Degrees of Categoricity of Computable Structures

Ekaterina B. Fokina[*], Iskander Kalimullin[†], & Russell Miller[‡]

September 22, 2009

Abstract

Defining the degree of categoricity of a computable structure $\mathcal{M}$ to be the least degree $\boldsymbol{d}$ for which $\mathcal{M}$ is $\boldsymbol{d}$-computably categorical, we investigate which Turing degrees can be realized as degrees of categoricity. We show that for all n, degrees d.c.e. in and above $\mathbf{0}^{(n)}$ can be so realized, as can the degree $\mathbf{0}^{(\omega)}$.

1 Introduction

The notion of *computable categoricity* has been part of computable model theory for more than fifty years, since Fröhlich and Shepherdson first produced an example of two computable fields which were isomorphic but not computably isomorphic. (See [2, Theorem 5.51], where the terms used are "explicitly presented" and "explicitly isomorphic.") Since then, the definition has been standardized and relativized to arbitrary Turing degrees $\boldsymbol{d}$, and has been the subject of much study.

Definition 1.1 *A computable structure $\mathcal{M}$ is $\boldsymbol{d}$-*computably categorical *if, for every computable structure $\mathcal{A}$ isomorphic to $\mathcal{M}$, there exists a $\boldsymbol{d}$-computable isomorphism from $\mathcal{M}$ onto $\mathcal{A}$. In case $\boldsymbol{d} = \mathbf{0}$, we simply say that $\mathcal{M}$ is* computably categorical.

Definition 1.2 *Let $\mathcal{M}$ be any computable structure. The* categoricity spectrum *of $\mathcal{M}$ is the set*

$$CatSpec(\mathcal{M}) = \{\boldsymbol{d} : \mathcal{M} \textit{ is } \boldsymbol{d}\textit{-computably categorical}\},$$

[*]The first author was partially supported by the Russian Foundation for Basic Research (grant RFBR 08-01-00336) and the State Maintenance Program for the Leading Scientific Schools of the Russian Federation (grant NSh-335.2008.1.). All three authors were supported by binational grant # DMS 0554841 from the National Science Foundation. The main question discussed in this paper was suggested by Sergey Goncharov.

[†]The second author was partially supported by the Russian Foundation for Basic Research (grant RFBR 05-01-00605) and RF President grant MK-4314.2008.1.

[‡]The third author was partially supported by grants numbered 80209-04 12, 67182-00-36, 68470-00 37, 69723-00 38 and 61467-00 39 from The City University of New York PSC-CUNY Research Award Program, and by grant #13397 from the Templeton Foundation.

the set of all Turing degrees capable of computing isomorphisms among arbitrary computable copies of $\mathcal{M}$. We say that a Turing degree $\boldsymbol{d}$ is the degree of categoricity of $\mathcal{M}$ *if $\boldsymbol{d}$ is the least degree in CatSpec($\mathcal{M}$). Finally, $\boldsymbol{d}$ is* categorically definable *if it is the degree of categoricity of some computable structure.*

This terminology is intended to recall the notions of the *spectrum of a structure* $\mathcal{A}$ (that is, the set of all Turing degrees of structures with domain ω which are isomorphic to $\mathcal{A}$), and the *degree of the isomorphism class of* $\mathcal{A}$, which was defined by Richter in [17] to be the least degree in the spectrum of $\mathcal{A}$, if such a degree exists. The terminology for structures has become confusing, since the simple phrase "degree of $\mathcal{A}$" usually means the Turing degree of the atomic diagram of $\mathcal{A}$, rather than the concept defined by Richter. For categoricity, however, no such confusion exists.

The focus of this paper will be the question of which Turing degrees are categorically definable. Since there are only countably many computable structures, clearly most Turing degrees are not categorically definable. Our main result, proven in several steps which culminate in Section 5, will be

Theorem 5.9 *If $\boldsymbol{d}$ is any Turing degree for which there exists an $m \in \omega$ such that $\mathbf{0}^{(m)} \leq_T \boldsymbol{d}$ but $\boldsymbol{d}$ is c.e. in $\mathbf{0}^{(m)}$, or even d.c.e. in $\mathbf{0}^{(m)}$, then $\boldsymbol{d}$ is categorically definable.*

In Section 6 we show that the nonarithmetical degree $\mathbf{0}^{(\omega)}$ is also categorically definable.

The concepts we use from computability and computable model theory are standard; we suggest [8] and [18] as references. For further background on the notion of computable categoricity, [5] is very useful. Much of the literature on this topic uses the term *autostability* in place of *computable categoricity.* The distinct terminology reflects the historical development of the subject, which was studied, largely independently, in both Russia and the West. We the present authors are grateful to those who came before us and helped to bridge that divide.

2 The Basic Construction

Theorem 2.1 *Let $\boldsymbol{d}$ be any c.e. degree. Then there exists a computable structure $\mathcal{B}$ with degree of categoricity $\boldsymbol{d}$. Moreover, an index for such a structure $\mathcal{B}$ is computable uniformly in the index e of any c.e. set $W_e \in \boldsymbol{d}$.*

Proof. Fix a c.e. set $A = W_e \in \boldsymbol{d}$ with a computable total 1-1 function h with range A. The domain of $\mathcal{B}$ will be ω, partitioned computably into

$$\{\alpha, \beta, \gamma, \delta\} \cup \{x_i : i \in \omega\} \cup \{y_i : i \in \omega\} \cup \{u_i : i \in \omega\}.$$

We view $\{x_i : i \in \omega\}$ and $\{y_i : i \in \omega\}$ as ω-chains, while $\{u_i : i \in \omega\}$ and $\{v_i : i \in \omega\}$ serve only as witness nodes. The language has a binary predicate P

and a ternary predicate R. In our $\mathcal{B}$, P holds of all pairs of each of the following forms

$$(x_i, x_{i+1}) \qquad (x_i, y_{i+1}) \qquad (y_i, x_{i+1}) \qquad (y_i, y_{i+1})$$

for every $i \in \omega$. Also, for each i, R holds of the triples

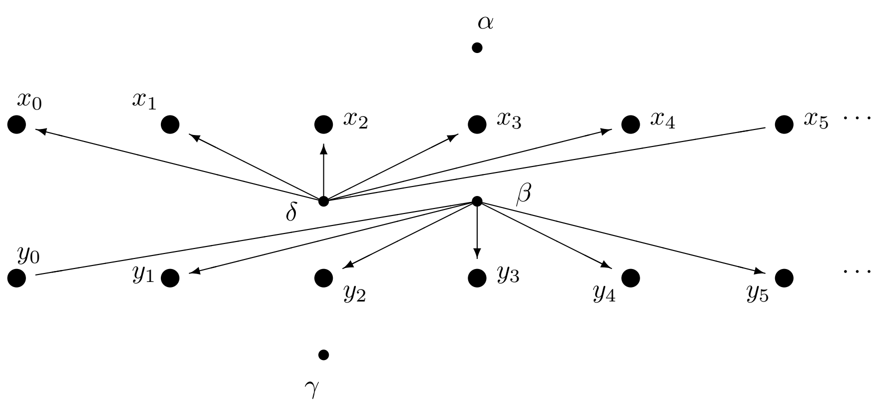
$$(\alpha, u_{6i}, x_{h(i)}) \ \ (\gamma, u_{6i+1}, y_{h(i)}) \ \ (\beta, u_{6i+2}, x_{h(i)}) \ \ (\delta, u_{6i+3}, y_{h(i)}).$$

Moreover, for all i, R holds of the triples

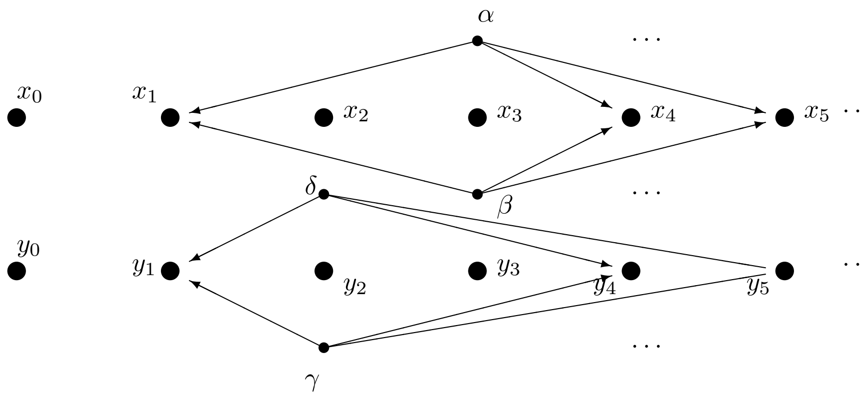
$$(\beta, u_{6i+4}, y_i) \ \text{ and } \ (\delta, u_{6i+5}, x_i).$$

Clearly these relations are all computable. Finally we have constant symbols c and d, which will be used to distinguish α from γ and β from δ. In our $\mathcal{B}$ we set $c^{\mathcal{B}} = \gamma$ and $d^{\mathcal{B}} = \delta$.

