ISSN 1066-369X, Russian Mathematics (Iz. VUZ), 2008, Vol. 52, No. 2, pp. 46–50. © Allerton Press, Inc., 2008.
Original Russian Text © M.N. Sabitova, 2008, published in Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika, 2008, No. 2, pp. 48–52.

On Representations of the Weil–Deligne Group

M. N. Sabitova[1]
[1]*Kazan State University, ul. Kremlyovskaya 18, Kazan, 420008 Russia*[1]

Received October 15, 2007

Abstract—We study admissible orthogonal and symplectic representations of the Weil–Deligne group $\mathcal{W}'(\overline{K}/K)$ of a local non-Archimedean field K. As an application of the obtained results we show that the root number of the tensor product of two admissible symplectic representations of $\mathcal{W}'(\overline{K}/K)$ is 1.

DOI: 10.3103/S1066369X08020072

1. INTRODUCTION

In this paper, we study admissible orthogonal and symplectic representations of the Weil–Deligne group $\mathcal{W}'(\overline{K}/K)$ of a non-Archimedean local field K. The basis of our investigation is the fact that each admissible indecomposable representation of the Weil–Deligne group has a unique irreducible subrepresentation. From the definition of admissible representations of the group $\mathcal{W}'(\overline{K}/K)$ it follows immediately that each admissible representation of the group $\mathcal{W}'(\overline{K}/K)$ is a direct sum of admissible indecomposable subrepresentations. In turn, it is known that each admissible indecomposable representation of the group $\mathcal{W}'(\overline{K}/K)$ is of the form $\alpha \otimes \mathrm{sp}(n)$, where α is an irreducible representation of the Weil group $\mathcal{W}(\overline{K}/K)$ of the field K, n is a positive integer, and the representation $\mathrm{sp}(n)$ is given by formula (3.2) on p. 49. One can easily show that $\alpha \otimes \mathrm{sp}(n)$ has a unique irreducible subrepresentation. In other words, the socle of the representation $\alpha \otimes \mathrm{sp}(n)$ is irreducible. Therefore, it is natural to begin the study of admissible orthogonal and symplectic representations of the group $\mathcal{W}'(\overline{K}/K)$ with the study of orthogonal and symplectic representations (of a group G over a field k) which can be written as direct sums of indecomposable subrepresentations with irreducible socles (see Theorem 2.1). As a result, we have obtained the following theorem.

Theorem 1.1. *If σ' is an admissible minimal symplectic or orthogonal representation of the group $\mathcal{W}'(\overline{K}/K)$, then either*

$$\sigma' \cong (\beta \otimes \mathrm{sp}(m)) \oplus \left(\beta^* \otimes \omega^{1-m} \otimes \mathrm{sp}(m)\right) \tag{1.1}$$

for some positive integer m and irreducible representation β of the group $\mathcal{W}(\overline{K}/K)$, or

$$\sigma' \cong \alpha \otimes \mathrm{sp}(n) \tag{1.2}$$

for some positive integer n and irreducible representation α of the group $\mathcal{W}(\overline{K}/K)$ such that the representation

$$\alpha \otimes \omega^{\frac{n-1}{2}} \begin{cases} \text{is symplectic} & \text{if } \sigma' \text{ is symplectic and } n \text{ is odd,} \\ \text{orthogonal} & \text{if } \sigma' \text{ is symplectic and } n \text{ is even,} \\ \text{orthogonal} & \text{if } \sigma' \text{ is orthogonal and } n \text{ is odd,} \\ \text{is symplectic} & \text{if } \sigma' \text{ is orthogonal and } n \text{ is even.} \end{cases} \tag{1.3}$$

Here a *minimal* symplectic (or orthogonal) representation is a representation which cannot be written in the form of an orthogonal sum of its nonzero invariant subrepresentations, β^* means the representation contragradient to a representation β, and ω is the one-dimensional representation of the group $\mathcal{W}(\overline{K}/K)$ defined by (3.1).

[1]E-mail: sabitova@math.uiuc.edu.

2. REPRESENTATIONS WITH IRREDUCIBLE SOCLES

Let k be a field and G a group. Recall that the *socle* of a representation $\sigma : G \to \mathrm{GL}_k(U)$ (or of a $k[G]$-module U) is the $k[G]$-submodule of U equal to the sum of all simple $k[G]$-submodules of U. If U has no simple submodules, the socle of U is assumed to be zero. We will denote the socle of a module U by $\mathrm{soc}(U)$. Note that the socle of a module U is simple if and only if U has only one simple submodule.

