(V_1^\perp + V_2)/V_2$ is fixed point free, it suffices check on the dual space. Via the pairing (2.2.1), we have

$$\big((V_1^\perp + V_2)/V_2\big)^* \cong V_2^\perp/(V_2^\perp \cap V_1).$$

But the latter injects into V_2/V_1, where 0 is the only $\tilde{\rho}_l(I_K)$-invariant point.

As for the third term in (4.2), we have $V_2^{\tilde{\rho}_l(I_K)} = V_1$ and hence $(V \otimes_\iota \mathbb{C})^{\rho(I_K)}$ is isomorphic to the complexification of $V/(V_1^\perp + V_2) \oplus V_1$. Since $\ker N_l = V_2$, we have $(\ker N)^{\rho(I_K)} = V_1 \otimes_\iota \mathbb{C}$. Hence $(V \otimes_\iota \mathbb{C})^{\rho(I_K)}/(\ker N)^{\rho(I_K)}$ is isomorphic to the complexification of $V/(V_1^\perp + V_2)$. The latter pairs perfectly with $V_2^\perp \cap V_1 = V_1^\perp \cap V_1 \cong \mathbb{V}_l(\mathcal{T}_{v_K})$ by (2.2.5). Finally, compatibility of the Galois actions of Φ and φ is immediate. □

Corollary 4.3. *Let* $\mathcal{H}(\mathcal{T}_{v_K}) = \mathrm{Hom}_\mathbb{Z}\left(\mathcal{T}_{v_K}(\overline{k}), \overline{k}^\times\right)$ *be the character group of the torus* $\mathcal{T}_{v_K}$ *and let* $\mathcal{H}_{\mathbb{Q}_l}(\mathcal{T}_{v_K}) = \mathcal{H}(\mathcal{T}_{v_K}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$. *Then*

$$\delta(\rho') = \det\left(-\varphi|_{\mathcal{H}_{\mathbb{Q}_l}(\mathcal{T}_{v_K})\otimes_\iota\mathbb{C}}\right).$$

Proof. Since the natural G_k-equivariant pairing $\mathcal{T}_{v_K}\left(\overline{k}\right) \times \mathcal{H}(\mathcal{T}_{v_K}) \to \overline{k}^\times$ induces a perfect pairing $\mathbb{V}_l(\mathcal{T}_{v_K}) \times \mathcal{H}_{\mathbb{Q}_l}(\mathcal{T}_{v_K}) \to \mathbb{Q}_l \otimes \omega_l$, the Proposition implies the claim. □

5. Root numbers of semistable Jacobians

Let $A = J(C)$ be the Jacobian of a smooth geometrically irreducible proper curve C over K such that the reduction $\mathcal{C}_{v_K}$ of C is a stable curve over k in the sense of [DM69, Def. 1.1]. By [DM69, Thm. 2.4], A has semistable reduction over K, so $F = K$ in our notation above. We have $W(\rho) = 1$ by Prop. 2.2.17. Hence the definition (2.2.11) and Cor. 4.3 give

$$W(A) = \delta(\rho') = \det\left(-\varphi|_{\mathcal{H}_{\mathbb{Q}_l}(\mathcal{T}_{v_K})\otimes_\iota\mathbb{C}}\right). \tag{5.1}$$

It is known that the connected component of the reduction of $J(C)$ over k is the Jacobian of the reduction of C, i.e., $\mathcal{A}^0_{v_K} = J(\mathcal{C}_{v_K}) = \mathrm{Pic}^0(\mathcal{C}_{v_K})$ by the results in [R70]. To simplify the notation throughout this subsection we will write $\mathcal{C}$ and $\mathcal{T}$ instead of $\mathcal{C}_{v_K}$ and $\mathcal{T}_{v_K}$, respectively. Let $\eta : \tilde{\mathcal{C}} \to \mathcal{C}$ be the normalization of $\mathcal{C}$, let $\tilde{\mathcal{C}}_1, \ldots, \tilde{\mathcal{C}}_s$ be irreducible components of $\tilde{\mathcal{C}}$, and let Σ be the set of singular points of $\mathcal{C}$ over $\overline{k}$, with $n = |\Sigma|$. We have the exact sequence ([BLR90, p. 247]):

$$1 \to \overline{k}^\times \longrightarrow \oplus_{i=1}^s \overline{k}^\times \longrightarrow \oplus_{i=1}^n \overline{k}^\times \to \mathcal{T}(\overline{k}) \to 1. \tag{5.2}$$

In particular, $\dim \mathcal{T} = n - s + 1$.