The idea of the construction is that, even though $\mathcal{B}$ is computable, it is built in a dynamic way. In the diagrams below, we draw an arrow from α to x_i to represent the truth of $(\exists u R(\alpha, u, x_i))$, and similarly for β, γ, δ, and for the nodes y_i. Thus the arrows form a c.e. relation, appearing during the construction as A is enumerated by h. Some arrows, however, are present right from the start. Our picture of $\mathcal{B}$ at stage 0 is the following:



The next picture shows the arrows which are added in the situation in which $1, 4, 5 \in A$ and $0, 2, 3 \notin A$. (For clarity we omit the pre-existing arrows drawn in the previous picture.)



Remembering that we started with arrows from β to all nodes y_i and from δ to all nodes x_i, we see now that the nodes divide into two cases, depending on A, as follows.

If $i \in A$, then there are arrows to x_i from α, β, and δ, and arrows to y_i from β, γ, and δ. Hence in this case, an automorphism g of $\mathcal{B}$ (in the reduct without the constants c and d) must interchange α with γ iff g interchanges x_i with y_i, and it is irrelevant whether g fixes β and δ or interchanges them.

If $i \notin A$, then the only arrow to x_i comes from δ, and the only arrow to y_i comes from β. The situation for an automorphism g (again ignoring c and d) is now the opposite: g must interchange β with δ iff g interchanges x_i with y_i, and it is irrelevant whether g fixes α and γ or interchanges them.

The structure $\hat{\mathcal{B}}$ will be an exact copy of $\mathcal{B}$ (with all elements represented by adding hats to the elements of $\mathcal{B}$), except that we define $c^{\hat{\mathcal{B}}} = \hat{\alpha}$ (and $d^{\hat{\mathcal{B}}} = \hat{\delta}$, just as in $\mathcal{B}$). Clearly there is a unique isomorphism f from $\mathcal{B}$ onto $\hat{\mathcal{B}}$, with $f(\alpha) = \hat{\gamma}$ and $f(\beta) = \hat{\beta}$, which must have $f(x_i) = \hat{y}_i$ and $f(y_i) = \hat{x}_i$ for each $i \in A$, and $f(x_i) = \hat{x}_i$ and $f(y_i) = \hat{y}_i$ for each $i \notin A$. (Since each witness element u_j is part of a unique triple satisfying R, this also lets us compute $f(u_j)$ for all j.) Intuitively, f interchanges the α-portion of the x_i-chain with the γ-portion of the y_i-chain, but leaves the β- and δ-portions fixed.

Now $f(\alpha) = \hat{\gamma}$, so for every $i \in A$ we have

$$(\exists u \in \mathcal{B}) R^{\mathcal{B}}(\alpha, u, x_i) \implies (\exists \hat{u} \in \hat{\mathcal{B}}) R^{\hat{\mathcal{B}}}(\hat{\gamma}, \hat{u}, f(x_i)) \implies f(x_i) \neq \hat{x}_i,$$

since we know $\neg(\exists \hat{u} \in \hat{\mathcal{B}}) R^{\hat{\mathcal{B}}}(\hat{\gamma}, \hat{u}, x_i)$. (In fact, $f(x_i)$ will equal $\hat{y}_i$, with an arrow from $\hat{\gamma}$ to $\hat{y}_i$.) On the other hand, $f(\beta) = \hat{\beta}$, so for $i \notin A$ we have

$$(\forall u \in \mathcal{B}) \neg R^{\mathcal{B}}(\beta, u, x_i) \implies (\forall \hat{u} \in \hat{\mathcal{B}}) \neg R^{\hat{\mathcal{B}}}(\hat{\beta}, \hat{u}, f(x_i)) \implies f(x_i) = \hat{x}_i,$$

since $(\exists \hat{u} \in \hat{\mathcal{B}})\ R^{\hat{\mathcal{B}}}(\hat{\beta}, u, \hat{y}_i)$. Thus $A \leq_T f$, as the theorem requires.

We also claim that for every computable $\mathcal{D} \cong \mathcal{B}$, an A-oracle can compute an isomorphism g from $\mathcal{B}$ onto $\mathcal{D}$. (In particular, this will show $f \equiv_T A$.) Clearly

$g(\gamma) = c^{\mathcal{D}}$ and $g(\delta) = d^{\mathcal{D}}$. Notice that since $P^{\mathcal{D}}$ is computable, we can induct on i and identify which two elements $a_i, b_i \in \mathcal{D}$ must be $g(x_i)$ and $g(y_i)$. $P^{\mathcal{D}}$ does not determine which of these is which, so now we use our A-oracle. If $i \notin A$, then

$$g(x_i) = a_i \iff (\exists u \in \mathcal{D})\ R^{\mathcal{D}}(d^{\mathcal{D}}, u, a_i) \iff \neg(\exists u \in \mathcal{D})\ R^{\mathcal{D}}(d^{\mathcal{D}}, u, b_i)$$

since $R^{\mathcal{B}}(d^{\mathcal{B}}, u, x_i)$ holds for some $u \in \mathcal{B}$. Similarly, if $i \in A$, then

$$g(y_i) = a_i \iff (\exists u \in \mathcal{D})\ R^{\mathcal{D}}(c^{\mathcal{D}}, u, a_i) \iff \neg(\exists u \in \mathcal{D})\ R^{\mathcal{D}}(c^{\mathcal{D}}, u, b_i).$$

So g on the elements x_i and y_i is computable from our A-oracle, and it is easy to extend g to the witness elements u in $\mathcal{B}$, since each u lies in a unique triple in $R^{\mathcal{B}}$. Thus this $\mathcal{B}$ is the computable structure required by Theorem 2.1.

For the uniformity claim, we note that this entire process was uniform in any index e such that $W_e = A$, assuming only that A is infinite. It can be made uniform for finite A as well: one only needs to ensure that the domain of $\mathcal{B}$ will be ω, and this can be done by building only finitely much of $\mathcal{B}$ at each stage, always adding the least available fresh element to the domain. Of course, for two distinct sets W_e of the same Turing degree, the two structures $\mathcal{B}$ built by this uniform process may be nonisomorphic. The theorem only states that there is a process, uniform in e, for building some $\mathcal{B}$ whose degree of categoricity is $\deg(W_e)$. ■

We remark that the structure $\mathcal{B}$ built in this proof can be made uniformly $\boldsymbol{d}$-computably categorical, by adding one more unary predicate P_0 to hold of the elements x_0 and y_0. (Without P_0, we need to know these elements to compute g from A.)

Also, we note that we can relativize the above proof to an X-oracle. This will be exploited in Section 5. It requires a generalization of Definition 1.2.

Definition 2.2 *Let $\boldsymbol{c}$ be the Turing degree of a structure $\mathcal{M}$. We define the* categoricity spectrum of $\mathcal{M}$ relative to $\boldsymbol{c}$*, written $CatSpec_{\boldsymbol{c}}(\mathcal{M})$, to be the set:*

$$\{\boldsymbol{d} : (\forall \mathcal{A} \cong \mathcal{M})[deg(\mathcal{A}) \leq \boldsymbol{c} \implies \exists \text{ an isomorphism } f \leq_T \boldsymbol{d} \text{ from } \mathcal{M} \text{ onto } \mathcal{A}]\}.$$

Thus $\boldsymbol{d} \in CatSpec_{\boldsymbol{c}}(\mathcal{M})$ iff all $\boldsymbol{c}$-computable copies of $\mathcal{M}$ are $\boldsymbol{d}$-computably isomorphic.