A representation σ of a group G over a field k is called *indecomposable* if it cannot be written as a direct sum of nonzero invariant subrepresentations. Otherwise we will say that σ is decomposable.

We will say that an orthogonal or symplectic representation of a group G over k is *minimal* if it cannot be written as an orthogonal sum of nonzero invariant subrepresentations. It is clear that each finite-dimensional orthogonal or symplectic representation is an orthogonal sum of minimal orthogonal or symplectic subrepresentations, respectively.

Theorem 2.1. *Let σ be a finite-dimensional minimal orthogonal or symplectic representation of a group G over a field k which can be written as a direct sum of indecomposable subrepresentations with irreducible socles. Let U be the space of the representation σ and $\langle\cdot\,,\cdot\rangle$ a nondegenerate invariant form on U. Then either σ is indecomposable, or $U \cong V \oplus V^*$, where V is an indecomposable submodule of U and V^* is the module contragradient to V. Moreover, an isomorphism of $k[G]$-modules $\lambda : V \oplus V^* \to U$ exists with the following property: if $\langle\cdot\,,\cdot\rangle' : (V \oplus V^*) \times (V \oplus V^*) \to k$ is the form on $V \oplus V^*$ defined by*

$$\langle x, y\rangle' = \langle \lambda(x), \lambda(y)\rangle, \quad x, y \in V \oplus V^*,$$

then the forms $\langle\cdot\,,\cdot\rangle'|_V$ and $\langle\cdot\,,\cdot\rangle'|_{V^*}$ are degenerate and $\langle\cdot\,,\cdot\rangle' : V \times V^* \to k$ is the standard form given by the relation

$$\langle u, f\rangle' = f(u), \quad u \in V, \ \ f \in V^*.$$

Proof. Let

$$U = U_1 \oplus \cdots \oplus U_s \oplus U_{s+1} \oplus \cdots \oplus U_{s+m},$$

where each U_i is an indecomposable submodule of the module U with simple socle. Assume that the first s modules $U_1, \ldots, U_s$ have maximal dimension n among all submodules U_i, so that $\dim U_1 = \cdots = \dim U_s = n$ and $\dim U_i < n$ for any $s+1 \le i \le s+m$; $\phi : U \to U^*$ is the isomorphism defined by the form $\langle\cdot\,,\cdot\rangle$, i.e., $\phi(u) = \langle u\,, \cdot\rangle$, $u \in U$; $\psi : U^* \longrightarrow U_1^* \oplus \cdots \oplus U_{s+m}^*$ denotes the natural isomorphism between $U^* = (U_1 \oplus \cdots \oplus U_{s+m})^*$ and $U_1^* \oplus \cdots \oplus U_{s+m}^*$. Let $\alpha_{ij} : U_i \longrightarrow U_j^*$, for each i and j, be the homomorphism of $k[G]$-modules defined by the diagram

$$\begin{array}{ccc} U & \xrightarrow{\ \psi\circ\phi\ } & U_1^* \oplus \cdots \oplus U_{s+m}^* \\ \big\uparrow & & \big\downarrow{\scriptstyle \pi_j} \\ U_i & \xrightarrow{\quad \alpha_{ij} \quad} & U_j^*, \end{array}$$

where π_j is the projection to the jth summand. If i exists such that α_{ii} is an isomorphism, then the form $\langle\cdot\,,\cdot\rangle|_{U_i}$ is nondegenerate. Consequently, the module U_i and its orthogonal complement in U (with respect to the form $\langle\cdot\,,\cdot\rangle$) are mutually orthogonal invariant subspaces of the space U. Since σ is minimal, it follows that $U = U_i$ and U is indecomposable.

Thus, we can assume that, for all i, the mapping α_{ii} is not an isomorphism, i.e., the form $\langle\cdot\,,\cdot\rangle|_{U_i}$ is degenerate for each i.

Let us show that among the mappings $\alpha_{11}, \alpha_{21}, \ldots, \alpha_{s1}$ there is at least one isomorphism. In fact, since each U_i has a unique simple submodule, it follows that each U_i^* has a unique maximal (proper) submodule. Therefore, if all α_{i1}, $1 \le i \le s+m$, are not surjective, then each $\alpha_{i1}(U_i)$ is contained in a unique maximal submodule of the module U_1^*, which contradicts the fact that the composition $\pi_1 \circ \psi \circ \phi$ is surjective. Thus, i exists such that α_{i1} is surjective and $1 \le i \le s$ (by the assumption on dimensions of the spaces $U_1, \ldots, U_s$). Without loss of generality we can assume that $i = 2$, i.e., the mapping α_{21}

is an isomorphism. Let us prove that, in this case, the form $\langle\cdot\,,\cdot\rangle|_{U_1\oplus U_2}$ is nondegenerate. Then the fact that the representation σ is minimal will imply that $U\cong U_1\oplus U_1^*$ and the form $\langle\cdot\,,\cdot\rangle$ has the property indicated in the assumptions of the theorem.

