A SIMPLIFICATION OF ROOT VALUATION DATA FOR CLASSICAL GROUPS

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Abstract. We study the root valuation strata of the adjoint quotient of the Lie algebra of a connected reductive group G over the field of complex numbers. Given a fixed maximal torus T of G and the corresponding Weyl group W each root valuation stratum corresponds to a pair (w, r) of an element w in W and a rational-valued function r on the set R of roots of T in G. We address the following question posed in a joint paper by Goresky, Kottwitz, and MacPherson. Suppose that for w, w' in W and a rational-valued function r on R the two root valuation strata corresponding to (w, r) and (w', r), respectively, are non-empty. Is it true that w and w' are conjugate in W (more precisely, in the stabilizer of r in W)? Goresky, Kottwitz, and MacPherson show that the answer is positive if r is a constant function. We show that the answer is positive for an arbitrary r if G is of classical type.

Introduction

Let G be a connected reductive group over $\mathbb{C}$ with a fixed maximal torus T and the corresponding Weyl group W. In order to study affine Springer fibers Goresky, Kottwitz, and MacPherson in their joint paper [GKM06] introduce the root valuation strata of the adjoint quotient $\mathbb{A} = \mathfrak{g}/G$ of the Lie algebra $\mathfrak{g}$ of G. Given the ring $\mathcal{O} = \mathbb{C}[[\epsilon]]$ of formal power series each root valuation stratum is a subset of $\mathbb{A}(\mathcal{O})' = \mathbb{A}(\mathcal{O}) \cap \mathbb{A}_{reg}(F)$, where $F = \mathbb{C}((\epsilon))$ is the field of formal Laurent power series and $\mathbb{A}_{reg}$ denotes the set of all elements in $\mathbb{A}$ that are images of regular semisimple elements in $\mathfrak{g}$ under the natural map. Furthermore, $\mathbb{A}(\mathcal{O})'$ is a (infinite) disjoint union of distinct root valuation strata. In [GKM06] the authors show that affine Springer fibers over points in the same root valuation stratum have the same dimension and it is expected that overall they have similar geometric characteristics.

Each root valuation stratum $\mathbb{A}(\mathcal{O})_{(w,r)}$ depends on a pair (w, r) of an element w in W and a $\mathbb{Q}$-valued function r on the set R of roots of T in G. More precisely, the correspondence is as follows. Let $\overline{F}$ be an algebraic closure of F, $\Gamma = \mathrm{Gal}(\overline{F}/F)$, and let τ denote a (non-canonical) topological generator of Γ. Let w be an element of W of order l and let $r : R \longrightarrow \mathbb{Q}_{\geq 0}$ be a function. For an l-th root $\epsilon^{1/l}$ of ϵ in $\overline{F}$ put $\mathcal{O}_l = \mathbb{C}[[\epsilon^{1/l}]]$ and define

$$\mathfrak{t}_w(\mathcal{O})_r := \left\{ u \in \mathfrak{t}(\mathcal{O}_l) \,\middle|\, w(\tau(u)) = u \text{ and } r(\alpha) = \mathrm{val}\, \alpha(u), \quad \forall \alpha \in R \right\},$$

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where $\mathfrak{t}$ denotes the Lie algebra of T and the valuation val on $\overline{F}$ extends the standard valuation on F via $\mathrm{val}(\epsilon^{1/l}) = 1/l$. By definition the root valuation stratum $\mathbb{A}(\mathcal{O})_{(w,r)}$ is the image of $\mathfrak{t}_w(\mathcal{O})_r$ under the map induced by the natural projection $\mathfrak{t} \twoheadrightarrow \mathbb{A}$, where $\mathbb{A}$ is identified with $\mathfrak{t}/W$. It is not difficult to see that the stratum $\mathbb{A}(\mathcal{O})_{(w,r)}$ depends only on the W-orbit of the pair (w, r), where W acts on itself by conjugation and on the set of functions $\{r : R \longrightarrow \mathbb{Q}_{\geq 0}\}$ in the natural way. In [GKM06] Goresky, Kottwitz, and MacPherson provide necessary and sufficient conditions for a root valuation stratum to be non-empty and prove that for a constant function r the non-emptyness of both $\mathbb{A}(\mathcal{O})_{(w,r)}$ and $\mathbb{A}(\mathcal{O})_{(w',r)}$ for $w, w' \in W$ implies that (w, r) and (w', r) are in the same W-orbit, i. e., more precisely, w and w' are conjugate under an element in the stabilizer of r in W. They ask whether this is true for an arbitrary r. The goal of this note is to show that the question has a positive answer in the case of an arbitrary r and a classical G. Namely, the main result of this paper is the following

Theorem. *Assume in addition that G is classical. Let $w_1, w_2 \in W$ and let $r : R \longrightarrow \mathbb{Q}_{\geq 0}$ be a function. Suppose that $\mathbb{A}(\mathcal{O})_{(w_1,r)}$ and $\mathbb{A}(\mathcal{O})_{(w_2,r)}$ are both non-empty. Then w_1 and w_2 are conjugate under an element in the stabilizer of r in W.*

Thus in the case under consideration non-empty root valuation strata depend only on functions r, which simplifies the original definition of Goresky, Kottwitz, and MacPherson.

Loosely speaking the proof of the theorem is based on the observation that for arbitrary G and r to show whether w_1 and w_2 are conjugate it is enough to show whether certain varieties are irreducible (see Lemma 3.6 and Remark 3.7 below). Each such variety is an open subset inside the quotient of a union of linear subspaces of the $\mathbb{C}$-span of a root subsystem $R' \subseteq R$ by the action of a subgroup H' in the Weyl group W' of R' ($W' \subseteq W$). Thus in order to analyze these varieties one needs to understand the action of H'. If G is a classical group, then due to the simplicity of W (and hence that of W') the action can be written explicitly and the irreducibility of the aforementioned varieties can be checked case by case with little modification between the cases when R is of type A, B, C, or D (Section 4 below). The group W is much more complicated in the case of an exceptional G and additional considerations are required.

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1. General facts and notation

In this paper we will keep the same notation as in [GKM06]. This section is devoted to reviewing this notation and some results that will be used later.

1.1. **The setup.** Let G be a connected reductive group over $\mathbb{C}$. We fix a maximal torus T of G with its Lie algebra $\mathfrak{t}$, the root system $R \subset X^*(T)$ of G associated to T, and the Weyl group W of T in G. We say that a subset R_1 of R is $\mathbb{Q}$-*closed* if $\alpha_1, \ldots, \alpha_t \in R_1$, $m_1, \ldots, m_t \in \mathbb{Q}$, and $\alpha = m_1\alpha_1 + \cdots + m_t\alpha_t \in R$ imply $\alpha \in R_1$.

By abuse of notation we will denote the differential of a root $\alpha \in R$ also by α. Later we will need the following result:

Lemma 1.1. *Let R_1 be a $\mathbb{Q}$-closed subset of R. If $u \in \mathfrak{t}$ and $\alpha(u) = 0$ for all $\alpha \in R_1$, then the Weyl group $W(R_1)$ of R_1 is contained in the stabilizer $W_u := \{w \in W \mid w(u) = u\}$ of u in W. If in addition $\alpha(u) \neq 0$ for all $\alpha \in R\backslash R_1$, then $W(R_1) = W_u$.*

Proof. The lemma is a consequence of Corollary 2.8, Lemma 3.7, Corollary 3.11, and Theorem 3.14 in [St75] (also [GKM06], Prop. 14.1.1(1)). □

1.2. **The base field F.** Let $F = \mathbb{C}((\epsilon))$ be the field of formal Laurent power series over $\mathbb{C}$ in an indeterminate ϵ with the ring of integers $\mathcal{O} = \mathbb{C}[[\epsilon]]$. For each $n \in \mathbb{N}$ let $\epsilon^{1/n} \in \overline{F}^{\times}$ (resp., $\xi_n \in \mathbb{C}^{\times}$) be a fixed n-th root of ϵ (resp., a primitive n-th root of unity) such that

$$\left(\epsilon^{\frac{1}{mn}}\right)^m = \epsilon^{1/n}, \quad (\xi_{mn})^m = \xi_n, \quad \forall m, n \in \mathbb{N}.$$

We put $F_n = \mathbb{C}((\epsilon^{1/n}))$, $\mathcal{O}_n = \mathbb{C}[[\epsilon^{1/n}]]$, and let τ_n denote the automorphism of F_n given by

$$\tau_n\left(\epsilon^{1/n}\right) = \xi_n \cdot \epsilon^{1/n}.$$

It is known that $\overline{F} = \bigcup_{n\in\mathbb{N}} F_n$. Thus the element $\tau_\infty \in \mathrm{Aut}(\overline{F})$ defined in such a way that its restriction to each F_n equals τ_n is a topological generator of $\Gamma = \mathrm{Gal}(\overline{F}/F)$, i.e., it determines an isomorphism $\widehat{\mathbb{Z}} \xrightarrow{\simeq} \Gamma$. We also fix the valuation val on $\overline{F}$ such that

$$\mathrm{val}(\epsilon^{1/n}) = 1/n, \quad \forall n \in \mathbb{N}, \quad \text{and} \quad \mathrm{val}(0) = +\infty.$$

1.3. **The definition of root valuation strata.** Let w be an element of W of order l and let $r : R \longrightarrow \mathbb{Q}_{\geq 0}$ be a function. Define

$$\mathfrak{t}_w(\mathcal{O}) := \left\{u \in \mathfrak{t}(\mathcal{O}_l) \,\middle|\, w\left(\tau_l(u)\right) = u\right\}$$

and

$$\mathfrak{t}_w(\mathcal{O})_r := \left\{u \in \mathfrak{t}_w(\mathcal{O}) \,\middle|\, r(\alpha) = \mathrm{val}\,\alpha(u), \quad \forall \alpha \in R\right\}.$$

By definition the root valuation stratum $\mathbb{A}(\mathcal{O})_{(w,r)}$ is the image of $\mathfrak{t}_w(\mathcal{O})_r$ under the map $\mathfrak{t}_w(\mathcal{O}) \longrightarrow \mathbb{A}(\mathcal{O})$ induced by the natural projection $\mathfrak{t} \twoheadrightarrow \mathbb{A}$, where $\mathbb{A}$ is identified with $\mathfrak{t}/W$ by the results of Springer and Steinberg (see [SS70]). Thus $\mathfrak{t}_w(\mathcal{O})_r$ is non-empty if and only if $\mathbb{A}(\mathcal{O})_{(w,r)}$ is non-empty, hence for our purposes it is enough to consider the sets $\mathfrak{t}_w(\mathcal{O})_r$.