The character group $\mathcal{H}(\mathcal{T}) = \mathrm{Hom}_{\mathbb{Z}}(\mathcal{T}(\overline{k}), \overline{k}^{\times})$ can be described by constructing a directed graph Γ, as follows. The set of vertices $\mathcal{V}$ is the set of irreducible components of $\tilde{\mathcal{C}}$. For each node of $\mathcal{C}$, say $z \in \Sigma$, fix an ordering, say (x, y) for the pair of points over z on $\tilde{\mathcal{C}}$. Suppose that x and y lie on the components $\tilde{\mathcal{C}}_i$ and $\tilde{\mathcal{C}}_j$, respectively. Let there be a directed edge e_z of Γ from the vertex $\tilde{\mathcal{C}}_i$ to the vertex $\tilde{\mathcal{C}}_j$. We allow the possibility of a loop if $i = j$. Denote the set of edges of Γ by $\mathcal{E}$. Let $C_0(\Gamma, \mathbb{Z})$ and $C_1(\Gamma, \mathbb{Z})$ be the free abelian groups on $\mathcal{V}$ and $\mathcal{E}$, respectively and define the boundary map γ by $\gamma(e_z) = \tilde{\mathcal{C}}_j - \tilde{\mathcal{C}}_i$. It follows from [BLR90, p. 247] that by dualizing (5.2), we obtain the exact sequence of G_k-modules:

$$0 \to \mathcal{H}(\mathcal{T}) \to C_1(\Gamma, \mathbb{Z}) \xrightarrow{\gamma} C_0(\Gamma, \mathbb{Z}) \xrightarrow{a} \mathbb{Z} \to 0, \tag{5.3}$$

in which a is the augmentation map induced by $a(\tilde{\mathcal{C}}_i) = 1$ for all i. Here G_k acts trivially on $\mathbb{Z}$, the Galois action on $C_0(\Gamma, \mathbb{Z})$ is induced from the Galois action on the set of irreducible components $\{\tilde{\mathcal{C}}_1, \ldots, \tilde{\mathcal{C}}_s\}$, the Galois action on $C_1(\Gamma, \mathbb{Z})$ is induced from the action on ordered pairs $(x, y) \mapsto (g(x), g(y))$ for $g \in G_k$ and we have the canonical action of G_k on $\mathcal{H}(\mathcal{T})$.

It is convenient to consider the set of singular k-points Σ_k of $\mathcal{C}$. Thus each element of Σ_k is an orbit, denoted $[z]$, of a node $z \in \Sigma$ under the action of G_k. Similarly, define $\mathcal{C}_k$, the set of irreducible k-components of $\tilde{\mathcal{C}}$. Let x and y be the points over z in $\tilde{\mathcal{C}}$ and let d be the size of the orbit $[z]$. Define a sign $\tau_{[z]}$ to be $+1$ if φ^d fixes x and y, while $\tau_{[z]} = -1$ if φ^d switches x and y.

Proposition 5.4. *Let $n_k = |\Sigma_k|$ be the number of singular k-points of $\mathcal{C}$ and let $s_k = |\mathcal{C}_k|$ be the number of irreducible k-components of $\tilde{\mathcal{C}}$. Then*

$$W(A) = (-1)^{n_k + s_k + 1} \prod_{[z] \in \Sigma_k} \tau_{[z]}.$$

Proof. By (5.1) it is enough to show that

$$\det\left(-\varphi|_{\mathcal{H}(\mathcal{T}) \otimes_{\mathbb{Z}} \mathbb{Q}}\right) = (-1)^{n_k + s_k + 1} \prod_{[z] \in \Sigma_k} \tau_{[z]}. \tag{5.5}$$

We tensor (5.3) with $\mathbb{Q}$ over $\mathbb{Z}$ and compute $\det(-\varphi)$ on each term in the exact sequence, beginning on the right side, where φ acts trivially on $\mathbb{Z}$. Hence

$$\det\left(-\varphi|_{\mathbb{Q}}\right) = -1. \tag{5.6}$$

Consider the orbit $\mathfrak{O}$ of a vertex of Γ under φ. Let S be the G_k-invariant submodule of $C_0(\Gamma, \mathbb{Z})$ generated by the elements of $\mathfrak{O}$ and let $t = |\mathfrak{O}|$. Then φ acts on $\mathfrak{O}$ as a cyclic permutation of length t and sign $(-1)^{t-1}$, so we have

$$\det(-\varphi|_{S \otimes_{\mathbb{Z}} \mathbb{Q}}) = (-1)^t \text{ sign } \varphi = (-1)^t \, (-1)^{t-1} = -1.$$