Assume that the form $\langle\cdot\,,\cdot\rangle|_{U_1\oplus U_2}$ is degenerate, i.e., $K=\ker(\langle\cdot\,,\cdot\rangle|_{U_1\oplus U_2})$ is not equal to zero. Let $p_i: U_1\oplus U_2\to U_i$, $i=1,2$, denotes the projection to the i-th summand. Since K is not zero, it follows that either $p_1(K)$, or $p_2(K)$ is not zero.

Consider the case when $p_1(K)\neq 0$. Denote by K_1 the kernel of the form $\langle\cdot\,,\cdot\rangle|_{U_1}$. Since, by assumptions, K_1 and $p_1(K)$ are not equal to zero, we have $\mathrm{soc}(U_1)\subseteq K_1$ and $\mathrm{soc}(U_1)\subseteq p_1(K)$. The following two cases are possible:

(1) $\mathrm{soc}(U_1)\subseteq K$,

(2) $\mathrm{soc}(U_1)\nsubseteq K$.

In case (1), for any $x\in\mathrm{soc}(U_1)$ and $y\in U_2$, we have $\langle x,y\rangle=0$, which implies that $\langle\cdot,y\rangle|_{\mathrm{soc}(U_1)}=0$, i.e., the element $\langle\cdot,y\rangle|_{U_1}=\alpha_{21}(y)$ belongs to the maximal submodule $(U_1/\mathrm{soc}(U_1))^*$ of U_1^*, which contradicts the fact that the homomorphism α_{21} is surjective.

In case (2), $x\in\mathrm{soc}(U_1)$ and $y\neq 0\in U_2$ exist such that $x+y\in K$. Taking into account that $\mathrm{soc}(U_1)\subseteq K_1$, for any $z\in U_1$, we have $\langle z,y\rangle=\langle z,x+y\rangle=0$, which contradicts the fact that the homomorphism α_{21} is injective.

Thus, $p_1(K)=0$. Consequently, $K\subseteq U_2$, which again contradicts the fact that the homomorphism α_{21} is injective. $\square$

Remark. If $k=\mathbb{C}$, then Theorem 2.1, with suitable changes, can be applied to finite-dimensional minimal unitary representations, where by a unitary representation we mean a $\mathbb{C}[G]$-module admitting a nondegenerate invariant Hermitian form (not necessarily positive definite). In more detail, if $\sigma: G\to \mathrm{GL}_{\mathbb{C}}(U)$ is a finite-dimensional minimal unitary representation, then, in the statement of Theorem 2.1, the $\mathbb{C}[G]$-module V^* should be replaced by the $\mathbb{C}[G]$-module $\check{V}$, where the G-module $\check{V}$ coincides with the G-module V^* and the multiplication by constants in $\check{V}$ is given by the formula

$$a\cdot\phi=\overline{a}\phi,\quad a\in\mathbb{C},\ \phi\in V^*.$$

3. PROOF OF THEOREM 1.1

Let K be a non-Archimedean local field with residue field k. Let $\overline{K}$ be a fixed separable algebraic closure of K, and let K^{unr} be a maximal unramified extension of K contained in $\overline{K}$. Let $I=\mathrm{Gal}(\overline{K}/K^{unr})$ be the inertia subgroup of the group $\mathrm{Gal}(\overline{K}/K)$, and let Φ be the preimage of the inverse Frobenius automorphism under the decomposition mapping

$$\pi:\mathrm{Gal}(\overline{K}/K)\longrightarrow\mathrm{Gal}(\overline{k}/k).$$

By a representation σ of the Weil group $\mathcal{W}(\overline{K}/K)$ we mean a continuous homomorphism

$$\sigma:\mathcal{W}(\overline{K}/K)\to\mathrm{GL}_{\mathbb{C}}(U),$$

where U is a finite-dimensional complex vector space (for the definition of the Weil group $\mathcal{W}(\overline{K}/K)$, see [1], § 1). Let $\omega:\mathcal{W}(\overline{K}/K)\longrightarrow\mathbb{C}^\times$ be the one-dimensional representation of the group $\mathcal{W}(\overline{K}/K)$ given by

$$\omega|_I=1,\quad \omega(\Phi)=q^{-1}, \tag{3.1}$$

where $q=\mathrm{card}\,(k)$.