1.4. **Conditions for $\mathfrak{t}_w(\mathcal{O})_r$ to be non-empty.** Let $r : R \longrightarrow \mathbb{Q}_{\geq 0}$ be a function that takes values in $\frac{1}{s}\mathbb{Z}$ for some $s \in \mathbb{N}$. For each $m \in \mathbb{Z}_{\geq 0}$ denote

$$R_m := \{\alpha \in R \,|\, r(\alpha) \geq m/s\}\,, \tag{1.1}$$

so that we have the chain

$$R = R_0 \supseteq R_1 \supseteq R_2 \supseteq \cdots .$$

Also, for $m \geq 1$ let

$$\begin{aligned} \mathfrak{a}_m &:= \{u \in \mathfrak{t} \,|\, \alpha(u) = 0, \ \forall \alpha \in R_m\}\,, \quad \text{and} \\ \mathfrak{a}_m^\sharp &:= \{u \in \mathfrak{a}_m \,|\, \alpha(u) \neq 0, \ \forall \alpha \in R_{m-1} \backslash R_m\}\,. \end{aligned}$$

For $w \in W$ of order l and each $i \in \mathbb{Z}_{\geq 0}$ we denote by $\mathfrak{t}(w,i)$ the set of all the eigenvectors of w in $\mathfrak{t}$ with the eigenvalue ξ_l^{-i} including the zero vector, i.e.,

$$\mathfrak{t}(w,i) := \left\{u \in \mathfrak{t} \,|\, w(u) = \xi_l^{-i} \cdot u\right\}.$$

Finally, we put

$$W_r := \left\{w \in W \,|\, r\left(w^{-1}(\alpha)\right) = r(\alpha), \quad \forall \alpha \in R\right\}.$$

Note that

$$W_r = \bigcap_{m \geq 0} \{w \in W \,|\, w(R_m) = R_m\}\,.$$

In the following lemma we summarize the results about strata $\mathfrak{t}_w(\mathcal{O})_r$ that will be used later.

Lemma 1.2. *If $\mathfrak{t}_w(\mathcal{O})_r$ is non-empty, then*
(1) $w^s = 1$;
(2) *r takes values in $\frac{1}{l}\mathbb{Z}$;*
(3) *each R_m is $\mathbb{Q}$-closed;*
(4) $w \in W_r$.
Also, the set $\mathfrak{t}_w(\mathcal{O})_r$ is non-empty if and only if $\mathfrak{t}(w,i) \cap \mathfrak{a}_{i+1}^\sharp$ is non-empty for all $i \geq 0$.

Proof. See [GKM06], Lemma 4.8.1, Proposition 4.8.2, and Corollary 4.8.4. □

2. Statement of the main result

2.1. **Main result.** For convenience we restate the main theorem of the note (Theorem in the introduction) using the notation of §1.

Theorem 2.1. *Assume that R is a reduced irreducible root system of type A, B, C, or D. Let $w_1, w_2 \in W$ and let $r : R \longrightarrow \mathbb{Q}_{\geq 0}$ be a function. Suppose that $\mathfrak{t}_{w_1}(\mathcal{O})_r$ and $\mathfrak{t}_{w_2}(\mathcal{O})_r$ are both non-empty. Then $w_1, w_2 \in W_r$ and they are conjugate by an element of W_r.*

Proof. See Lemma 2.5, Lemma 3.6, and §4. □

Remark 2.2. Theorem 2.1 in the case of an arbitrary R and a constant function r is proved in Proposition 4.9.1 and the discussion after it on p. 10 of [GKM06].

2.2. **Reduction of Theorem 2.1 to a question about root systems.** In this section we show that Theorem 2.1 follows from a certain result about root systems (Theorem 2.3 below).

Let V be a finite-dimensional vector space over $\mathbb{C}$ and let R be a reduced root system in the dual vector space V^*. Let W be the Weyl group of R. We identify V^* with V via a fixed W-invariant scalar product $(\cdot\,,\cdot)$ on V. Assume that we have the following chain:

$$R = R_0 \supseteq R_1 \supseteq R_2 \supseteq \cdots \supseteq R_k \supseteq R_{k+1} = \emptyset,$$

where $k \geq 0$ and each R_i is a $\mathbb{Q}$-closed subset of R. It is easy to see that each R_i is a root system in the vector space it spans. For each $i \in \{0, 1, \ldots, k\}$ we denote by W_i the subgroup of W consisting of all elements $w \in W$ such that $w(R_i) = R_i$, and by $W(R_i)$ the Weyl group of R_i considered as a subgroup of W. We also put $W_{k+1} = W$ and $W(R_{k+1}) = \{1\}$. Let

$$W_r := \bigcap_{i=0}^{k} W_i.$$

Also, let μ_l denote the group of all l-th roots of unity for a natural number l and let ζ_i be an arbitrary fixed element from μ_l for each $i \in \{0, 1, \ldots, k\}$.

Theorem 2.3 below is a slightly more general reformulation of Theorem 2.1 in terms of root systems than Theorem 2.1 itself.

Theorem 2.3. *Suppose that R is a reduced irreducible root system of type A, B, C, or D. If there exist $w_1, w_2 \in W_r$ and $\{u_{1i}\}_{i=0}^k, \{u_{2i}\}_{i=0}^k \in V$ such that*

$$w_1(u_{1i}) = \zeta_i \cdot u_{1i}, \quad w_2(u_{2i}) = \zeta_i \cdot u_{2i}, \quad \text{for each } i, \tag{2.1}$$

and

$$\alpha(u_{ji}) = 0 \quad \text{for any } \alpha \in R_{i+1}, \tag{2.2}$$

$$\alpha(u_{ji}) \neq 0 \quad \text{for any } \alpha \in R_i \backslash R_{i+1},\ j = 1, 2,\ i \in \{0, 1, \ldots, k\}, \tag{2.3}$$

then w_1 is conjugate to w_2 in W_r.

Proof. See Lemma 3.6 and §4. □

Remark 2.4. If $k = 0$, then Theorem 2.3 for an arbitrary root system R (not necessarily of classical type) is a well-known result of Springer on regular elements of Weyl groups (see [Sp74]). As was shown by Goresky, Kottwitz, and MacPherson this result implies Theorem 2.1 for an arbitrary R in the case of a *constant* function r ([GKM06], Prop. 4.9.1 and the discussion after it).

Lemma 2.5. *Theorem* 2.3 *implies Theorem* 2.1.

Proof. First, note that in Theorem 2.1 without loss of generality we can assume that R is a root system in $\mathfrak{t}^*$. Let $\mathfrak{t}_{w_1}(\mathcal{O})_r$ and $\mathfrak{t}_{w_2}(\mathcal{O})_r$ be both non-empty. Then by Lemma 1.2(4) we have $w_1, w_2 \in W_r$ and it is an easy consequence of Lemma 1.2(1), (2) that w_1, w_2 have the same order, say l. For each $m \geq 0$ let R_m be defined by (1.1). Then by Lemma 1.2(3) each R_m is $\mathbb{Q}$-closed. Finally, by Lemma 1.2 we have $\mathfrak{t}(w_j,i)\cap\mathfrak{a}^\sharp_{i+1}\neq\emptyset$ for all $i\geq 0$, which implies that for some $\zeta_i\in\mu_l$ and $u_{ji}\in\mathfrak{t}$ $(i\geq 0,\ j=1,2)$ the conditions (2.1) – (2.3) hold for the chain $R=R_0\supseteq R_1\supseteq R_2\cdots$. Thus, Theorem 2.1 follows from Theorem 2.3. □

3. Chains of root systems

In this section we show that Theorem 2.3 follows from a *general conjugacy theorem* (Theorem 3.1 below) together with one statement about root systems (Proposition 3.5 below).