Note that each such orbit corresponds to a component of $\tilde{\mathcal{C}}$ defined over k. Hence the number of distinct orbits is s_k. By decomposing $C_0(\Gamma, \mathbb{Z})$ according to these distinct orbits, we find that

$$\det\left(-\varphi|_{C_0(\Gamma, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}}\right) = (-1)^{s_k}. \tag{5.7}$$

Consider the orbit $\mathfrak{O}' = [z]$ of a node $z \in \Sigma$ of $\mathcal{C}$, under the action of φ and let $d = |\mathfrak{O}'|$. The Galois conjugates of the associated edge e_z on the graph Γ generate a G_k-invariant submodule S_1 of $C_1(\Gamma, \mathbb{Z})$ whose rank is d and on which φ^d acts as multiplication by $\tau_{[z]}$. Hence $\det(-\varphi|_{S_1\otimes_{\mathbb{Z}}\mathbb{Q}}) = -\tau_{[z]}$. Each orbit $[z]$ corresponds to an element of Σ_k and since there are n_k distinct orbits, we have

$$\det\left(-\varphi|_{C_1(\Gamma,\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Q}}\right) = (-1)^{n_k} \prod_{[z]\in\Sigma_k} \tau_{[z]}. \tag{5.8}$$

Thus (5.5) follows from (5.6) – (5.8). □

Corollary 5.9. *Let $k' \subset \overline{k}$ be the quadratic extension of k and let $n_{k'} = |\Sigma_{k'}|$ be the number of singular k'-points of $\mathcal{C}$. Then*

$$\det\left(-\varphi|_{\mathcal{H}(\mathcal{T})\otimes_{\mathbb{Z}}\mathbb{Q}}\right) = (-1)^{n_{k'}+n_k+s_k+1}.$$

Proof. For each $[z]$ in Σ_k, there are one or two elements over it in $\Sigma_{k'}$, corresponding to whether $\tau_{[z]} = -1$ or $+1$. Hence

$$\prod_{[z]\in\Sigma_k} \tau_{[z]} = (-1)^{n_{k'}}. \qquad \square$$

6. Examples

In this section we give examples of genus 2 hyperelliptic curves C over $\mathbb{Q}$ for which we can compute global root number of the Jacobians $A = J(C)$ using the results above. Refer to §2.2 for the following notation. In each example, $J(C)$ has additive reduction at $p = 3$, i.e., for $K = \mathbb{Q}_3$, the connected component $\mathcal{A}^0_{v_K} = \mathcal{U}$ is unipotent. Moreover, A requires a wildly ramified extension F of $\mathbb{Q}_3$ to achieve semistable reduction, which is then totally toroidal. Thus $\mathcal{A}^0_{v_F} = \mathcal{T}_{v_F}$ is a torus. If $\bar{\rho}^{av}_l$ is the representation in (2.2.15), then $\bar{\rho}^{av} = \bar{\rho}^{av}_l \otimes_\iota \mathbb{C}$ is trivial and its root number $W(\bar{\rho}^{av}) = 1$. Since $\mathcal{T}_{v_K}$ is trivial, Cor. 4.3 gives $\delta(\rho') = 1$. By the definition (2.2.11) of the local root number and Prop. 2.2.17, we have

$$W_3 = W(A_{/\mathbb{Q}_3}) = (\det \bar{\rho}^{tor})(-1), \tag{6.1}$$

which we evaluate by Prop. 3.4. At other primes $q \neq 3$ the curves C have stable reduction over $\mathbb{Q}_q$ and so $A = J(C)$ is semistable over $\mathbb{Q}_q$ by [BLR90, Example 8]. In that case, we evaluate $W_q = W(A_{/\mathbb{Q}_q})$ by Prop. 5.4. In particular, $W(A_{/\mathbb{Q}_q}) = 1$ for prime q at which A has good reduction.