Corollary 2.3 *Fix an oracle $X \subseteq \omega$, and let $\boldsymbol{d}$ be any Turing degree containing a set W which is c.e. relative to X and satisfies $X \leq_T W$. Then there exists an X-computable structure $\mathcal{B}$ for which $CatSpec_{deg(X)}(\mathcal{B})$ contains precisely those Turing degrees which compute W (that is, the upper cone of degrees $\geq_T \boldsymbol{d}$). Moreover, an index for $\mathcal{B}$ is computable uniformly from the index e of any set $W_e^X \in \boldsymbol{d}$ serving as W, under the usual numbering of X-c.e. sets.* ■

The relativization of the proof is immediate, but one subtlety can be missed. If the degree $\boldsymbol{d}$ failed to compute X, then the two structures $\mathcal{B}$ and $\hat{\mathcal{B}}$ we built would still have the property that every isomorphism between them computes W, but we would not be able to show that $\boldsymbol{d} \in \mathrm{CatSpec}_{\deg(X)}(\mathcal{B})$. Indeed, since the $\mathcal{B}$ we built is rigid, the following lemma shows this to be impossible.

Lemma 2.4 *Let $\mathcal{B}$ be a rigid structure, of Turing degree $\boldsymbol{c}$, and suppose $\boldsymbol{c} \not\leq_T \boldsymbol{d}$. Then $\boldsymbol{d} \notin CatSpec_{\boldsymbol{c}}(\mathcal{B})$.*

Proof. Let f be a permutation of ω of degree precisely $\boldsymbol{c}$. Define a structure $\mathcal{A} \cong \mathcal{B}$ so that f is an isomorphism between them: $x \in R^{\mathcal{A}}$ iff $f(x) \in R^{\mathcal{B}}$, etc. Then this "pullback structure" $\mathcal{A}$ is also $\boldsymbol{c}$-computable, but by rigidity the unique isomorphism between them is f, which is not $\boldsymbol{d}$-computable since $\boldsymbol{c} \not\leq_T \boldsymbol{d}$. ∎

The structure $\mathcal{B}$ built here is similar in some ways to those used by Goncharov in [3] to show that all finite computable dimensions are possible, and to those built by Cholak, Goncharov, Khoussainov, and Shore in [1] to show that computable categoricity need not persist under expansions by constants. However, by a theorem of Goncharov in [4], if two computable structures $\mathcal{B}$ and $\mathcal{C}$ are $\mathbf{0}'$-computably isomorphic but not computably isomorphic, then the computable dimension of $\mathcal{B}$ must be infinite. This is the case with the $\mathcal{B}$ built in Theorem 2.1, so its computable dimension is ω.

3 D.C.E. Degrees

Theorem 3.1 *Let $\boldsymbol{d}$ be any d.c.e. degree. Then there exists a computable structure $\mathcal{B}$ with degree of categoricity $\boldsymbol{d}$.*

Proof. Fix a d.c.e. set $D = A - B \in \boldsymbol{d}$, where $B \subseteq A$ are c.e. sets, and let h be a computable total 1-1 function h with range A. Define the c.e. set $C = h^{-1}(B) = \{x : h(x) \in B\}$ and the d.c.e. set $E = D \oplus C = \{2x : x \in D\} \cup \{2x+1 : x \in C\}$. It is easy to see that $D \equiv_T E$ and

$$D \leq_T X \iff \overline{E} \text{ is c.e. in } X$$

for every $X \subseteq \omega$ (since C is c.e. and D is c.e. in C). So without loss of generality we can assume from the beginning that

$$D \leq_T X \iff \overline{D} \text{ is c.e. in } X$$

for every $X \subseteq \omega$.

Define first a computable structure $\mathcal{A}$ with universe partitioned computably into five pieces:

$$\{x_i : i \in \omega\} \cup \{a_i : i \in \omega\} \cup \{b_i : i \in \omega\} \cup \{c_i : i \in \omega\} \cup \{d_i : i \in \omega\}.$$

The language has one binary predicate P. In the structure $\mathcal{A}$, P holds of all pairs of each of the following forms

$$(x_i, x_{i+1}) \ \ (x_i, a_i) \ \ (x_i, b_i) \ \ (x_i, c_i) \ \ (x_i, d_i) \ \ (a_i, b_i) \ \ (b_i, c_i) \ \ (c_i, d_i) \ \ (d_i, a_i)$$

for every $i \in \omega$. We can view this as the ω-chain $\{x_i : i \in \omega\}$, each node x_i of which is connected with all nodes of the cycle $\{a_i, b_i, c_i, d_i\}$ of length 4.

Keeping $D = A - B$, let $\mathcal{B}$ be the substructure of $\mathcal{A}$ on the universe

$$\{x_i : i \in \omega\} \cup \{a_i : i \in A\} \cup \{b_i : i \in B\} \cup \{c_i : i \in \omega\} \cup \{d_i : i \in \omega\}.$$

Note that the universe of $\mathcal{B}$ is c.e. so we can fix a computable $1-1$ function which enumerates the elements of $\mathcal{B}$. Thus $\mathcal{B}$ is computably isomorphic to its preimage under this function, so we may treat $\mathcal{B}$ itself as computable.

We claim that for every computable $\mathcal{D} \cong \mathcal{B}$ there is a D-computable isomorphism $g : \mathcal{B} \to \mathcal{D}$. Since $\mathcal{B}$ and $\mathcal{D}$ are computable we can computably find $g(x_i)$ for each $i \in \omega$. If $i \in D$ then $\mathcal{B}$ contains exactly three elements from the cycle $\{a_i, b_i, c_i, d_i\}$, and in this case we can uniquely define $g(a_i)$, $g(c_i)$ and $g(d_i)$. If $i \notin D$ then $\mathcal{B}$ has either two elements $\{c_i, d_i\}$, or the full cycle $\{a_i, b_i, c_i, d_i\}$. We define $g(c_i)$ and $g(d_i)$ as soon as we find two elements of $\mathcal{D}$ that can serve as their images. If later we find that $i \in A$ (so that $a_i \in \mathcal{B}$), then also $i \in B$, so $b_i \in \mathcal{B}$, and we will be able to extend the isomorphism g by the symmetry of the 4-cycle. Thus, $\mathcal{B}$ is D-computably categorical.

Now consider another substructure $\mathcal{C}$ of $\mathcal{A}$, with universe

$$\{x_i : i \in \omega\} \cup \{b_i : i \in A\} \cup \{a_i : i \in B\} \cup \{c_i : i \in \omega\} \cup \{d_i : i \in \omega\}.$$

Again the universe is c.e., so we can identify $\mathcal{C}$ with the corresponding preimage under a computable $1-1$ function. It is easy to see that $\mathcal{B} \cong \mathcal{C}$.

Let $f : \mathcal{B} \to \mathcal{C}$ be an isomorphism. Then for each $i \in \omega$,

$$i \notin D \iff i \in B \text{ or } f(c_i) = c_i$$

Indeed, if $i \in \overline{D} - B = \overline{A}$ then a_i and b_i lie in neither $\mathcal{B}$ nor $\mathcal{C}$, so that $f(c_i) = c_i$. For the reverse direction we note that if $a_i \in \mathcal{B}$ then either $b_i \in \mathcal{B}$ (so that $i \in B \subseteq \overline{D}$), or $f(c_i) = b_i \neq c_i$. Finally, if $a_i \notin \mathcal{B}$ then $i \in \overline{A} \subseteq \overline{D}$.

Hence, $\overline{D}$ is c.e in f, and by our choice of D above, we have $D \leq_T f$. Thus the degree of categoricity of $\mathcal{B}$ is exactly $\boldsymbol{d}$. ■

This too relativizes, just as Theorem 2.1 did.