3.1. **General conjugacy theorem.** Let X be a separated algebraic variety over $\mathbb{C}$, and let Z and G be finite groups acting on X by morphisms. For $z\in Z$ (resp., $g\in G$) and $x\in X$ we denote by $z\cdot x$ (resp., $g(x)$) the action of z (resp., of g) on x. Assume that the actions of Z and G commute, i.e.,

$$g(z\cdot x)=z\cdot g(x),\quad \forall g\in G,\, z\in Z,\, x\in X.$$

Let H be a normal subgroup in G and denote $X^\circ:=\{x\in X\,|\,H_x=\{1\}\}$, where H_x stands for the stabilizer of x in H. Let $Y=X/H$ be the quotient considered as a topological space with the quotient topology of the Zariski topology on X and let $f:X\twoheadrightarrow Y$ denote the quotient map. Since H is normal in G and the actions of Z and G on X commute, we have the induced actions of Z and G/H on Y. Let $\pi:G\twoheadrightarrow G/H$ denote the projection map and for $g\in G, z\in Z, \sigma\in G/H$ let

$$X(g,z):=\{x\in X\,|\,g(x)=z\cdot x\}\,,\quad Y(\sigma,z):=\{y\in Y\,|\,\sigma(y)=z\cdot y\}\,.$$

Theorem 3.1. *Let $z\in Z$ and $\sigma\in G/H$ be such that $Y(\sigma,z)\cap f(X^\circ)$ is irreducible. If there exist $g_1,g_2\in G$, $x_1,x_2\in X^\circ$ such that*

$$g_i(x_i)=z\cdot x_i,\quad i=1,2,$$

and $\pi(g_1)=\pi(g_2)=\sigma$, then there exists $h\in H$ such that $g_1=hg_2h^{-1}$.

Proof. Since X is separated, $X(g,z')$ is closed for all $g\in G, z'\in Z$. Also, for $g\neq g'$ in $\pi^{-1}(\sigma)$ we have $X(g,z)\cap X(g',z)\cap X^\circ=\emptyset$. Therefore

$$f^{-1}(Y(\sigma,z))\cap X^\circ=\coprod_{g\in\pi^{-1}(\sigma)}X(g,z)\cap X^\circ,$$

and each $X(g,z)\cap X^\circ$ is open in $f^{-1}(Y(\sigma,z))\cap X^\circ$. Since the restriction of f to $f^{-1}(Y(\sigma,z))\cap X^\circ$ is also an open map, we see that $f\left(X(g_1,z)\cap X^\circ\right)$ and $f\left(X(g_2,z)\cap X^\circ\right)$ intersect, being non-empty open subsets of the irreducible space $Y(\sigma,z)\cap f(X^\circ)$. Hence there exist $u,v\in X^\circ$ and $h\in H$ such that $u=h(v)$, $g_1(u)=z\cdot u$, and $g_2(v)=z\cdot v$. Combining these three equalities and taking into account that $\pi(g_1)=\pi(g_2)$, we get $g_1=hg_2h^{-1}$. □

Remark 3.2. This argument is a slightly more general version of Theorem 4.1 in [BS07].

Remark 3.3. In Theorem 3.1 the condition "$Y(\sigma,z)\cap f(X^\circ)$ is irreducible" can be replaced by a weaker condition "$Y(\sigma,z)\cap f(X^\circ)$ is connected." Indeed, for each $g\in\pi^{-1}(\sigma)$ the set $f(X(g,z)\cap X^\circ)$ is open in $Y(\sigma,z)\cap f(X^\circ)$ and $\mathcal{A} := \{f(X(g,z)\cap X^\circ)\}_{g\in\pi^{-1}(\sigma)}$ is a cover of $Y(\sigma,z)\cap f(X^\circ)$. Since by assumption $Y(\sigma,z)\cap f(X^\circ)$ is connected, any two elements $U,V\in\mathcal{A}$ can be joined by a chain of elements $U_1,\dots,U_n\in\mathcal{A}$, i.e.,

$$U=U_1,\ V=U_n,\ \text{and}\ U_i\cap U_{i+1}\neq\emptyset,\ \ \forall i\in\{1,\dots,n-1\}$$

(see e.g., [P89], p. 313, Lemma 1). Taking into account that for $g_1,g_2\in\pi^{-1}(\sigma)$ the sets $f(X(g_1,z)\cap X^\circ)$, $f(X(g_2,z)\cap X^\circ)$ intersect if and only if g_1,g_2 are conjugate by an element of H, this implies the claim.

Note that the converse to Theorem 3.1 does not hold in general. However, there is a partial converse.

Proposition 3.4. *In the notation of Theorem* 3.1 *assume in addition that X is an affine space and G, Z act linearly on X. Let $z\in Z$ and $\sigma\in G/H$ be such that the set*

$$M := \{g\in G\,|\,X(g,z)\cap X^\circ\neq\emptyset\}\cap\pi^{-1}(\sigma)$$

is non-empty. If any two elements of M are conjugate by an element of H, then the space $Y(\sigma,z)\cap f(X^\circ)$ is irreducible.

Proof. Indeed, it is easy to see that for all $z'\in Z$ and $\sigma'\in G/H$ we have

$$Y(\sigma',z')\cap f(X^\circ)=\bigcup_{g'\in\pi^{-1}(\sigma')} f(X(g',z')\cap X^\circ),$$

which under the hypotheses on σ and z in Proposition 3.4 implies

$$Y(\sigma,z)\cap f(X^\circ)=f(X(g,z)\cap X^\circ)\ \text{ for any } g\in M. \tag{3.1}$$

Note that X° is open in X and $X(g,z)$ is irreducible as a linear subspace of X. This implies that $X(g,z)\cap X^\circ$ is irreducible and hence $Y(\sigma,z)\cap f(X^\circ)$ is also irreducible by (3.1). □

3.2. **Reduction of Theorem 2.3.** In this subsection we use the notation of §2.2. For each $i\in\{0,1,\dots,k\}$ we put

$$V_i := \{u\in V\,|\,\alpha(u)=0,\ \forall\alpha\in R_{i+1}\}\quad\text{and}\quad H_i := (W_r\cap W(R_i))\,/\,(W_r\cap W(R_{i+1})).$$

By Lemma 1.1 the action of W_r on V_i induces the action of H_i on V_i and we denote $Y_i=V_i/H_i$. Let $f_i: V_i\twoheadrightarrow Y_i$ be the quotient map and let

$$V_i^\circ=\{u\in V_i\,|\,\alpha(u)\neq 0,\ \forall\alpha\in R_i\backslash R_{i+1}\},\quad Y_i^\circ=f_i(V_i^\circ).$$

By Lemma 1.1 the set V_i° is contained in the set of all points in V_i with trivial stabilizer in H_i. Note that since $W_r\cap W(R_i)$ is a normal subgroup of W_r, we also have the induced action of W_r on Y_i. The action of μ_l on V by multiplication induces the action of μ_l on each Y_i, because the actions of H_i and μ_l commute. For $\sigma\in W_r$ and $z\in\mu_l$ we put

$$Y_i(\sigma,z)=\{y\in Y_i\ |\,\sigma(y)=z\cdot y\},\quad Y_i^\circ(\sigma,z)=Y_i(\sigma,z)\cap Y_i^\circ=\{y\in Y_i^\circ\ |\,\sigma(y)=z\cdot y\}.$$

We consider Y_i as a topological space with the quotient topology of the standard Zariski topology on V_i and we endow $Y_i(\sigma, z)$ with the induced topology.

Proposition 3.5. *Suppose that R is a reduced irreducible root system of type A, B, C, or D. Then for each $\sigma \in W_r$ and $z \in \mu_l$ the space $Y_i^\circ(\sigma, z)$ is irreducible.*

Proof. See Section 4 below. □

Lemma 3.6. *Proposition* 3.5 *implies Theorem* 2.3.

Proof. Suppose the assumptions of Theorem 2.3 hold. We will prove by induction on i that if Proposition 3.5 holds, then for each $i \in \{0, 1, \dots, k+1\}$ the images of w_1, w_2 in $W_r/\left(W_r \cap W(R_i)\right)$ are conjugate. Clearly, the claim for $i = k+1$ implies Theorem 2.3.

Obviously, the claim holds for $i = 0$. Assume that it holds for some $i \in \{1, 2, \dots, k\}$, i.e., w_1 and w_2 are conjugate in $W_r/\left(W_r \cap W(R_i)\right)$. We need to show that this implies that w_1, w_2 are conjugate in $W_r/\left(W_r \cap W(R_{i+1})\right)$. First, note that without loss of generality we can assume that the images of w_1 and of w_2 coincide in $W_r/\left(W_r \cap W(R_i)\right)$.

We will apply Theorem 3.1 to prove the induction step. If $R_i = \emptyset$, then there is nothing to prove. Suppose R_i is not empty. In the notation of Theorem 3.1 we take $X = V_i$, $G = W_r/\left(W_r \cap W(R_{i+1})\right)$, and $H = H_i$, so that G/H can be identified with $W_r/\left(W_r \cap W(R_i)\right)$. Also, in the notation of Theorem 3.1 we take $Y = Y_i$ and $Z = \mu_l$. Clearly, $u_{ji} \in X$ for $j = 1, 2$ by (2.2). Lemma 1.1 together with equations (2.2) and (2.3) imply that the stabilizer of u_{ji} in H is trivial. Since $Y_i^\circ(w_j, \zeta_i)$ is irreducible by assumption, using equation (2.1) we see that w_1, w_2 are conjugate in $W_r/\left(W_r \cap W(R_{i+1})\right)$ by Theorem 3.1. □

Remark 3.7. Note that in the proof of Lemma 3.6 we have not used the fact that R is of classical type, hence Proposition 3.5 for an arbitrary R implies Theorem 2.3 for an arbitrary R.

4. Proof of Proposition 3.5

In this section we prove Proposition 3.5. This together with Lemmas 2.5 and 3.6 finishes the proofs of Theorems 2.1, 2.3 and more importantly provides evidence for the positive answer to the question by Goresky, Kottwitz, and MacPherson ([GKM06], §1, p. 3) mentioned in this paper's introduction. We keep the notation of §3.2.