Next we discuss some polynomials needed to construct the curves C. Let

$$f(x) = x^3 + 3ax^2 + 3bx + 3c, \quad a, b, c \in \mathbb{Z},\ 3 \nmid c. \tag{6.2}$$

Then $3^d \,\|\, \operatorname{disc}(f)$, with $d = \begin{cases} 3, & \text{if } 3 \nmid b, \\ 4, & \text{if } 3 \mid b \text{ and } 3 \nmid a, \\ 5, & \text{if } 3 \mid b \text{ and } 3 \mid a. \end{cases}$

Fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$, let $L \subset \overline{\mathbb{Q}}$ be the splitting field of f and factor $f(x) = (x - r_1)(x - r_2)(x - r_3)$ with $r_1, r_2, r_3 \in L$. Since f is Eisenstein at 3, the inertia subgroup $I_v(L/\mathbb{Q})$ of $\mathrm{Gal}(L/\mathbb{Q})$ at a place v over 3 in L contains an element σ of order 3. Furthermore, $I_v(L/\mathbb{Q})$ is the cyclic group of order 3 if d is even and $I_v(L/\mathbb{Q})$ is the symmetric group S_3 if d is odd.

Thanks to the action of σ, the various differences of roots of f have the same v-adic valuation, say $m = v(r_i - r_j)$ for all $i \neq j$. Since

$$\prod_{1\le i<j\le 3} (r_i - r_j)^2 = \operatorname{disc}(f),$$

we find that $m = \frac{1}{6}\, v(\mathrm{disc}(f)) = \frac{1}{6}\, v(3)\, d = \begin{cases} d & \text{if } d \text{ is odd,} \\ d/2 & \text{if } d \text{ is even.} \end{cases}$

Fix an element t in the ring of integers $\mathcal{O}_L$ such that $v(t) = 1$ and write

$$r_2 - r_1 = t^m u \quad \text{and} \quad r_3 - r_1 = t^m u'.$$

Then u and u' embed as v-adic units in the completion L_v. Furthermore, u and u' are distinct modulo t, since $t^m(u' - u) = r_3 - r_2$ also has v-adic valuation m. By stretching and translation, we have

$$f(t^m X + r_1) = t^m X(t^m X - t^m u)(t^m X - t^m u') = t^{3m} X(X - u)(X - u'). \tag{6.3}$$

Proposition 6.4. *Let $g(x) = \sum_{j=0}^{5} c_j x^j \in \mathbb{Z}[x]$ with $9|c_0$ and $3|c_1, c_2, c_3$. Given $f(x)$ as in (6.2), with $3^d \| \operatorname{disc}(f)$, suppose that $f(x)^2 + 4\cdot 3^{d-1} g(x)$ has distinct roots in $\overline{\mathbb{Q}}$. Then the Jacobian $A = J(C)$ of the hyperelliptic genus 2 curve C defined by*

$$C\colon\ y^2 = f(x)^2 + 4\cdot 3^{d-1} g(x)$$

has additive reduction over $\mathbb{Q}_3$ and totally toroidal reduction over L_v for $v|3$. Furthermore, L_v is the minimal Galois extension of $\mathbb{Q}_3$ over which A becomes semistable. The local root number is given by $W_3 = (-1)^d$.

Proof. By (6.3), the change of variables $x = t^m X + r_1$, $y = t^{3m} Y$ leads to a model C' of C over L_v of the form

$$C' : Y^2 = X^2(X - u)^2(X - u')^2 + 4t\, G(X), \tag{6.5}$$

in which the coefficients of $G(X) = 3^{d-1} g(t^m X + r_1)/t^{6m+1}$ are v-integral because of the congruences imposed on the coefficients of $g(x)$. Indeed, let $e = 2$ if d is odd and $e = 1$ if d is even. Then $v(3) = 3e$ and $v(t^m) = m > v(r_1) = e$. For the desired integrality, it suffices to choose $\alpha_j = \mathrm{ord}_3(c_j)$ satisfying $t^{6m+1} \mid 3^{d-1} c_j (r_1)^j$ for all j. Thus

$$\alpha_j \geq \frac{6m + 1 - je}{3e} - d + 1,$$

which leads to the congruences stated in the Proposition.