Corollary 3.2 *Fix any $X \subseteq \omega$, and let $\boldsymbol{d}$ be any Turing degree containing a set W which is d.c.e. relative to X and satisfies $X \leq_T W$. Then there exists an X-computable structure $\mathcal{B}$ for which $CatSpec_{deg(X)}(\mathcal{B})$ contains precisely those Turing degrees which compute W.* ■

4 Standard Theories

In [11], Hirschfeldt, Khoussainov, Shore, and Slinko showed how to make several computability-theoretic properties of arbitrary nontrivial computable structures (in computable languages) carry over to computable models of specific theories. For example, for any computable structure $\mathcal{M}$, they built a computable directed graph $\mathcal{G}$ with the same computable dimension and the same spectrum as $\mathcal{M}$, and their results also covered spectra of relations and persistence of computable categoricity. They did not consider the degree of categoricity of a computable structure, but nevertheless their method of coding $\mathcal{M}$ into $\mathcal{G}$ does preserve the categoricity spectrum: $\text{CatSpec}(\mathcal{M}) = \text{CatSpec}(\mathcal{G})$, and so $\mathcal{M}$ has a degree of categoricity iff $\mathcal{G}$ does, in which case those degrees are equal. The same holds for their coding of an arbitrary $\mathcal{M}$ into other structures, and when one examines those codings, the following proposition becomes clear.

Proposition 4.1 *For every computable structure $\mathcal{M}$, there exist a directed graph, a symmetric irreflexive graph, a lattice, an integral domain (of arbitrary characteristic), a commutative semigroup, and a two-step nilpotent group with the same categoricity spectrum as $\mathcal{M}$. In particular, every possible degree of categoricity is categorically definable via a model of any of those theories.* ■

This also shows that any categoricity spectrum for a computable structure in an infinite computable signature can be realized in a finite signature as well, even with just a single binary relation, as for graphs. So, if it is simpler or more intuitive to use an infinite computable signature to create a particular categoricity spectrum, there is no reason not to do so.

Certain theories are omitted from the main theorem of [11], and indeed that theorem is known to be false for linear orders, Boolean algebras, and trees, by results of Richter in [17]. It is not known whether the theorem holds for fields or not. Here we take a small step in that direction.

Theorem 4.2 *Let $\boldsymbol{d}$ be any c.e. degree. Then there exists a computable algebraic field $\mathcal{F}$ with degree of categoricity $\boldsymbol{d}$. Moreover, an index for $\mathcal{F}$ is computable uniformly in the index e of any c.e. set $W_e \in \boldsymbol{d}$.*

Proof. Fix an infinite c.e. set $W \in \boldsymbol{d}$, and fix a presentation $\mathbb{Q}$ of the field of rational numbers. Let $p_n \in \omega$ denote the n-th prime number, and let $q_n = \tilde{q}_n \in \mathbb{Q}$ be the result of adding the multiplicative identity to itself p_n times in $\mathbb{Q}$. That is, q_n is the element of $\mathbb{Q}$ representing the n-th prime p_n.

We build two isomorphic fields F and $\tilde{F}$ in stages, with F_0 being the field of characteristic 0 generated by the square roots of all the primes q_n in $\mathbb{Q}$. $\tilde{F}_0$ is the same field as F_0, but we write $\tilde{q}_n$ for the n-th prime in $\tilde{F}_0$. The idea is that the unique isomorphism f from F to $\tilde{F}$ should satisfy $f(\sqrt{q_n}) = -\sqrt{\tilde{q}_n}$ iff $n \in W$. (So, for those $n \notin W$, we must have $f(\sqrt{q_n}) = \sqrt{\tilde{q}_n}$.) As long as n appears not to be in the c.e. set W, we will ensure that $\sqrt{q_n}$ and $\sqrt{\tilde{q}_n}$ both satisfy an existential property (the existence of a root of a certain polynomial with coefficients in $\mathbb{Q}[\sqrt{q_n}]$) which $-\sqrt{q_n}$ and $-\sqrt{\tilde{q}_n}$ do not (yet) satisfy. If we ever see n enter W, then we will adjoin more elements to the fields so that all of $\pm\sqrt{q_n}$ and $\pm\sqrt{\tilde{q}_n}$ satisfy this property, but so that $\sqrt{q_n}$ and $-\sqrt{\tilde{q}_n}$ now satisfy a separate existential property which $-\sqrt{q_n}$ and $\sqrt{\tilde{q}_n}$ do not satisfy.

(Of course, F_0 contains two square roots of q_n. For clarity, let us specify that $-\sqrt{q_n}$ will always denote the lesser of these two, under the $<$ relation on the domain ω of F_0, and that $\sqrt{q_n}$ will denote the greater. Likewise, $-\sqrt{\tilde{q}_n}$ will denote the lesser square root of $\tilde{q}_n$ in $\tilde{F}_0$.)

The existential properties we use are the existence of roots of polynomials given by the following proposition, which is proven in [15, Prop. 2.15].

Proposition 4.3 (Miller) *For any fixed prime q, let E be the field $\mathbb{Q}[\sqrt{q}]$. Then for every odd prime number d, there exists a polynomial $h(X) \in E[X]$ of degree d, with image $h^-(X) \in E[X]$ under the automorphism of E mapping $\sqrt{q}$ to $-\sqrt{q}$, such that The splitting field of h over E has Galois group isomorphic to S_d, the symmetric group on the d roots of h, and the same holds for h^-. The splitting field of $h(X)$ over the splitting field of $h^-(X)$ also has Galois group isomorphic to S_d (and vice versa). (It follows that h and h^- are both irreducible in the polynomial ring $E[X]$, of course, and likewise over the splitting fields of each other.) Moreover, uniformly in q, d, and any computable presentation of E, it is computable whether an arbitrary $h(X) \in E[X]$ satisfies these properties.* ■

At stage $2s+1$ of the construction, we choose a prime number d_s bigger than any prime used so far, and apply this proposition to find a polynomial h_s of degree d_s satisfying these properties over the field $E_s = \mathbb{Q}[\sqrt{q_s}]$. To each of F_{2s} and $\tilde{F}_{2s}$ we adjoin one root r (respectively $\tilde{r}$) of $h_s(X)$, setting $F_{2s+1} = F_{2s}[r]$ and $\tilde{F}_{2s+1} = \tilde{F}_{2s}[\tilde{r}]$. Notice that since $d_s = [E_s[r] : E_s]$, this forces d_s to divide $[F_{2s+1} : F_0]$, and since the large prime d_s does not divide $[F_{2s} : F_0]$, it must divide $[F_{2s+1} : F_{2s}]$. Indeed $[F_{2s+1} : F_{2s}] = d_s$, since r is a root of the polynomial h_s of degree d_s, and so $h_s(X)$ is the minimal polynomial of r over F_{2s}. (This justifies us in not specifying r to be any particular root of $h_s(X)$; we must have $F_{2s+1} \cong F_{2s}[X]/(h_s(X))$.) By induction, F_{2s} and $\tilde{F}_{2s}$ were isomorphic, and clearly now F_{2s+1} and $\tilde{F}_{2s+1}$ are isomorphic as well, but only via an isomorphism mapping $\sqrt{q_s}$ to $\sqrt{\tilde{q}_s}$.

At stage $2s+2$, we check whether $W_{s+1} = W_s$. (We arrange our enumeration so that $W_{s+1} \subseteq \{0, \ldots, s\}$, and so that at most one element may enter W at stage $s+1$.) If $W_{s+1} = W_s$, we do nothing. If not, then for the unique $n \in (W_{s+1} - W_s)$, we adjoin to each of F_{2s+1} and $\tilde{F}_{2s+1}$ a root of $h_n^-(X)$. Then we choose a prime number d'_n greater than any yet used, and find a polynomial $j_n(X)$ of degree d'_n satisfying Proposition 4.3 over the field E_n. We adjoin one root of this new $j_n(X)$ to F_{2s+1}, and one root of $j_n^-(X)$ to $\tilde{F}_{2s+1}$. This completes the constructions of F_{2s+2} and $\tilde{F}_{2s+2}$: they are still isomorphic (by an argument similar to the above; see [15] for details), but every isomorphism between them now maps $\sqrt{q_n}$ to $-\sqrt{\tilde{q}_n}$.