By a representation σ' of the Weil–Deligne group $\mathcal{W}'(\overline{K}/K)$ we mean a continuous homomorphism

$$\sigma':\mathcal{W}'(\overline{K}/K)\to\mathrm{GL}_{\mathbb{C}}(U),$$

where U is a finite-dimensional complex vector space; it is also assumed that the restriction of σ' to the subgroup $\mathbb{C}$ of the group $\mathcal{W}'(\overline{K}/K)$ is complex analytic (for the definition of the Weil–Deligne group $\mathcal{W}'(\overline{K}/K)$, see [1], § 3).

It is known that there exists a bijection between the representations of the group $\mathcal{W}'(\overline{K}/K)$ and the pairs (σ, N), where $\sigma : \mathcal{W}(\overline{K}/K) \to \mathrm{GL}_{\mathbb{C}}(U)$ is a representation of the group $\mathcal{W}(\overline{K}/K)$ and N is a nilpotent endomorphism on U such that

$$\sigma(g)N\sigma(g)^{-1} = \omega(g)N, \quad g \in \mathcal{W}(\overline{K}/K).$$

In what follows we identify σ' with the corresponding pair (σ, N) and write $\sigma' = (\sigma, N)$. In this case, a representation σ of the group $\mathcal{W}(\overline{K}/K)$ is identified with the representation $(\sigma, 0)$ of the group $\mathcal{W}'(\overline{K}/K)$ ([1], § 3).

For a positive integer n, we denote by $\mathrm{sp}(n) = (\sigma, N)$ the special representation of the group $\mathcal{W}'(\overline{K}/K)$ of dimension n, i.e., the representation of this group in $\mathbb{C}^n$ (with the standard basis $e_0, \dots, e_{n-1}$) given by

$$\begin{aligned} \sigma(g)e_i &= \omega(g)^i e_i, \quad 0 \le i \le n-1, \;\; g \in \mathcal{W}(\overline{K}/K), \\ Ne_j &= e_{j+1}, \qquad 0 \le j \le n-2, \\ Ne_{n-1} &= 0. \end{aligned} \tag{3.2}$$

We will say that a representation $\sigma' = (\sigma, N)$ of the group $\mathcal{W}'(\overline{K}/K)$ is *admissible* if the representation σ is semisimple ([1], § 5).

Applying Theorem 2.1 to admissible minimal unitary, orthogonal, and symplectic representations of the group $\mathcal{W}'(\overline{K}/K)$ and taking into account the Remark to Theorem 2.1, we obtain

Corollary. Let σ' be an admissible minimal unitary, orthogonal, or symplectic representation of the group $\mathcal{W}'(\overline{K}/K)$. Let U be the space of the representation σ', and let $\langle \cdot\,, \cdot \rangle$ be a nondegenerate invariant form on U. Then either $\sigma' \cong \alpha \otimes \mathrm{sp}(n)$ for some irreducible representation α of the group $\mathcal{W}(\overline{K}/K)$ and a positive integer n, or $U \cong V \oplus \widetilde{V}$, where $V \cong \beta \otimes \mathrm{sp}(m)$ for a some irreducible representation β of the group $\mathcal{W}(\overline{K}/K)$ and a positive integer m, $\widetilde{V} = V^*$ if the form $\langle \cdot\,, \cdot \rangle$ is bilinear, and $\widetilde{V} = \check{V}$ if the form $\langle \cdot\,, \cdot \rangle$ is sesquilinear. What is more, there exists an isomorphism of $\mathbb{C}[\mathcal{W}'(\overline{K}/K)]$-modules $\lambda : V \oplus \widetilde{V} \to U$ with the following property: If $\langle \cdot\,, \cdot \rangle' : \left(V \oplus \widetilde{V}\right) \times \left(V \oplus \widetilde{V}\right) \to \mathbb{C}$ is the form on $V \oplus \widetilde{V}$ defined by

$$\langle x, y \rangle' = \langle \lambda(x), \lambda(y) \rangle, \quad x, y \in V \oplus \widetilde{V},$$

then the forms $\langle \cdot\,, \cdot \rangle'|_V$ and $\langle \cdot\,, \cdot \rangle'|_{\widetilde{V}}$ are degenerate and $\langle \cdot\,, \cdot \rangle' : V \times \widetilde{V} \to \mathbb{C}$ is the standard form given by