4.1. **Reductions of Proposition 3.5.** Denote

$$A_r = \bigcap_{m \geq 0} \left\{ \phi \in \mathrm{Aut}(V) \mid \phi(R_m) = R_m \right\},$$

so that $W_r = A_r \cap W$. Note that W_r is a normal subgroup of A_r. For an arbitrary $i \in \{0, 1, \dots, k\}$ let $R_i = S_1 \cup \cdots \cup S_t$ be a decomposition of R_i into W_r-orbits of its irreducible components, i.e., each S_j is a W_r-orbit of some irreducible component of R_i and R_i is a direct sum of root systems $S_1, \dots, S_t$. Then

$$\begin{aligned} W(R_i) &= W(S_1) \times \cdots \times W(S_t), \\ W_r \cap W(R_i) &= W_r \cap W(S_1) \times \cdots \times W_r \cap W(S_t), \\ R_{i+1} &= (R_{i+1} \cap S_1) \cup \cdots \cup (R_{i+1} \cap S_t), \quad \text{and} \\ V_i &= U_1 \oplus \cdots \oplus U_t \oplus U_{t+1}, \end{aligned}$$

where $U_j = \{u \in \mathrm{Span}_{\mathbb{C}}(S_j) \,|\, \alpha(u) = 0, \forall \alpha \in R_{i+1} \cap S_j\}$, $j \in \{1, 2, \dots, t\}$, and U_{t+1} is the orthogonal complement to $\mathrm{Span}_{\mathbb{C}}(R_i)$ in V. For each $j \in \{1, 2, \dots, t\}$ denote $U_j^\circ = \{u \in U_j \,|\, \alpha(u) \neq 0, \forall \alpha \in S_j \backslash (R_{i+1} \cap S_j)\}$. Then $V_i^\circ \cong U_1^\circ \times \cdots \times U_t^\circ \times U_{t+1}$ and $Y_i^\circ(\sigma, z)$ breaks into a direct product of analogous spaces corresponding to $U_1, \dots, U_{t+1}$, i.e.,

$$Y_i^\circ(\sigma, z) \cong (U_1^\circ / W_r \cap W(S_1))\,(\sigma, z) \times \cdots \times (U_t^\circ / W_r \cap W(S_t))(\,\sigma, z) \times U_{t+1}(\sigma, z).$$

(Since each $U_j^\circ / W_r \cap W(S_j)$ and U_{t+1} are W_r- and μ_l-invariant.) Note that U_{t+1} is a vector space and both groups W_r and μ_l act on it by linear automorphisms, hence $U_{t+1}(\sigma, z)$ is irreducible as a vector subspace of U_{t+1}. Thus it is enough to show that for each $j \in \{1, 2, \dots, t\}$ the space $\left(U_j^\circ / W_r \cap W(S_j)\right)(\sigma, z)$ is irreducible. (It also shows that we can think of V_i equivalently as a subspace of V or of $\mathrm{Span}_{\mathbb{C}}(R_i)$.) In other words, without loss of generality we can assume that R_i is isotypic and W_r acts transitively on its set of irreducible components.

Let R be a reduced irreducible root system with a basis of simple roots Δ. Then there is $g \in W(R)$ such that $g(R_i)$ has a basis of simple roots that is a subset of Δ. By considering Dynkin diagrams this implies that there is at most one irreducible component of R_i that is not of type A. Together with the assumption in the previous paragraph we conclude that without loss of generality we can assume that R_i is either irreducible or is a direct sum of irreducible root systems of type A on which W_r acts transitively.

Remark 4.1. Note that the group $W_r \cap W(R_i)$ plays the role of W_r for the chain

$$R_i \supseteq R_{i+1} \supseteq \cdots \supseteq R_k \supseteq R_{k+1} = \emptyset \tag{4.1}$$

and the image of W_r in the group of automorphisms of $\mathrm{Span}_{\mathbb{C}}(R_i)$ is contained in the group $\bigcap_{j \geq i} \{\phi \in \mathrm{Aut}(\mathrm{Span}_{\mathbb{C}}(R_i)) \,|\, \phi(R_j) = R_j\}$, which is the analogue of the group A_r for the chain (4.1). Denoting R_i by R, R_{i+1} by R_1 and so on, we see that to prove Proposition 3.5 it is enough to show that $Y_0^\circ(\sigma, z)$ is irreducible. Here R is either irreducible or a direct sum of irreducible root systems of type A on which $A_r \cap W(R')$ acts transitively, where $W(R')$ is the Weyl group of a reduced irreducible root system R' of classical type that contains R, $\sigma \in A_r \cap W(R')$, $z \in \mu_l$, and $R_1 \neq R$.

Let R' be as before considered as a root system contained in $E' = \mathbb{C}^n$ with the standard basis $\epsilon'_1, \dots, \epsilon'_n$ and let Δ' denote the standard basis of simple roots in R'. Since R' is reduced irreducible of classical type, according to [Bou68] (Chapter VI) without loss of generality we can assume that we have the following cases.

(1) R' has type A_{n-1} ($n \geq 2$) and

$$\begin{array}{rcl} R' & = & \left\{\epsilon'_i - \epsilon'_j \,|\, i \neq j,\, 1 \leq i \leq n,\, 1 \leq j \leq n\right\}, \\ \Delta' & = & \left\{\epsilon'_i - \epsilon'_{i+1} \,|\, 1 \leq i \leq n-1\right\}, \\ W(R') & = & \mathcal{S}_n. \end{array}$$

Here $\mathcal{S}_n$ denotes the symmetric group on n elements with the action on $\mathbb{C}^n$ by permutations of the standard coordinates. The action of $\mathcal{S}_n$ on $\mathrm{Span}_{\mathbb{C}}(R')$ is induced by that on $\mathbb{C}^n$.

(2) R' has type B_n ($n \geq 2$) and

$$\begin{array}{rcl} R' & = & \left\{\pm\epsilon'_i \,(1 \leq i \leq n),\, \pm\epsilon'_i \pm \epsilon'_j \,(1 \leq i < j \leq n)\right\}, \\ \Delta' & = & \left\{\epsilon'_i - \epsilon'_{i+1},\, \epsilon'_n \,|\, 1 \leq i \leq n-1\right\}, \\ W(R') & = & \mathcal{S}_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n. \end{array}$$

Here $\mathrm{Span}_{\mathbb{C}}(R') = \mathbb{C}^n$ with the action of $\mathcal{S}_n$ as in (1) and $(\mathbb{Z}/2\mathbb{Z})^n$ acts on $\mathbb{C}^n$ by multiplying each standard coordinate by ± 1.

(3) R' has type C_n ($n \geq 2$) and

$$\begin{array}{rcl} R' & = & \left\{\pm 2\epsilon'_i \,(1 \leq i \leq n),\, \pm\epsilon'_i \pm \epsilon'_j \,(1 \leq i < j \leq n)\right\}, \\ \Delta' & = & \left\{\epsilon'_i - \epsilon'_{i+1},\, 2\epsilon'_n \,|\, 1 \leq i \leq n-1\right\}, \text{ and} \\ W(R') & = & \mathcal{S}_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n. \end{array}$$

Here $\mathcal{S}_n$ and $(\mathbb{Z}/2\mathbb{Z})^n$ act as in (2).

(4) R' has type D_n ($n \geq 3$) and

$$\begin{array}{rcl} R' & = & \left\{\pm\epsilon'_i \pm \epsilon'_j \,|\, 1 \leq i < j \leq n\right\}, \\ \Delta' & = & \left\{\epsilon'_i - \epsilon'_{i+1},\, \epsilon'_{n-1} + \epsilon'_n \,|\, 1 \leq i \leq n-1\right\}, \text{ and} \\ W(R') & = & \mathcal{S}_n \ltimes K(n). \end{array}$$

Here

$$K(n) = \left\{(\lambda_1, \lambda_2, \dots, \lambda_n) \in (\mathbb{Z}/2\mathbb{Z})^n \mid \prod_i \lambda_i = 1\right\},$$

the actions of $\mathcal{S}_n$ and $(\mathbb{Z}/2\mathbb{Z})^n$ on $\mathrm{Span}_{\mathbb{C}}(R') = \mathbb{C}^n$ are as in (2), and the action of $K(n)$ is induced by that of $(\mathbb{Z}/2\mathbb{Z})^n$.