The fields F and $\tilde{F}$ thus built will be computable, algebraic, and isomorphic to each other. However, in the argument above, we saw that for $n \notin W$, $h_n(X)$ has a root in F and also in $\tilde{F}$, whereas $h_n^-(X)$ has no root in either field, so that $\sqrt{q_n}$ in F must map to $\sqrt{\tilde{q}_n}$ in $\tilde{F}$. On the other hand, for $n \in W$, both these polynomials had roots in both fields, and $j_n(X)$ had a root in F but not in $\tilde{F}$, whereas $j_n^-(X)$ had a root in $\tilde{F}$ but not in F. Hence any isomorphism $f : F \to \tilde{F}$ must satisfy

$$f(\sqrt{q_n}) = \sqrt{\tilde{q}_n} \iff n \notin W,$$

and therefore must compute W. So $\mathrm{CatSpec}(F) \subseteq \{\boldsymbol{d} : \deg(W) \leq_T \boldsymbol{d}\}$.

Conversely, suppose that F' is any computable field isomorphic to F. There is only one possible isomorphism from the rationals in F to those in F', and with a W-oracle, we can determine how to extend this to an isomorphism from F into F', as follows. If $n \notin W$, then wait for a root of either $h_n(X)$ or $h_n^-(X)$ to appear in F' (where again we define $-\sqrt{q_n'}$ to be the lesser square root of q_n' in F'), and when that root appears, we know which of $\pm\sqrt{q_n'}$ must be the image in F' of $\sqrt{q_n}$. (We also map the root of $h_n(X)$ to the root of $h_n(X)$ or $h_n^-(X)$ in F', whichever appeared.) On the other hand, if $n \in W$, then we run the construction of F until the stage when n enters W, thus determining the polynomial $j_n(X)$. We now do the same with $j_n(X)$, finding an appropriate root in F' to be the image of the root of $j_n(X)$ from F; this dictates the image in F' of $\sqrt{q_n}$, which in turn dictates the images in F' of the roots of $h_n(X)$ and $h_n^-(X)$. Thus we have computed the images in F' of all generators of F, using a W-oracle, and extending the map to all of F is simple. Moreover, any embedding of an algebraic field into a field isomorphic to itself must in fact be an isomorphism (see e.g. [15, Lemma 2.10]). Therefore, $\deg(W)$ must lie in the categoricity spectrum of F, and since every isomorphism from F onto $\tilde{F}$ computes W, it must be the degree of categoricity of F.

Finally, it is clear that this construction is uniform in any index e such that $W = W_e$. On the other hand, if W_e and W_i both have degree $\boldsymbol{d}$, the fields constructed for these two sets are generally not isomorphic, and this is often the case even when $W_e = W_i$. ■

It is shown in [15] that all algebraic fields are $\mathbf{0}''$-categorical (indeed, categorical in any degree which is PA relative to $\mathbf{0}'$), so unlike Theorem 2.1, Theorem 4.2 does not carry over to arbitrarily large arithmetic degrees. Possibly for arbitrary fields, as opposed to algebraic fields, it would carry over. (A field is *algebraic* if it is an algebraic extension of its prime field, either $\mathbb{Q}$ or $\mathbb{Z}_p$.) Indeed, [15] provides the first examples we know of structures with no degree of categoricity: it shows that there exists a computable field with a splitting algorithm which is not computably categorical, and that the categoricity spectrum of such a field must contain degrees $\boldsymbol{d}_0$ and $\boldsymbol{d}_1$ with $\boldsymbol{d}_0 \wedge \boldsymbol{d}_1 = \mathbf{0}$. Subsequently, it builds another computable field whose categoricity spectrum has no least degree and does not contain $\mathbf{0}'$.

5 Marker's Construction

Fix a finite language L with no function symbols, and let $\mathcal{A} = (A, P_0^{n_0}, \ldots, P_m^{n_m})$ be a structure of L. We assume that for every P of this structure the sets P and $A^k \setminus P$ are infinite, where k is the arity of P. For each k-ary predicate P of this structure we define $\exists$- and $\forall$-extensions of P, following the work of Marker in [14].

Marker's $\exists$-extension of P is a $(k+1)$-ary predicate denoted by $P_\exists$ with the following properties. Let X be an infinite set disjoint with A. Then $P_\exists$ satisfies the following conditions:

1. If $P_\exists(a_1, a_2, \ldots, a_k, a_{k+1})$ then $P(a_1, \ldots, a_k)$ and $a_{k+1} \in X$.

2. For every $a_{k+1} \in X$ there exists a unique tuple $(a_1, \ldots, a_k) \in A^k$ such that $P_\exists(a_1, a_2, \ldots, a_k, a_{k+1})$.

3. If $P(a_1, \ldots, a_k)$ then there exists a unique a such that $P_\exists(a_1, a_2, \ldots, a_k, a)$.

Marker's $\forall$-extension of the predicate P is a $(k+1)$-ary predicate $P_\forall$ with the following properties. Let X be an infinite set disjoint with A. Then $P_\forall$ satisfies the following conditions:

1. If $P_\forall(a_1, a_2, \ldots, a_k, a_{k+1})$ then $a_1, \ldots, a_k \in A$ and $a_{k+1} \in X$.

2. For all $(a_1, \ldots, a_k) \in A$ there exists at most one $a_{k+1} \in X$ such that $\neg P_\forall(a_1, a_2, \ldots, a_k, a_{k+1})$.

3. $P(a_1, \ldots, a_k)$ iff for every $a_{k+1} \in X$ we have $P_\forall(a_1, a_2, \ldots, a_k, a_{k+1})$.

4. For every $a_{k+1} \in X$ there exists a unique tuple $(a_1, \ldots, a_k) \in A^k$ such that $\neg P_\forall(a_1, a_2, \ldots, a_k, a_{k+1})$.

The set X in an $\exists$- or $\forall$-extension is called a **fellow of** P.

Definition 5.1 *Let $\mathcal{A} = (A, P_0^{n_0}, \ldots, P_m^{n_m})$ be a structure.*

1. *$\mathcal{A}_\exists$ is a structure $(A \cup X_0 \ldots \cup X_m, P_0^{n_0+1}, \ldots, P_m^{n_m+1}, X_0, \ldots, X_m)$, where each $P_i^{n_i+1}$, $i = 0, \ldots, m$, is a Marker's $\exists$-extension of $P_i^{n_i}$ such that fellows X_i of distinct predicates are pairwise disjoint sets.*

2. *$\mathcal{A}_\forall$ is a structure $(A \cup X_0 \ldots \cup X_m, P_0^{n_0+1}, \ldots, P_m^{n_m+1}, X_0, \ldots, X_m)$, where each $P_i^{n_i+1}$, $i = 0, \ldots, m$, is a Marker's $\forall$-extension of $P_i^{n_i}$ such that fellows X_i of distinct predicates are pairwise disjoint sets.*

We now prove the properties we will need to lower the complexity of a structure while preserving its categoricity spectrum.

Theorem 5.2 *Let $\mathcal{A}_*$ and $\mathcal{B}_*$, for $* \in \{\exists, \forall\}$, be the Marker's extensions of the models $\mathcal{A}$ and $\mathcal{B}$. Then they satisfy the following properties:*

1. *The set A (the domain of the original model $\mathcal{A}$) is definable without quantifiers in the extension $\mathcal{A}_*$.*

2. *Every isomorphism between $\mathcal{A}$ and $\mathcal{B}$ can be extended to an isomorphism between $\mathcal{A}_*$ and $\mathcal{B}_*$, and moreover the extension is unique.*

Proof. Let $X_0, X_1, \ldots, X_m$ be all the fellows needed to define the predicates $P_0^{n_0+1}, \ldots, P_m^{n_m+1}$ in any of Marker's extensions. The predicate $\bigwedge_{j=0}^{m} \neg X_i(x)$ defines the original domain of the structure $\mathcal{A}$. Clearly, in the model $\mathcal{A}_\exists$, the predicate $P_i^{n_i}$ is definable by the formula $\exists x P_i^{n_i+1}(x_1, \ldots, x_{n_i}, x)$. Similarly, the formula $\forall x P_i^{n_i+1}(x_1, \ldots, x_{n_i}, x)$ defines the predicate $P_i^{n_i}$ in the model $\mathcal{A}_\forall$.