Clearly, C' defines a stable curve [DM69, Def. 1.1] over the ring of integers $\mathcal{O}_{L_v}$ of L_v and the reduction $\mathcal{C}'$ of C' over the residue field k of $\mathcal{O}_{L_v}$ consists of two projective lines intersecting transversally at three points. We claim that the connected component $\mathcal{A}_v^0$ of the special fiber of the Néron model $\mathcal{A}$ at $v = v_L$ is totally toroidal. Indeed, over an algebraic closure $\overline{k}$ of k, $\mathcal{A}_v^0$ is isomorphic to $\mathrm{Pic}^0(\mathcal{C}'/\overline{k})$ (see [R70]). The dimension of the toric part $\mathcal{T}_v$ of $\mathcal{A}_v^0$ is the first Betti number of $\mathcal{C}'$ and the dimension of the abelian variety part $\mathcal{B}_v$ is the sum of the genera of the irreducible components of $\mathcal{C}'$, cf. [BLR90, Example 8, Cor. 12].

Over $\mathbb{Z}$, the curve C has a model of the form

$$y^2 = (x^3 + e_1 x^2 + e_2 x + e_3)^2 + e_4 x^2 + e_5 x + e_6, \quad \forall e_i \in \mathbb{Z},$$

such that $3\|e_3$; $3|e_1, e_2$ and $9|e_4, e_5, e_6$. Liu's algorithm [L94, pp. 151–152] implies that 3 divides the degree of the minimal Galois extension of $\mathbb{Q}_3$ over which C has stable reduction. Furthermore, the closed fiber $\mathcal{F}$ of a minimal model of C over $\mathbb{Z}_3$ has type $[\mathrm{III}_N]$ for some natural number N (see [NU73]). In particular, $\mathcal{F}$ has no cycles and its irreducible components are projective lines. As was explained at the end of the previous paragraph, this implies that the reduction of $A_{/\mathbb{Q}_3}$ must be additive. Finally, by the Deligne-Mumford theorem ([DM69, Thm. 2.4]) on stable curves and their Jacobians, L_v is the minimal Galois extension over which A becomes semistable.

If d is odd, the inertia group $I_v(L/\mathbb{Q})$ is the symmetric group S_3 and $\bar{\rho}^{tor}$ is its 2-dimensional irreducible representation. If d is even, $I_v(L/\mathbb{Q})$ is cyclic of order 3 and $\bar{\rho}^{tor} \cong \mu \oplus \mu^*$ where μ^* is a character of order 3. Hence $W_3 = (-1)^d$ by (6.1) and Prop. 3.4. □

Our global examples of genus 2 curves C have the form

$$C\colon\ y^2 + Q(x)y = P(x), \quad P, Q \in \mathbb{Z}[x], \tag{6.6}$$

with $F = Q^2 + 4P$ of degree 6. Then the discriminant of C is $\Delta_C = 2^{-12}\operatorname{disc}(F)$, cf. [L96, p. 4581].

Lemma 6.7. *Suppose that the reduction $\bar{F}$ of F at an odd prime q has the form*

$$\bar{F}(x) = (x-r)^2\bar{h}(x), \quad r \in \mathbb{F}_q, \quad \bar{h}(r) \neq 0 \text{ in } \mathbb{F}_q,$$

where $\bar{h}$ is a separable polynomial of degree 3 or 4 in $\mathbb{F}_q[x]$. In terms of the Legendre symbol, the local root number of $A = J(C)$ is $W_q = -\left(\frac{\bar{h}(r)}{q}\right)$. In particular, this holds when $\operatorname{ord}_q(\Delta_C) = 1$.

Proof. The equation $y^2 = F(x)$ defines a minimal Weierstrass model of C over $\mathbb{Z}_q$ (see [L96]). The reduction $\mathcal{C}$ of C at q has one ordinary double point, corresponding to $x = r$ and r is in $k = \mathbb{F}_q$ by uniqueness of this node. The blow-up of $\mathcal{C}$ at $x = r$ is a curve of genus 1. This implies that A has semistable reduction at q, with abelian variety and toric parts both of dimension one. The points over $x = r$ on the normalization are rational or not, according to whether or not the slopes of the tangents to $\mathcal{C}$ at $x = r$ are in k, i.e., whether or not $\bar{h}(r)$ is a square in k. Prop. 5.4 applies with $n_k = 1$ and, since $\mathcal{C}$ is irreducible, $s_k = 1$. Hence $W_q = -\tau_{[r]} = -\left(\frac{\bar{h}(r)}{q}\right)$. □

The numerical examples of curves in the tables below have minimal models over $\mathbb{Z}$ of the form (6.6), where $P = 3^{d-1}g$ and $Q = f$, with $f(x)$, $g(x)$ and d as in Prop. 6.4. That Proposition and Lemma 6.7 give the local root numbers. The global root number $W_{\mathbb{Q}}(A)$ is their product. We write G_v and I_v for the decomposition and inertia groups in $\operatorname{Gal}(L/\mathbb{Q})$ at $v|3$ in L. Magma [M97] was used to find the rank of $A(\mathbb{Q})$.