Note that there exists $g \in W(R')$ such that the system $g(R)$ has a basis of simple roots Δ that is a subset of Δ'. Since $g : V \longrightarrow g(V)$ induces an isomorphism between the space $Y_0^\circ(\sigma, z)$ corresponding to the chain

$$C : R \supseteq R_1 \supseteq \cdots \supseteq R_k \supseteq R_{k+1} = \emptyset \tag{4.2}$$

and the space $Y_0^\circ(g\sigma g^{-1}, z)$ corresponding to the chain $g(C)$, without loss of generality we can assume that R itself has the basis Δ. Hence R can be considered as a root system contained in the space E spanned by some subset F of $\{\epsilon'_1, \dots, \epsilon'_n\}$ that is invariant under the action of $A_r \cap W(R')$. Since every element of $W(R')$ permutes the vectors $\epsilon'_1, \dots, \epsilon'_n$ and possibly multiplies them by ± 1, we conclude that $A_r \cap W(R')$ is contained in the group $\tilde{A}_r$ consisting of elements $\phi \in \mathrm{Aut}_{\mathbb{C}}(E)$ such that $\phi(R_i) = R_i$ for all $i \geq 0$, ϕ permutes vectors in F and multiplies them by ± 1. Note that we have $W_r \subseteq \tilde{A}_r$ and if R is irreducible, then as in cases (1) – (4) above we can assume that $\mathrm{Span}_{\mathbb{C}}(R) = E$ if R is of type B, C, or D and $\mathrm{Span}_{\mathbb{C}}(R)$ is a hyperplane in E if R is of type A. Suppose $F = \{\epsilon_1, \ldots, \epsilon_a\}$, then according to [Bou68] (Chapter VI) we have

$$\begin{array}{lllll} \tilde{A}_r & \subseteq & W(R) \times (\mathbb{Z}/2\mathbb{Z}) & \text{if} \quad R = A_l, & a = l+1, \\ \tilde{A}_r & = & W_r & \text{if} \quad R = B_l \text{ or } C_l, & a = l, \\ \tilde{A}_r & \subseteq & \mathcal{S}_l \ltimes (\mathbb{Z}/2\mathbb{Z})^l & \text{if} \quad R = D_l, & a = l. \end{array} \tag{4.3}$$

Here in the case $R = A_l$ the only non-trivial element of $\mathbb{Z}/2\mathbb{Z}$ takes ϵ_i to $-\epsilon_{a+1-i}$, $i \in \{1, \ldots, a\}$, and in the case $R = D_l$ the group $(\mathbb{Z}/2\mathbb{Z})^l$ acts by multiplying each element ϵ_i by ± 1, $i \in \{1, \ldots, l\}$. In what follows we will assume that $\sigma \in \tilde{A}_r$.

Remark 4.2. Note that since $Y_0 = V_0/W_r$ is a quotient of a vector space by the action of a finite group, Y_0 is an irreducible affine variety in an affine space $\mathbb{A}^s$ and the embedding $Y_0 \hookrightarrow \mathbb{A}^s$ is a homeomorphism given by homogeneous W_r-invariant polynomial functions on V_0. This implies that the action of z on Y_0 is induced by a linear map on $\mathbb{A}^s$. It turns out that in some cases, e.g., if $R = A_l, B_l$, or C_l, the variety Y_0 is actually the whole affine space $\mathbb{A}^s$. Hence to show that in this case $Y_0^\circ(\sigma, z)$ is irreducible it is enough to show that σ acts on Y_0 by a linear automorphism. Indeed, $Y_0(\sigma, z)$ is then a linear subspace of $\mathbb{A}^s$, hence irreducible and $Y_0^\circ(\sigma, z)$ is irreducible as an open subset of $Y_0(\sigma, z)$.

4.2. **Case of an irreducible** R**.** Assume first that R is irreducible, $\sigma \in \tilde{A}_r$, and $z \in \mu_l$. Let $E = \mathbb{C}^a$ with the standard basis $\{\epsilon_1, \ldots, \epsilon_a\}$, where $a = l+1$ if $R = A_l$ and $a = l$ if $R = B_l, C_l$, or D_l. We have $V = E$ if $R = B_l, C_l$, or D_l and

$$V = \{(x_1, \ldots, x_{l+1}) \in E \,|\, x_1 + \cdots + x_{l+1} = 0\}$$

if $R = A_l$. (Recall that V denotes $\mathrm{Span}_{\mathbb{C}}(R)$.) Without loss of generality we can assume that for any R under consideration

$$V_0 = \{u \in E \,|\, \alpha(u) = 0, \, \forall \alpha \in R_1\}.$$

(Indeed, this follows from an argument analogous to one in the first paragraph on p. 9 of the present paper.) If $R_1 = \emptyset$, then $V_0 = E$ and $W_r = W(R)$, so that Y_0 is an affine space by the well-known result due to Chevalley (see [Bou68], p. 107). Furthermore, σ acts linearly on Y_0 as can be seen using (4.3). Thus $Y_0^\circ(\sigma, z)$ is irreducible by Remark 4.2.

Suppose $R_1 \neq \emptyset$. Let $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ denote the standard basis of simple roots in R and let $R_1 = S_1 \cup \cdots \cup S_t$ be a decomposition of R_1 into $\tilde{A}_r$-orbits of its irreducible components, i.e., each S_j is an $\tilde{A}_r$-orbit of some irreducible component of R_1 and R_1 is a direct sum of root systems $S_1, \ldots, S_t$. (We use the same letters $S_1, \ldots, S_t$ to denote objects related to a possibly different chain of root systems than in the previous sections, but since we will not return to S_i's introduced in §4.1 that should not cause confusion.) As in Remark 4.1 without loss of generality we can assume that R_1 has the basis of simple roots Δ_1 that is a subset of Δ.

Consider the case when $R \neq A_l$ and $\alpha_l \in \Delta_1$. Then without loss of generality we can assume that $\alpha_l \in S_t$ and there are the following possibilities:

(a) $\alpha_{l-1} \in \Delta_1$ and $R = B_l$ or C_l,
(b) $\alpha_{l-1} \notin \Delta_1$ and $R = B_l$ or C_l,
(c) $R = D_l$, $\alpha_{l-1} \in \Delta_1$, $\alpha_{l-2} \in \Delta_1$,
(d) $R = D_l$, $\alpha_{l-1} \in \Delta_1$, $\alpha_{l-2} \notin \Delta_1$,
(e) $R = D_l$, $\alpha_{l-1} \notin \Delta_1$.

In the case (a) the element α_{l-1} is not orthogonal to $\alpha_l \in S_t$, hence $\alpha_{l-1} \in S_t$. Since by assumption all the irreducible components of S_t are isomorphic to each other, this implies that S_t is an irreducible root system of type B or C and S_t has a basis $\{\alpha_k, \ldots, \alpha_{l-1}, \alpha_l\}$ for some $k \leq l-1$.

In the case (b) the element α_l is orthogonal to every $\alpha \in R_1$, hence S_t is a direct sum of irreducible root systems of type A_1. Since $\tilde{A}_r$ acts transitively on the set of irreducible components of S_t and $\tilde{A}_r = W_r$, we conclude that $S_t = \{\pm\alpha_l\}$.

In the case (c) the element α_{l-2} is not orthogonal to α_{l-1} and $\alpha_l \in S_t$, hence $\alpha_{l-1}, \alpha_{l-2} \in S_t$. Thus S_t is an irreducible system of type D and S_t has a basis $\{\alpha_k, \ldots, \alpha_{l-2}, \alpha_{l-1}, \alpha_l\}$ for some $k \leq l-2$.

In the case (d) without loss of generality we can assume that $\alpha_{l-1} \in S_{t-1}$ or $\alpha_{l-1} \in S_t$. We put $S'_t = S_{t-1} \cup S_t$ in the first case and $S'_t = S_t$ in the second, so that $\alpha_{l-1}, \alpha_l \in S'_t$.

Finally, in the case (e) denote by ϕ the element of $\mathrm{Aut}_{\mathbb{C}}(V)$ that permutes α_{l-1} and α_l, and $\phi(\alpha_i) = \alpha_i$ for any $i \notin \{l-1, l\}$. Then $\phi(R) = R$, $\alpha_{l-1} = \phi(\alpha_l) \in \phi(R_1)$ and $\alpha_l = \phi(\alpha_{l-1}) \notin \phi(R_1)$. Working with the chain $\phi(C)$ instead of the chain C, this implies that without loss of generality we can assume that $\alpha_l \notin R_1$.

It follows from the cases (a)—(e) considered above that without loss of generality we can assume that S_1 is either empty or has a basis $\{\epsilon_i - \epsilon_{i+1} \,|\, i \in I_1\}, \ldots, S_{t-1}$ is either empty or has a basis $\{\epsilon_i - \epsilon_{i+1} \,|\, i \in I_{t-1}\}$. Finally, if $R \neq A_l$ and $\alpha_l \in R_1$, then we can assume that S_t (or S'_t in the case (d)) has a basis $\{\alpha_i \,|\, i \in I_t\}$ for some subset I_t of $\{1, 2, \ldots, a\}$ that contains l. Here $I_t = \{l-k, l-k+1, \ldots, l\}$ in cases (a)—(c) and $l-1 \in I_t$ in case (d). Note that $\{i, i+1 \,|\, i \in I_1\}, \ldots, \{i, i+1 \,|\, i \in I_{t-1}\}$, $\{i, i+1 \,|\, i \in I_t,\, i \neq l\}$ are disjoint subsets of $\{1, 2, \ldots, a\}$. Denote

$$\begin{aligned} X_1 &= \mathrm{Span}_{\mathbb{C}}\{\epsilon_i, \epsilon_{i+1} \,|\, i \in I_1\}, \\ &\cdots \\ X_{t-1} &= \mathrm{Span}_{\mathbb{C}}\{\epsilon_i, \epsilon_{i+1} \,|\, i \in I_{t-1}\}, \\ X_t &= \mathrm{Span}_{\mathbb{C}}\left\{\epsilon_i, \epsilon_{i+1} \,|\, i \in I_t,\, i \neq l\right\}, \end{aligned} \tag{4.4}$$

and let U_t denote the orthogonal complement to $X_1 \oplus \cdots \oplus X_t$ in E with respect to the standard scalar product. Then

$$V_0 = U_1 \oplus \cdots \oplus U_t \oplus U_{t+1}, \tag{4.5}$$

where

$$U_i = \{u \in X_i \,|\, \alpha(u) = 0, \, \forall \alpha \in S_i\}, \quad i \in \{1, \ldots, t-1\}, \tag{4.6}$$

and $U_{t+1} = \{u \in X_t \,|\, \alpha(u) = 0, \, \forall \alpha \in S_t \, (\text{or } S'_t \text{ in the case (d)})\}$. It is easy to see that

$$U_{t+1} = \{(0, \ldots, 0)\}. \tag{4.7}$$

Indeed, it is clear in the cases (a)—(c) when $I_t = \{l-k, l-k+1, \ldots, l\}$, where $k \geq 0$ for $R = B_l, C_l$ and $k \geq 1$ for $R = D_l$. In the case (d) we have $l-1 \in I_t$ and $R = D_l$. Since $\tilde{A}_r$ acts transitively on the irreducible components of S_{t-1} and S_t, for any $\alpha_i = \epsilon_i - \epsilon_{i+1} \in S'_t$ there is $\phi \in \tilde{A}_r$ such that $\alpha_i = \phi(\alpha_l)$ or $\alpha_i = \phi(\alpha_{l-1})$. Then $\epsilon_i + \epsilon_{i+1}$ equals $\pm\phi(\alpha_{l-1})$ or $\pm\phi(\alpha_l)$ and hence belongs to S'_t, which implies (4.7).