Let $f : \mathcal{A} \to \mathcal{B}$ be an isomorphism. We want to extend f to an isomorphism h of $\mathcal{A}_\exists$ and $\mathcal{B}_\exists$. Take an $x \in X^{\mathcal{A}}$, where X is the fellow of a predicate P. Then there exists a unique tuple $(a_1, \ldots, a_k)$ in A so that $P_\exists(a_1, \ldots, a_k, x)$. Note that $\mathcal{A} \models P(a_1, \ldots, a_k) \Rightarrow \mathcal{B} \models P(f(a_1), \ldots, f(a_k))$. By the definition of $P_\exists$, therefore, there is a unique $y \in X^{\mathcal{B}}$ such that $\mathcal{B}_\exists \models P_\exists(f(a_1), \ldots, f(a_k), y)$. Set $h(x) = y$. It is not hard to see that h is an isomorphism from $\mathcal{A}_\exists$ onto $\mathcal{B}_\exists$ that extends f. The isomorphism f can be extended to an isomorphism h' of $\mathcal{A}_\forall$ and $\mathcal{B}_\forall$ in exactly the same manner. ■

Let $\mathcal{A}$ be a structure and w be a word over the alphabet $\{\exists, \forall\}$. We define $\mathcal{A}_w$ by recursion. If w is an empty string then $\mathcal{A}_w = \mathcal{A}$. If $w = w'\exists$ or $w = w'\forall$ and $\mathcal{B} = \mathcal{A}_{w'}$ then $\mathcal{A}_{w'\exists} = \mathcal{B}_\exists$ and $\mathcal{A}_{w'\forall} = \mathcal{B}_\forall$. An easy induction then yields the following corollary:

Corollary 5.3 *Let $\mathcal{A}, \mathcal{B}$ be structures and w be a word over the alphabet $\{\exists, \forall\}$. Then*

1. *The model $\mathcal{A}$ is definable in $\mathcal{A}_w$.*
2. *Every isomorphism between $\mathcal{A}$ and $\mathcal{B}$ can be extended to an isomorphism between $\mathcal{A}_w$ and $\mathcal{B}_w$, and moreover the extension is unique.* ■

Our next goal is to show that $\mathcal{A}_{\exists\forall}$ is less complex than $\mathcal{A}$ itself from a computability-theoretic point of view. The following definition and lemmas can be found in [6]; see also [13]. We will need them for the proof of the main result of this section.

Definition 5.4 *A* one-to-one representation *of a set $A \subseteq \omega$ is a set $Q \subset \omega^3$ satisfying:*

1. *For every $n \in \omega$, $\exists a \forall b Q(n, a, b)$ if and only if $n \in A$;*
2. *For every $n \in \omega$, $\exists a \forall b Q(n, a, b)$ if and only if $\exists^{=1} a \forall b Q(n, a, b)$;*
3. *For every b there exists a unique pair $\langle n, a\rangle$ such that $\neg Q(n, a, b)$;*
4. *For every pair $\langle n, a\rangle$ either $\exists^{=1} b \neg Q(n, a, b)$ or $\forall b Q(n, a, b)$;*
5. *For every a there exists a unique n such that $\forall b Q(n, a, b)$.*

Here $\exists^{=1} x P(x)$ means that there exists a unique x satisfying P.

Lemma 5.5 (see [6].) *Let A be a coinfinite Σ^0_2-set with an infinite computable subset S such that $A \setminus S$ is infinite. Then A has a computable one-to-one-representation.* ■

Of course, if either A is cofinite or $A \setminus S$ is finite, then A is computable (and infinite), in which case it is easy to build a computable one-to-one representation of A. Our construction will use the following relativized version of the lemma.

Lemma 5.6 *Let A be a $\Sigma^{0,X}_2$-set with an infinite computable subset S. Then there exists a set $Q \leq_T X$ which is a one-to-one-representation of A and contains an infinite computable subset of its own.*

Proof. The proof of Lemma 5.5 relativizes, yielding the desired Q, and the infinite computable subset of Q can be $\{\langle n,a,b\rangle : b \in \omega\}$ for any fixed $n \in A$ and the corresponding a from part (1) of Definition 5.4. ■

A $\mathbf{0}^{(m+1)}$-computable structure $\mathcal{A}$ whose predicates $P^{\mathcal{A}}$ satisfy the hypotheses of Lemma 5.6 for $X = \emptyset^{(m)}$, i.e. with all $P^{\mathcal{A}}$ containing a computable infinite subset, will be called a $\mathbf{0}^{(m+1)}$*-acceptable structure.* This is a property of the presentation $\mathcal{A}$, not of its isomorphism type.

Proposition 5.7 *Fix $m \in \omega$. For every $\mathbf{0}^{(m+1)}$-acceptable structure $\mathcal{A}$ of finite signature, the structure $\mathcal{A}_{\exists\forall}$ from Definition 5.1 has a $\mathbf{0}^{(m)}$-acceptable presentation (which we will denote hereafter just by $\mathcal{A}_{\exists\forall}$) such that for all $\mathbf{0}^{(m+1)}$-computable $\mathcal{A}$ and $\mathcal{B}$:*

- *every isomorphism $f : \mathcal{A}_{\exists\forall} \to \mathcal{B}_{\exists\forall}$ restricts to an isomorphism $g : \mathcal{A} \to \mathcal{B}$ with $g \leq_T f \leq_T g \oplus \emptyset^{(m+1)}$; and*
- *every isomorphism $g : \mathcal{A} \to \mathcal{B}$ extends to an isomorphism $f : \mathcal{A}_{\exists\forall} \to \mathcal{B}_{\exists\forall}$ with $g \leq_T f \leq_T g \oplus \emptyset^{(m+1)}$.*

Proof. For each predicate symbol P in the signature of $\mathcal{A}$, say of arity k, we do the following. The signature of $\mathcal{A}_{\exists}$ has a $(k+1)$-ary symbol $P_{\exists}$ and a unary X_P, and the signature of $\mathcal{A}_{\exists\forall}$ has a $(k+2)$-ary symbol $P_{\exists\forall}$, a binary $X_{P\forall}$, and two unary symbols $X_{P_{\exists}}$ and X_{X_P}. We build our copy $\mathcal{C}$ of $\mathcal{A}_{\exists\forall}$ with the original domain $\{a_0, a_1, \ldots\}$ of $\mathcal{A}$ and three disjoint computable copies of ω. One of these copies, with elements $b_0, b_1, \ldots$, constitutes the set $X^{\mathcal{C}}_{P_{\exists}}$; the second, with elements $c_0, c_1, \ldots$, is $X^{\mathcal{C}}_{X_P}$, and the third, with elements $d_0, d_1, \ldots$, is denoted by $X^{\mathcal{C}}_P$, although the symbol X_P is not in the signature. The important symbol is defined in $\mathcal{C}$ by:

$$P^{\mathcal{C}}_{\exists\forall} = \{(a_i, d_j, b_k) \in \mathcal{A}^k \times X^{\mathcal{C}}_P \times X^{\mathcal{C}}_{P_{\exists}}) : (i,j,k) \in Q\},$$

where Q is the $\mathbf{0}^{(m)}$-computable predicate given by Lemma 5.6 for the set $\{n : a_n \in P^{\mathcal{A}}\}$, which must be $\Sigma_2^{0,\emptyset^{(m)}}$ since it is $\mathbf{0}^{(m+1)}$-computable. The acceptability of $\mathcal{A}$ allows us to apply the lemma, and the lemma in turn ensures that $P^{\mathcal{C}}_{\exists\forall}$ contains an infinite computable subset. Finally, $X^{\mathcal{C}}_{P\forall}$ is just the diagonal subset of $X^{\mathcal{C}}_P \times X^{\mathcal{C}}_{X_P}$, i.e. the set $\{(d_n, c_n) : n \in \omega\}$.