$$\langle u, f \rangle' = f(u), \quad u \in V, \;\; f \in \widetilde{V}.$$

Proof. It is clear that σ' is a direct sum of admissible indecomposable subrepresentations. What is more, it is known that each admissible indecomposable representation of the group $\mathcal{W}'(\overline{K}/K)$ is isomorphic to the representation $\alpha \otimes \mathrm{sp}(n)$ for some irreducible representation α of the group $\mathcal{W}(\overline{K}/K)$ and a positive integer n ([2], proposition 3.1.3). It is easy to verify that if W is the space of the representation α, then $W \otimes \mathbb{C}e_{n-1}$ is the unique simple submodule of the space $W \otimes \mathbb{C}^n$ of the representation $\alpha \otimes \mathrm{sp}(n)$, i.e., $\mathrm{soc}(\alpha \otimes \mathrm{sp}(n))$ is irreducible. Thus, Theorem 2.1 and Remark to it can be applied to admissible minimal unitary, orthogonal, and symplectic representations of the group $\mathcal{W}'(\overline{K}/K)$. □

Proof of Theorem 1.1. From Corollary it follows that if a representation σ' is decomposable, then

$$\sigma' \cong (\beta \otimes \mathrm{sp}(m)) \oplus (\beta \otimes \mathrm{sp}(m))^*$$

for some positive integer m and irreducible representation β of the group $\mathcal{W}(\overline{K}/K)$. It is easy to verify that

$$(\beta \otimes \mathrm{sp}(m))^* \cong \beta^* \otimes \omega^{1-m} \otimes \mathrm{sp}(m)$$

([1], § 3, proposition (iii)), which implies (1.1).

Assume now that a representation σ' is in the form (1.2). Note that the representation $\alpha \otimes \omega^{\frac{n-1}{2}}$ is either orthogonal, or symplectic. In fact, since σ' is self-contragradient, we have

$$\alpha \otimes \mathrm{sp}(n) \cong (\alpha \otimes \mathrm{sp}(n))^* \cong \alpha^* \otimes \omega^{1-n} \otimes \mathrm{sp}(n).$$

By virtue of the uniqueness of decomposition of an admissible representation of the group $\mathcal{W}'(\overline{K}/K)$ into irreducible subrepresentations ([1], § 5, corollary 2), we have $\alpha \cong \alpha^* \otimes \omega^{1-n}$ or, which is equivalent, $\alpha \otimes \omega^{\frac{n-1}{2}} \cong \left(\alpha \otimes \omega^{\frac{n-1}{2}}\right)^*$. Since the representation $\alpha \otimes \omega^{\frac{n-1}{2}}$ of the group $\mathcal{W}(\overline{K}/K)$ is irreducible, it follows that $\alpha \otimes \omega^{\frac{n-1}{2}} \cong \rho \otimes \omega^s$ for some irreducible representation ρ of the group $\mathcal{W}(\overline{K}/K)$ with finite image and $s \in \mathbb{C}$ ([3], proposition 4.10). Thus, $\rho \otimes \omega^s \cong \rho^* \otimes \omega^{-s}$. Therefore, ω^s has finite image (one can verify this, for example, calculating the determinant). Consequently, the representation $\alpha \otimes \omega^{\frac{n-1}{2}}$ has finite image. Since it is self-contragradient and irreducible, it is either orthogonal, or symplectic. In addition, the representation ([1], § 7)

$$\omega^{\frac{1-n}{2}} \otimes \mathrm{sp}(n) \begin{cases} \text{is orthogonal} & \text{if } n \text{ is odd}, \\ \text{is symplectic} & \text{if } n \text{ is even}. \end{cases}$$

By Corollary, this proves Theorem 1.1. □

REFERENCES

1. D. E. Rohrlich, “Elliptic Curves and the Weil–Deligne Group,” in *Elliptic Curves and Related Topics. CRM Proc. Lecture Notes* (Providence, Amer. Math. Soc., 1994, Vol. 4), pp. 125–157.
2. P. Deligne, “Formes Modulaires et Représentations de GL(2),” in *Modular Functions of One Variable* (Springer-Verlag, New York, 1973, Vol. 2), pp. 55–105.
3. P. Deligne, “Les Constantes des Équations Fonctionnelles des Fonctions L,” in *Modular Functions of One Variable* (Springer-Verlag, New York, 1973, Vol. 2), pp. 501–595.