Clearly, all X_i's and U_i's are W_r-invariant and if $R = A_l, B_l$, or C_l, then

$$V_0/W_r \cong U_1/W_r \times \cdots \times U_t/W_r \tag{4.8}$$

and hence we have

$$Y_0(\sigma, z) \cong (U_1/W_r)\,(\sigma, z) \times \cdots \times (U_t/W_r)\,(\sigma, z), \tag{4.9}$$

since each U_i/W_r is $\tilde{A}_r$- and μ_l-invariant. The formula (4.8) is a consequence of the following easily verified fact. Suppose W is the Weyl group of A_l, B_l, or C_l, i.e., $W = \mathcal{S}_a$ or $W = \mathcal{S}_l \ltimes (\mathbb{Z}/2\mathbb{Z})^l$. Let $V_1 = \operatorname{Span}_{\mathbb{C}}(\epsilon_1, \epsilon_2, \ldots, \epsilon_s)$ and $V_2 = \operatorname{Span}_{\mathbb{C}}(\epsilon_{s+1}, \epsilon_{s+2}, \ldots, \epsilon_a)$. If for $i = 1, 2$ and $w \in W$ we have $w(V_i) = V_i$, then $\tilde{w} \in \operatorname{Aut}(V)$ given by

$$\tilde{w}|_{V_1} = w, \quad \tilde{w}|_{V_2} = \mathrm{id}$$

also belongs to W. If $R = D_l$, then (4.8) does not necessarily hold.

Our next step is to understand the action of W_r on each U_i. Thus in the next few paragraphs we assume that R_1 is isotypic and $\tilde{A}_r$ (hence A_r) acts transitively on its set of irreducible components. Also, according to the results of the previous paragraphs we only need to consider the case when all the irreducible components of R_1 are of type A. Let $R_1 = M_1 \cup \cdots \cup M_n$ be a decomposition of R_1 into its irreducible components, where $M_1, \ldots, M_n$ are of the same type A_k. Without loss of generality we can assume that

M_1 has the basis $\alpha_1, \ldots, \alpha_k$,
M_2 has the basis $\alpha_{k+2}, \ldots, \alpha_{2k+1}$,
$\ldots$
M_n has the basis $\alpha_{(n-1)k+n}, \ldots, \alpha_{nk+n-1}$,

where $nk+n-1 \le l-1$, if $R = B_l, C_l$, or D_l and $nk+n-1 \le l$, if $R = A_l$. As before, denote

$$\begin{aligned} Z_1 &= \mathrm{Span}_{\mathbb{C}}\left(\epsilon_1, \dots, \epsilon_{k+1}\right), \\ Z_2 &= \mathrm{Span}_{\mathbb{C}}\left(\epsilon_{k+2}, \dots, \epsilon_{2k+2}\right), \\ &\dots \\ Z_n &= \mathrm{Span}_{\mathbb{C}}\left(\epsilon_{(n-1)k+n}, \dots, \epsilon_{n(k+1)}\right), \\ Z_{n+1} &= \mathrm{Span}_{\mathbb{C}}\left(\epsilon_{n(k+1)+1}, \dots, \epsilon_a\right). \end{aligned}$$

Then $E = Z_1 \oplus \cdots \oplus Z_n \oplus Z_{n+1}$. Put $Z = \{(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n) \mid \forall x_i \in \mathbb{C}\}$, where each $\bar{x}_i$ denotes a $(k+1)$-tuple $(x_i, x_i, \dots, x_i)$. We have $V_0 = Z \oplus Z_{n+1}$. Since we already included the "remainder" U_t in V_0 (see (4.5)), in this case we only need to consider Z/W_r. Let $\mathcal{S}_n$ denote the symmetric group on n elements viewed as the subgroup of W that permutes the components $M_1, \dots, M_n$ and acts trivially on Z_{n+1}. We will show that without loss of generality we can assume that $\mathcal{S}_n$ is contained in W_r (i.e., $\mathcal{S}_n$ leaves the chain (4.2) invariant). This will follow from the assumption that $\tilde{A}_r$ acts *transitively* on the set $\{M_1, \dots, M_n\}$.

Lemma 4.3. *If $\tilde{A}_r$ acts transitively on the set $\{M_1, \dots, M_n\}$, then without loss of generality we can assume that $\mathcal{S}_n \subseteq \tilde{A}_r$.*

Proof. First, note that there exists $\lambda_1 \in W(M_1)$ such that $\lambda_1(M_1 \cap R_2)$ has a basis of simple roots that is a subset of the standard basis of M_1. Analogously, there exists $\lambda_2 \in W\left(\lambda_1(M_1 \cap R_2)\right)$ such that $\lambda_2\left(\lambda_1(M_1 \cap R_3)\right)$ has a basis that is a subset of the standard basis of $\lambda_1(M_1 \cap R_2)$ and so on. Repeating this process for each component M_i we see that there exists $\beta \in W(R_1)$ such that the chain $\beta(C)$ consisting of the systems $\beta(R_i)$, $0 \le i \le k+1$, looks like the one on Figure 1.

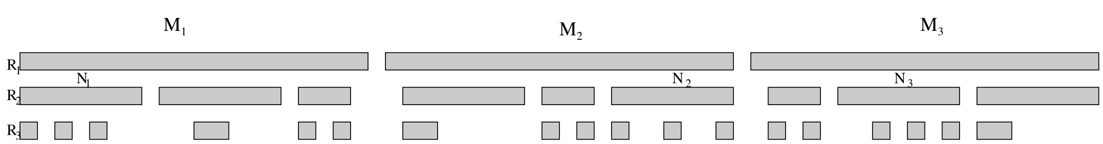

FIGURE 1

(Here each rectangle depicts an irreducible system of type A, the first line of rectangles depicts the system R_1, the second line depicts the system R_2, and so on.) As before without loss of generality we can assume that C itself has the form described above.

Since $\tilde{A}_r$ acts transitively on the set $\{M_1, \dots, M_n\}$, for each irreducible component N_1 of $M_1 \cap R_2$ and for each $i \in \{2, \dots, n\}$ there exists an irreducible component N_i of $M_i \cap R_2$ such that $N_1 \cap R_j \cong N_i \cap R_j$ for all $j \ge 1$ (see Figure 1). This implies that there exists $\phi_i \in W(M_i)$ such that $(1i)(N_1) = \phi_i(N_i)$ for all $i \ge 2$, where $(1i) \in \mathcal{S}_n$ denotes a transposition. Let $\phi = \phi_2\phi_3 \cdots \phi_n$. Then $\phi(M_j) = M_j$ for each j, $\phi \in W(R_1)$, and $\phi(R) = R$. Also, by construction $\phi(\alpha) = \alpha$ for any $\alpha \in M_1$ and $(1j)(N_1) = \phi(N_j)$ for all $j \in \{2, \ldots, n\}$. In other words, using permutations of the basis vectors ϵ_i that are contained in Z_2 we can rearrange blocks inside M_2 so that $M_2 \cap R_2$ looks like $M_1 \cap R_2$ and so on. Applying this process to other irreducible components of $M_1 \cap R_2$ and then if necessary to irreducible components of $N_1 \cap R_3$ and so on, we see that there exists $\psi \in W(R_1)$ such that $\delta(\psi(R_i)) = \psi(R_i)$ for all $i \geq 1$ and $\delta \in \mathcal{S}_n$. Since $\psi \in W(R_1)$, arguing as above without loss of generality we can assume that $\delta(R_i) = R_i$ for any $\delta \in \mathcal{S}_n$ and i. □

Let $X_1 = Z_1 \oplus \cdots \oplus Z_n$. Note that the image of $\tilde{A}_r$ in $\mathrm{Aut}_{\mathbb{C}}(X_1)$ is contained in the subgroup

$$A(R_1) = \{\phi \in \mathrm{Aut}_{\mathbb{C}}(X_1) \,|\, \phi(R_1) = R_1\}$$

and $\mathcal{S}_n \subseteq A(R_1)$. Let $A(M_1) = \{\phi \in \mathrm{Aut}_{\mathbb{C}}(Z_1) \,|\, \phi(M_1) = M_1\}$. We consider $A(M_1)$ as a subgroup of $A(R_1)$ in the usual way, i.e., by letting it act trivially on each Z_i, $i \neq 1$. Doing the same for each component we get a subgroup $A(M_1)^n$ in $A(R_1)$ and it is easy to see that $A(R_1) = \mathcal{S}_n \ltimes A(M_1)^n$. (Moreover, $A(R_1)$ is generated by $\mathcal{S}_n$ and $A(M_1)$.) Since $\mathcal{S}_n \subseteq W_r \subseteq \tilde{A}_r \subseteq A(R_1)$, we have

$$\begin{aligned} \tilde{A}_r &= \mathcal{S}_n \ltimes \left(\tilde{A}_r \cap A(M_1)^n\right), \\ W_r &= \mathcal{S}_n \ltimes \left(W_r \cap A(M_1)^n\right). \end{aligned}$$

Taking into account that M_1 is of type A by (4.3) we get

$$\tilde{A}_r \cap A(M_1)^n \subseteq \left(W(M_1) \times \mathbb{Z}/2\mathbb{Z}\right)^n.$$

Here $\mathbb{Z}/2\mathbb{Z}$ is considered as a subgroup of $A(M_1)$ whose the only non-trivial element w_0 takes ϵ_i to $-\epsilon_{k+2-i}$, $i \in \{1, \ldots, k+1\}$. We also extend w_0 to an element of $\mathrm{Aut}_{\mathbb{C}}(E)$ by letting it act trivially on all ϵ_j, $j \notin \{1, \ldots, k+1\}$.