It is clear from Definitions 5.4 and 5.1 that $\mathcal{A}_{\exists\forall}$ is isomorphic to the structure $\mathcal{C}$ built by applying this process separately to each of the finitely many symbols of the signature of $\mathcal{A}$. Moreover, $\mathcal{C}$ is $\mathbf{0}^{(m)}$-acceptable. Hereafter we write $\mathcal{A}_{\exists\forall}$ to denote this specific presentation $\mathcal{C}$ of the isomorphism type defined in Definition 5.1.

Now any isomorphism f from this $\mathcal{A}_{\exists\forall}$ onto some $\mathcal{B}_{\exists\forall}$ restricts to an isomorphism g from $\mathcal{A}$ onto $\mathcal{B}$, with $\mathcal{A}$ and $\mathcal{B}$ viewed as the (computable) subsets denoted $\mathcal{A}$ and $\mathcal{B}$ within the domains. Of course, the restriction is computable from the original isomorphism. Conversely, suppose $g : \mathcal{A} \to \mathcal{B}$ is an isomorphism, with $\mathcal{A}$ and $\mathcal{B}$ both $\mathbf{0}^{(m+1)}$-computable. We claim that we can extend g

to an isomorphism f from $\mathcal{A}_{\exists\forall}$ onto $\mathcal{B}_{\exists\forall}$, using only g and a $\mathbf{0}^{(m+1)}$-oracle. All the distinct copies of ω that were put together to form the domain of $\mathcal{A}_{\exists\forall}$ are computable, so we know that each must map onto the corresponding computable subset in $\mathcal{B}_{\exists\forall}$. The extension of g is then defined using the $\mathbf{0}^{(m)}$-computable relations $P_{\exists\forall}^{\mathcal{A}_{\exists\forall}}$ and $P_{\exists\forall}^{\mathcal{B}_{\exists\forall}}$: part (3) of Definition 5.4 allows us to find the (unique possible) image of each element of $X_{P_\exists}^{\mathcal{A}_{\exists\forall}}$, and using part (5) we may likewise determine the unique possible image of each element of $X_P^{\mathcal{A}_{\exists\forall}}$. (For the latter, part (5) requires the entire $\mathbf{0}^{(m+1)}$-oracle, so as to decide universal questions about the $\mathbf{0}^{(m)}$-computable relation $P_{\exists\forall}^{\mathcal{A}_{\exists\forall}}$.) Elements of $X_{P\forall}^{\mathcal{A}_{\exists\forall}}$ are easily mapped into $\mathcal{B}_{\exists\forall}$ since this was defined as the diagonal relation.

Furthermore, the processes described above are inverse to each other, by Corollary 5.3. This yields the Turing reductions named in the Proposition. ■

To continue our analysis of categoricity spectra of computable structures, we now need to refer to Definition 2.2 and show how, for a $\mathbf{0}^{(m+1)}$-computable structure $\mathcal{M}$, we can transfer $\text{CatSpec}_{\mathbf{0}^{(m+1)}}(\mathcal{M})$ and make it the categoricity spectrum of a less complex structure.

Corollary 5.8 *Let $\mathcal{M}$ be any $\mathbf{0}^{(m+1)}$-acceptable structure, and assume that all $\boldsymbol{d} \in \mathit{CatSpec}_{\mathbf{0}^{(m+1)}}(\mathcal{M})$ satisfy $\mathbf{0}^{(m+1)} \leq_T \boldsymbol{d}$. Then*

$$\mathit{CatSpec}_{\mathbf{0}^{(m+1)}}(\mathcal{M}) = \mathit{CatSpec}_{\mathbf{0}^{(m)}}(\mathcal{M}_{\exists\forall}).$$

Proof. Let σ, $\sigma_\exists$ and $\sigma_{\exists\forall}$ be the signatures of $\mathcal{M}$, $\mathcal{M}_\exists$ and $\mathcal{M}_{\exists\forall}$, respectively.

First let $\boldsymbol{d} \in \text{CatSpec}_{\mathbf{0}^{(m)}}(\mathcal{M}_{\exists\forall})$, and take any $\mathbf{0}^{(m+1)}$-computable $\mathcal{A} \cong \mathcal{M}$. Then the $\mathbf{0}^{(m)}$-computable presentations $\mathcal{A}_{\exists\forall}$ and $\mathcal{M}_{\exists\forall}$ given by Proposition 5.7 are $\boldsymbol{d}$-computably isomorphic by assumption, and so $\mathcal{A}$ and $\mathcal{M}$ are $\boldsymbol{d}$-computably isomorphic by the proposition. Thus $\boldsymbol{d} \in \text{CatSpec}_{\mathbf{0}^{(m+1)}}(\mathcal{M})$.

Conversely, let $\boldsymbol{d} \in \text{CatSpec}_{\mathbf{0}^{(m+1)}}(\mathcal{M})$, so that by assumption $\mathbf{0}^{(m+1)} \leq_T \boldsymbol{d}$. Fix any $\mathbf{0}^{(m)}$-computable $\mathcal{B}$ isomorphic to the $\mathcal{M}_{\exists\forall}$ given by the proposition, say via an isomorphism $h : \mathcal{M}_{\exists\forall} \to \mathcal{B}$. Let A be the image $h(\mathcal{M})$, a subset of the domain of $\mathcal{B}$, which is existentially definable within $\mathcal{B}$ because h is an isomorphism and the subset $\mathcal{M}$ is $\exists$-definable within $\mathcal{M}_{\exists\forall}$. ($\mathcal{M}$ is definable in $\mathcal{M}_\exists$ by the conjunction of the negations of the fellows, as noted in Theorem 5.2, and the fellows themselves are then $\forall$-definable in $\mathcal{M}_{\exists\forall}$, so that the conjunction of their negations is $\exists$-definable there.) Next, for any predicate P in σ, define $P^{\mathcal{A}}$ on A to be the image of $P^{\mathcal{M}}$ under h. Now $P^{\mathcal{M}}$ is Δ^0_2-definable over predicates in $\sigma_{\exists\forall}$, and since h is an isomorphism, this means that $P^{\mathcal{A}}$ must be Δ^0_2-definable from the predicates of $\mathcal{B}$. Since $\mathcal{B}$ is $\mathbf{0}^{(m)}$-computable, $\mathcal{A}$ is therefore $\mathbf{0}^{(m+1)}$-computable, and isomorphic to $\mathcal{M}$. By assumption there is a $\boldsymbol{d}$-computable isomorphism f between them.

Now each element $y \in X_P^{\mathcal{M}_\exists}$ of a fellow X_P (in $\sigma_\exists$) of a predicate P from the signature σ is uniquely determined by the tuple $\vec{y}$ which it witnesses to be in $P^{\mathcal{M}}$. By computing $f(y_1), \ldots, f(y_k)$, therefore, we can map any such x to the witness in $\mathcal{B}$ for the corresponding tuple from the subset $\mathcal{A}$ of $\mathcal{B}$. (The predicate X_P is not in the signature $\sigma_{\exists\forall}$, but $X_P^{\mathcal{M}_\exists}$ is definable in $\mathcal{M}_{\exists\forall}$ without quantifiers, and the same definition defines the corresponding set within $\mathcal{B}$. Since $\boldsymbol{d} \geq_T \mathbf{0}^{(m+1)}$, a $\boldsymbol{d}$-oracle allows us to determine these sets.) Thus we extend f to an isomorphism from the subset $\mathcal{M}_{\exists}$ of $\mathcal{M}_{\exists\forall}$ into $\mathcal{B}$.