Table 1. $Q = f = x^3 - 3x^2 + 3$, $\operatorname{disc} f = 3^4$, $G_v = I_v = \mathbb{Z}/3$

P	Δ_C	local root numbers	$W_{\mathbb{Q}}$	rank $A(\mathbb{Q})$
$81x$	$3^{21} \cdot 17 \cdot 4931$	$W_3 = 1$, $W_{17} = W_{4931} = -1$	1	2
$-81x$	$3^{21} \cdot 86201$	$W_3 = 1$, $W_{86201} = -1$	-1	1
$27x^5$	$3^{22} \cdot 53^2 \cdot 73$	$W_3 = W_{53} = 1$, $W_{73} = -1$	-1	1
$-81x^5$	$3^{25} \cdot 13 \cdot 561359$	$W_3 = W_{13} = 1$, $W_{561359} = -1$	-1	1

Table 2. $Q = f = x^3 - 3x - 3$, $\operatorname{disc} f = -3^3 \cdot 5$, $G_v = I_v = S_3$

P	Δ_C	local root numbers	$W_{\mathbb{Q}}$	rank $A(\mathbb{Q})$
81	$-3^{18} \cdot 5^2 \cdot 13 \cdot 349$	$W_{13} = W_{349} = 1$, $W_3 = W_5 = -1$	1	0
-81	$-3^{18} \cdot 13 \cdot 17 \cdot 19 \cdot 23$	$W_{17} = 1, W_3 = -1$, $W_{13} = W_{19} = W_{23} = -1$	1	0
$27x$	$3^{16} \cdot 97 \cdot 491$	$W_{491} = 1$, $W_3 = W_{97} = -1$	1	2
$-27x$	$3^{16} \cdot 19 \cdot 733$	$W_{19} = W_{733} = 1$, $W_3 = -1$	-1	1
$27x^3$	$3^{18} \cdot 13^2 \cdot 251$	$W_{13} = W_{251} = 1$, $W_3 = -1$	-1	1
$-27x^3$	$3^{18} \cdot 7 \cdot 73 \cdot 163$	$W_7 = W_{163} = 1$, $W_3 = W_{73} = -1$	1	2
$-9x^5$	$3^{17} \cdot 11 \cdot 23 \cdot 331$	$W_{11} = W_{23} = 1$, $W_{331} = 1, W_3 = -1$	-1	1

Table 3. $Q = f = x^3 - 6x^2 + 9x + 3$, $\operatorname{disc} f = -3^4 \cdot 7$, $G_v = S_3$, $I_v = \mathbb{Z}/3$

P	Δ_C	local root numbers	$W_{\mathbb{Q}}$	rank $A(\mathbb{Q})$
243	$-3^{23} \cdot 109 \cdot 1021$	$W_3 = W_{109} = 1$, $W_{1021} = -1$	-1	1
-243	$-3^{23} \cdot 13 \cdot 71 \cdot 107$	$W_3 = W_{107} = 1$, $W_{13} = W_{71} = -1$	1	0
$-81x^3$	$-3^{23} \cdot 13^2 \cdot 2423$	$W_3 = W_{13} = 1$, $W_{2423} = 1$	1	2
$27x^5$	$3^{22} \cdot 37 \cdot 73 \cdot 131$	$W_3 = W_{73} = 1$, $W_{37} = W_{131} = -1$	1	2
$81x^5$	$3^{25} \cdot 47 \cdot 4691$	$W_3 = W_{47} = 1$, $W_{4691} = -1$	-1	1

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Department of Mathematics, Fordham University, Bronx, NY 10458
E-mail address: brumer@fordham.edu

Department of Mathematics, Queens College, Flushing, NY 11367; The Graduate Center CUNY, New York, NY 10016
E-mail address: kkramer@qc.cuny.edu

Department of Mathematics, Queens College, Flushing, NY 11367; The Graduate Center CUNY, New York, NY 10016
E-mail address: maria.sabitova@qc.cuny.edu