Let $R = A_l$. Then $W_r \cap A(M_1)^n \subseteq W(M_1)^n$ and hence W_r acts on Z (naturally identified with $\mathbb{C}^n$) as $\mathcal{S}_n$ and Z/W_r is an affine space. By (4.3) $\sigma \in \tilde{A}_r$ permutes vectors ϵ_i, $1 \leq i \leq l+1$, and possibly multiplies all of them by -1. Since $\sigma(X_1) = X_1$ (resp., $\sigma(U_t) = U_t$), σ permutes vectors ϵ_j inside X_1 (resp., inside U_t) and possibly multiplies all of them by -1. Recall that W_r acts on the "remainder" $U_t \cong \mathbb{C}^{n_t}$ as $\mathcal{S}_{n_t}$, hence U_t is an affine space and σ acts linearly on both Z/W_r and U_t/W_r. This together with (4.9) and Remark 4.2 finishes the proof that $Y_0^\circ(\sigma, z)$ is irreducible in the case when $R = A_l$.

Let $R = B_l, C_l$, or D_l. Suppose first that there is $\lambda \in W(M_1)$ such that $\lambda w_0 \in \tilde{A}_r$. Since $\mathcal{S}_n \subseteq \tilde{A}_r$, this implies that for any $\alpha_1, \ldots, \alpha_n \in \mathbb{Z}/2\mathbb{Z}$ there exist $\lambda_1, \ldots, \lambda_n \in W(M_1)$ such that $\lambda = (\lambda_1\alpha_1, \ldots, \lambda_n\alpha_n) \in \tilde{A}_r$. Clearly, if $\alpha_1 \cdots \alpha_n = 1$, then $\lambda \in W_r$. Let F and G denote the images of $W_r \cap A(M_1)^n$ and of $\tilde{A}_r \cap A(M_1)^n$, respectively, in $\mathrm{Aut}_{\mathbb{C}}(Z)$. Then we have

$$K(n) \subseteq F \subseteq G \subseteq (\mathbb{Z}/2\mathbb{Z})^n.$$

Since $K(n)$ is a subgroup of $(\mathbb{Z}/2\mathbb{Z})^n$ of index 2, we have either $F = G$ or $F = K(n)$ and $G = (\mathbb{Z}/2\mathbb{Z})^n$. If $F = G$, then W_r acts on $Z \cong \mathbb{C}^n$ as $\mathcal{S}_n \ltimes K(n)$ or as $\mathcal{S}_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$ and σ acts on Z/W_r trivially. Thus in this case Z/W_r is an affine space with the trivial action of σ. If $F = K(n)$ and $G = (\mathbb{Z}/2\mathbb{Z})^n$, then W_r acts on $Z \cong \mathbb{C}^n$ as $\mathcal{S}_n \ltimes K(n)$. Hence Z/W_r is an affine space $\mathbb{A}^n$ with the isomorphism induced by the map

$$\phi = (\phi_1, \dots, \phi_n) : \mathbb{C}^n \longrightarrow \mathbb{A}^n$$

given by

$$y_j = \phi_j(x_1, \dots, x_n) = \sum_{\tau \in \mathcal{S}_n} x_{\tau(1)}^2 x_{\tau(2)}^2 \cdots x_{\tau(j)}^2, \quad 1 \le j \le n-1, \tag{4.10}$$

$$y_n = \phi_n(x_1, \dots, x_n) = x_1 x_2 \cdots x_n. \tag{4.11}$$

It follows that σ acts on $Z/W_r \cong \mathbb{A}^n$ via

$$(y_1, \dots, y_{n-1}, y_n) \mapsto (y_1, \dots, y_{n-1}, \pm y_n). \tag{4.12}$$

Suppose now that for any $\lambda \in W(M_1)$ we have $\lambda w_0 \notin \tilde{A}_r$. This implies

$$\tilde{A}_r \cap A(M_1)^n \subseteq W(M_1)^n$$

and hence both W_r and $\tilde{A}_r$ act on $Z \cong \mathbb{C}^n$ as $\mathcal{S}_n$. Thus Z/W_r is again an affine space with the trivial action of σ. Thus we have proved that each $(U_i/W_r)(\sigma, z)$, $1 \le i \le t-1$, is irreducible, if $R = B_l, C_l$ or D_l.

We now show that $(U_t/W_r)(\sigma, z)$ is also irreducible. Indeed, if $R = B_l$ or C_l, then W_r acts on $U_t \cong \mathbb{C}^{n_t}$ as $\mathcal{S}_{n_t} \ltimes (\mathbb{Z}/2\mathbb{Z})^{n_t}$. Thus U_t/W_r is an affine space and σ acts trivially on it, because $\tilde{A}_r = W_r$. This together with (4.9) and Remark 4.2 finishes the proof that $Y_0^\circ(\sigma, z)$ is irreducible in the case when $R = B_l$ or C_l.

Let $R = D_l$. Then W_r acts on U_t as either $\mathcal{S}_{n_t} \ltimes (\mathbb{Z}/2\mathbb{Z})^{n_t}$ or $\mathcal{S}_{n_t} \ltimes K(n_t)$. If $\sigma \in \tilde{A}_r$, then by (4.3) the action of σ on $U_t/W_r \cong \mathbb{A}^{n_t}$ is given by (4.12) with n replaced by n_t. Thus $(U_t/W_r)(\sigma, z)$ is irreducible in this case as well.

Next we continue working on the case $R = D_l$, since in that case we do not always have the decomposition (4.9).

4.3. The case $R = D_l$.

In what follows we assume that $R = D_l$. If $S_1, \dots, S_{t-1}$ are all empty, then $S_t \neq \emptyset$ and by (4.5), (4.7) we have

$$V_0 = \{(x_1, \dots, x_{k-1}, 0, \dots, 0) \mid \forall x_i \in \mathbb{C}\}.$$

It can be checked that W_r acts on the first $k-1$ coordinates of V_0 as either $\mathcal{S}_{k-1} \ltimes (\mathbb{Z}/2\mathbb{Z})^{k-1}$ or $\mathcal{S}_{k-1} \ltimes K(k-1)$, hence $Y_0 = V_0/W_r$ is an affine space in this case. Also, by (4.3) any $\sigma \in \tilde{A}_r$ permutes $x_1, \dots, x_{k-1}$ and multiplies them by ± 1, which induces a linear map on Y_0 (see (4.12)). Hence $Y_0^\circ(\sigma, z)$ is irreducible by Remark 4.2.

Assume now that there is at least one non-empty S_i for some $1 \le i \le t-1$. Clearly, in this case without loss of generality we can assume that all $S_1, \dots, S_{t-1}$ are not empty. As was proved in the previous subsection W_r acts on each $U_i \cong \mathbb{C}^{n_i}$, $1 \le i \le t-1$, as either $\mathcal{S}_{n_i}$, or $\mathcal{S}_{n_i} \ltimes (\mathbb{Z}/2\mathbb{Z})^{n_i}$, or $\mathcal{S}_{n_i} \ltimes K(n_i)$ for some n_i. Also, W_r acts on $U_t \cong \mathbb{C}^{n_t}$ as either $\mathcal{S}_{n_t} \ltimes (\mathbb{Z}/2\mathbb{Z})^{n_t}$ or $\mathcal{S}_{n_t} \ltimes K(n_t)$ and $U_{t+1} = \{(0, \dots, 0)\}$ by (4.7). Thus by (4.5)

$$V_0 \cong U_1 \oplus \cdots \oplus U_t \cong \mathbb{C}^{n_1} \oplus \cdots \oplus \mathbb{C}^{n_t}.$$

It can be easily verified that if W_r acts on U_1 as $\mathcal{S}_{n_1}$, or as $\mathcal{S}_{n_1} \ltimes K(n_1)$, then $V_0/W_r \cong (U_1/W_r) \times (U_2 \times \cdots \times U_t)/W_r$. Applying the same argument to all U_i's we get

$$V_0/W_r \cong \left(\prod_{i=1}^{s} U_{k_i}\right) \Big/ W_r \times \prod_{j \notin \{k_1,\ldots,k_s\}} (U_j/W_r)\,, \tag{4.13}$$

where for each $i \in \{1,\ldots,s\}$ the group W_r acts on U_{k_i} as $\mathcal{S}_{n_{k_i}} \ltimes (\mathbb{Z}/2\mathbb{Z})^{n_{k_i}}$, and for each j the variety U_j/W_r is an affine space. Put $X = \prod_{i=1}^{s} U_{k_i}$ and consider the variety X/W_r. Assume that X/W_r cannot be decomposed any further in the way described above into direct products, i.e., $X/W_r \ncong \left(\prod_{j\in J_1} U_j\right)/W_r \times \left(\prod_{j\in J_2} U_j\right)/W_r$ for any nonempty subsets J_1, J_2 such that $J_1 \coprod J_2 = \{k_1,\ldots,k_s\}$. Then the image of W_r in $\mathrm{Aut}_{\mathbb{C}}(X)$ equals

$$\left\{ \prod_{i=1}^{s}(\sigma_i;\lambda_1^i,\ldots,\lambda_{n_{k_i}}^i) \in \prod_{i=1}^{s} \mathcal{S}_{n_{k_i}} \ltimes (\mathbb{Z}/2\mathbb{Z})^{n_{k_i}} \;\Big|\; \prod_{i,j} \lambda_j^i = 1 \right\}.$$

For each i let $Q_i = \mathcal{S}_{n_{k_i}} \ltimes K(n_{k_i})$. The group Q_i acts on U_{k_i} and we consider Q_i as a subgroup of $\mathrm{Aut}_{\mathbb{C}}(X)$ by letting it act trivially on each U_j, $j \neq k_i$. Also, denote $Q = Q_1 \times Q_2 \times \cdots \times Q_s$. Then Q is a normal subgroup of W_r (or rather of the image of W_r in $\mathrm{Aut}_{\mathbb{C}}(X)$, but to simplify the notation we will not distinguish between these two). We have

$$X/W_r = (X/Q)/(W_r/Q) = \left(\prod_{i=1}^{s} (U_{k_i}/Q_i)\right) \Big/ (W_r/Q).$$

Recall that each U_{k_i}/Q_i is isomorphic to an affine space $\mathbb{A}^v$ with the isomorphism given by (4.10) and (4.11). Thus by (4.12) the induced action of W_r/Q on $\mathbb{A}^v$ has the form

$$(y_1,\ldots,y_{v-1},y_v) \mapsto (y_1,\ldots,y_{v-1},\pm y_v).$$

This implies that X/W_r is isomorphic to a direct product $\mathbb{A}^\beta \times \mathbb{A}^s/K(s)$. By (4.13) the variety $Y_0 = V_0/W_r$ has the same form as X/W_r and since $K(s)$ is not a reflection group for $\mathbb{A}^s$, this shows that Y_0 is not necessarily an affine space when $R = D_l$.

Let $\sigma \in \tilde{A}_r$. Note that

$$Y_0(\sigma,z) \cong (X/W_r)(\sigma,z) \times \prod_{j \notin \{k_1,\ldots,k_s\}} (U_j/W_r)(\sigma,z),$$

since each U_i/W_r is $\tilde{A}_r$- and μ_l-invariant. Recall that each $(U_j/W_r)(\sigma,z)$ is irreducible (see §4.2) and since $Y_0^\circ(\sigma,z)$ is an open subset of $Y_0(\sigma,z)$, it is enough to show that the projection of $Y_0^\circ(\sigma,z)$ onto $(X/W_r)(\sigma,z)$ is contained in an irreducible subset of $(X/W_r)(\sigma,z)$. Furthermore, by the results of the previous paragraph $X/W_r \cong \mathbb{A}^\beta \times \mathbb{A}^s/K(s)$ and it is easy to see that both $\mathbb{A}^\beta$ and $\mathbb{A}^s/K(s)$ are σ- and μ_l-invariant with linear actions of σ and z on $\mathbb{A}^\beta$. (In fact, σ acts trivially on $\mathbb{A}^\beta$.) This implies in turn that it is enough to show that the projection of $Y_0^\circ(\sigma,z)$ onto

$$P = \{\bar{u} \in \mathbb{A}^s/K(s) \,|\, \sigma(\bar{u}) = z \cdot \bar{u},\ u \in \mathbb{A}^s\}$$

is contained in an irreducible subset of P. For $u = (u_1, \dots, u_s) \in \mathbb{A}^s$ we have

$$\sigma^{-1}(z \cdot \bar{u}) = \overline{\sigma^{-1}(z \cdot u)}, \quad \sigma^{-1}(z \cdot u) = (z_1 u_1, \dots, z_s u_s),$$

for some $z_1, \dots, z_s \in \mathbb{C}^\times$. Thus $\bar{u} \in P$ if and only if

$$u_1 = \lambda_1 z_1 u_1,\ u_2 = \lambda_2 z_2 u_2\ , \dots, u_s = \lambda_s z_s u_s,$$

for some $\lambda_i \in \mathbb{Z}/2\mathbb{Z}$ such that $\prod_i \lambda_i = 1$. We claim that P is either irreducible and hence both $Y_0(\sigma, z)$ and $Y_0^\circ(\sigma, z)$ are irreducible, or P is the image of the union of the coordinate hyperplanes in $\mathbb{A}^s$ under the quotient map $\mathbb{A}^s \twoheadrightarrow \mathbb{A}^s/K(s)$. Indeed, if there exists $z_i \neq \pm 1$, then $u_i = 0$ and hence P is irreducible as a linear subspace of the affine space $\mathbb{A}^{s-1}/(\mathbb{Z}/2\mathbb{Z})^{s-1}$. Thus assume $z_i = \pm 1$ for all $i \in \{1, \dots, s\}$. If $\prod_i z_i = 1$, then P coincides with $\mathbb{A}^s/K(s)$, which is an irreducible variety. If $\prod_i z_i = -1$, then $\prod_i (\lambda_i z_i) = -1$ and hence at least one u_j equals zero. This shows that P is contained in the image of the union of the coordinate hyperplanes in $\mathbb{A}^s$ and since we have the reverse inclusion, the claim follows. Since the images $H_1, \dots, H_s$ of the coordinate hyperplanes do not coincide, this shows, in particular, that $Y_0(\sigma, z)$ is not necessarily irreducible. However, the projection of $Y_0^\circ(\sigma, z)$ onto P is irreducible, since it is contained in some H_i. Indeed, recall that $\mathbb{A}^s = \{(x_1, \dots, x_s)\}$, where each x_i is the last standard coordinate of an element in $U_{k_i}/Q_i \cong \mathbb{A}^{n_{k_i}}$, $1 \leq i \leq s$, $s \in \{1, \dots, t\}$. Here each U_{k_i}, $1 \leq k_i \leq t-1$, corresponds to the system S_{k_i}, which has only type A irreducible components. Thus the projection of

$$V_0^\circ = \{u \in V_0 \mid \ \alpha(u) \neq 0, \forall \alpha \in R \backslash R_1\}$$

onto $U_{k_i} \cong \mathbb{C}^{n_{k_i}}$ is contained in the set $\{(x_1, \dots, x_{n_{k_i}}) \mid \forall x_j \neq 0\}$ and hence the image of V_0° in $\mathbb{A}^{n_{k_i}}$ is contained in $\{(x_1, \dots, x_{n_{k_i}}) \mid x_{n_{k_i}} \neq 0\}$. This implies that the image of V_0° in $\mathbb{A}^s$ can intersect at most one coordinate hyperplane (corresponding to the "remainder" U_t) and the claim follows.

4.4. **Case of an isotypic R.** Assume now that R is reducible, isotypic, each irreducible component of R is of type A, and $\tilde{A}_r$ acts transitively on the set of irreducible components of R. As we have proved above in this case Y_0 is an affine space and as usual it is enough to show that $\tilde{A}_r$ acts linearly on Y_0. Let $R = T_1 \cup T_2 \cup \dots \cup T_m$ be a decomposition of R into irreducible components. Since by assumption $\tilde{A}_r$ acts transitively on the set $\{T_1, \dots, T_m\}$, by Lemma 4.3 without loss of generality we can assume that $\mathcal{S}_m \subseteq \tilde{A}_r$. Thus as in the previous section we get

$$\tilde{A}_r = \mathcal{S}_m \ltimes (\tilde{A}_r \cap A(T_1)^m).$$

We have $Y_0 = Y_0^{(1)} \times \dots \times Y_0^{(m)}$, where each $Y_0^{(i)}$ is the analogue of Y_0 for the system T_i and $\mathcal{S}_m$ acts on Y_0 by permutations of $Y_0^{(1)}, \dots, Y_0^{(m)}$. Since each T_i is of type A, by the results of §4.2 each $Y_0^{(i)}$ is an affine space and clearly $\mathcal{S}_m$ acts linearly on Y_0. Thus to show that $Y_0^\circ(\sigma, z)$ is irreducible it is enough to show that $\tilde{A}_r \cap A(T_1)^m$ also acts linearly on Y_0. Note that

$$\tilde{A}_r \cap A(T_1)^m = (\tilde{A}_r \cap A(T_1)) \times (\tilde{A}_r \cap A(T_2)) \times \dots \times (\tilde{A}_r \cap A(T_m)),$$

where each $\tilde{A}_r \cap A(T_i)$ acts linearly on $Y_0^{(i)}$ by the results of §4.2. This implies irreducibility of $Y_0(\sigma, z)$ and hence of $Y_0^\circ(\sigma, z)$ for all $\sigma \in \tilde{A}_r$, $z \in \mu_l$. This finishes the proof of Proposition 3.5 and hence by Lemmas 2.5 and 3.6 proves Theorem 2.3 and Theorem 2.1.

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