For elements of $\mathcal{M}_{\exists\forall}$ not in the subset $\mathcal{M}_{\exists}$, a similar procedure holds: each such element is the witness to the failure of a unique predicate in $\mathcal{M}_{\exists}$ to hold for a unique tuple from $\mathcal{M}_{\exists}$, and the witnessing is defined by a predicate in $\sigma_{\exists\forall}$. So, just by considering the fellows in $\sigma_{\exists\forall}$ of the predicates of $\sigma_{\exists}$, we may extend f to an isomorphism g from all of $\mathcal{M}_{\exists\forall}$ into $\mathcal{B}$. By Theorem 5.2, g must have all of $\mathcal{B}$ in its image. Moreover, g is computable from a $\boldsymbol{d}$-oracle, proving that $\boldsymbol{d} \in \text{CatSpec}_{\mathbf{0}^{(m)}}(\mathcal{M}_{\exists\forall})$. ∎

Theorem 5.9 *For any $m \in \omega$, let $\boldsymbol{d}$ be any Turing degree c.e. or even d.c.e. in $\mathbf{0}^{(m)}$. Assume $\boldsymbol{d} \geq_T \mathbf{0}^{(m)}$. Then there is a computable structure $\mathcal{B}$ such that* $\mathbf{d}$ *is the degree of categoricity of $\mathcal{B}$.*

Proof. By taking the join $A \oplus \omega$ if necessary, we may assume that $A \in \boldsymbol{d}$ contains an infinite computable subset. By Corollaries 2.3 and 3.2, there exists a $\mathbf{0}^{(m)}$-acceptable structure $\mathcal{M}$ such that $\text{CatSpec}_{\mathbf{0}^{(m)}}(\mathcal{M})$ is the upper cone above $\boldsymbol{d}$. But now we can simply apply Corollary 5.8 repeatedly until we have a computable structure $\mathcal{B} = \mathcal{M}_{(\exists\forall)\cdots(\exists\forall)}$ with $\text{CatSpec}(\mathcal{B}) = \text{CatSpec}_{\mathbf{0}^{(m)}}(\mathcal{M})$. Moreover, since Corollaries 2.3 and 3.2 both built $\mathbf{0}^{(m)}$-computable structures $\hat{\mathcal{M}} \cong \mathcal{M}$ such that all isomorphisms between them compute $\boldsymbol{d}$, the same must be true of $\mathcal{B}$ and $\hat{\mathcal{M}}_{(\exists\forall)\cdots(\exists\forall)}$, by Proposition 5.7. ∎

6 Non-arithmetical Degrees

We can extend Theorem 5.9 beyond the arithmetical degrees, but only just barely: we know of exactly one non-arithmetical degree which can be the degree of categoricity of a computable structure. Predictably, that degree is $\mathbf{0}^{(\omega)}$.

Theorem 6.1 *There exists a computable structure $\mathcal{A}$ with degree of categoricity $\mathbf{0}^{(\omega)}$.*

Proof. $\mathcal{A}$ is the cardinal sum of the computable structures built in Theorem 5.9 for the degrees $\mathbf{0}^{(n)}$, over all $n \in \omega$. Notice that the construction of Theorem 5.9 can be carried out *uniformly* in n and e, where $W_e^{\emptyset^{(n)}}$ is a set c.e. in $\mathbf{0}^{(n)}$ which computes $\emptyset^{(n)}$. In particular, we can choose e such that $W_e^{\emptyset^{(n)}} = \emptyset^{(n)}$, and build the corresponding structure $\mathcal{A}_n$, with degree of categoricity $\mathbf{0}^{(n)}$, uniformly in n.

The cardinal sum $\mathcal{A}$ of these uniformly presented structures has domain ω, partitioned effectively into countably many infinite subsets, with each subset identified by a unary relation symbol R_n in the language of the structure. The set $R_n = \{\langle n, i\rangle : i \in \omega\}$ then becomes the domain of the structure $\mathcal{A}_n$, built using the second coordinate of each pair in R_n. Since the symbols R_n are in the language, they must be computable in any computable $\mathcal{A}' \cong \mathcal{A}$, allowing us to compute, below a $\mathbf{0}^{(\omega)}$-oracle, the isomorphism between $R_n^{\mathcal{A}}$ and $R_n^{\mathcal{A}'}$. Thus $\mathcal{A}$

is $\mathbf{0}^{(\omega)}$-computably categorical. On the other hand, we can also build, uniformly in n, structures $\mathcal{A}'_n \cong \mathcal{A}_n$ such that the unique isomorphism between $\mathcal{A}_n$ and $\mathcal{A}'_n$ computes $\emptyset^{(n)}$. Now any isomorphism f between $\mathcal{A}$ and the cardinal sum $\mathcal{A}'$ of these $\mathcal{A}'_n$ must restrict to an isomorphism from $\mathcal{A}_n$ onto $\mathcal{A}'_n$, and the resulting computation of $\emptyset^{(n)}$ can be done uniformly in n. Thus the degree of f must be $\geq_T \mathbf{0}^{(\omega)}$, and so $\mathbf{0}^{(\omega)}$ itself must be the degree of categoricity of $\mathcal{A}$. ∎

It remains open whether any other hyperarithmetical degrees can be degrees of categoricity of computable structures. Our reason for giving the full proof of Theorem 2.1 above, despite its subsequent generalization by Theorem 3.1 with a different proof, is our belief that the construction proving Theorem 2.1 will lend itself more readily to degrees above $\mathbf{0}^{(\omega)}$.

To go beyond the hyperarithmetical degrees, we would need different methods from those used here. All degrees of categoricity found in this paper are strong, under the following definition.

Definition 6.2 *If $\boldsymbol{d}$ is the degree of categoricity of a computable structure $\mathcal{A}$, and there exist computable $\mathcal{B}$ and $\mathcal{C}$ isomorphic to $\mathcal{A}$ such that every isomorphism from $\mathcal{B}$ onto $\mathcal{C}$ has Turing degree $\geq_T \boldsymbol{d}$, then we say that $\boldsymbol{d}$ is the* strong degree of categoricity *of $\mathcal{A}$.*

The proof of Theorem 5.9 actually showed that every degree d.c.e. in some $\mathbf{0}^{(n)}$ and above that $\mathbf{0}^{(n)}$ is a strong degree of categoricity, and Theorem 6.1 did the same for the degree $\mathbf{0}^{(\omega)}$. Likewise, the fields built in Theorem 4.2 all had strong degrees of categoricity. It remains open whether any degree can be the degree of categoricity of a computable $\mathcal{A}$ without being the strong degree of categoricity for $\mathcal{A}$ – or better yet, without being the strong degree of categoricity for any computable structure. The next theorem shows that this would be our only hope for finding degrees of categoricity beyond the hyperarithmetical.

Theorem 6.3 *If $\boldsymbol{d}$ is the strong degree of categoricity of a computable structure, then $\boldsymbol{d}$ is hyperarithmetical.*

Proof. The Perfect Set Theorem is the key to this proof. Suppose that $\boldsymbol{d}$ is the degree of categoricity of $\mathcal{A}$, and that there is a computable $\mathcal{B} \cong \mathcal{A}$ such that every isomorphism from $\mathcal{A}$ onto $\mathcal{B}$ computes $\boldsymbol{d}$. Being an isomorphism is arithmetically definable, so the sets in $\boldsymbol{d}$ are precisely the sets S satisfying both of the arithmetical conditions:

- $(\exists e)[\Phi_e^S$ is an isomorphism from $\mathcal{A}$ onto $\mathcal{B}]$; and
- $(\forall i)[$If Φ_i^S is an isomorphism from $\mathcal{A}$ onto $\mathcal{B}$, then $(\exists j) S = \Phi_j^{\Phi_i^S}$.

The Effective Perfect Set Theorem (first proven by Harrison in [10]; see also [16, Thm. 4F.1]) then shows that $\boldsymbol{d}$ must be hyperarithmetical. ∎

7 Questions

While we are proud of the results in this paper, we also admit that they raise more questions than they answer. First of all, we have examined c.e. and d.c.e. degrees, but have gone no further in the Ershov hierarchy than that. We have no proof that it is impossible to go further, but the methods used here do not have an obvious generalization to n-c.e. sets, let alone ω-c.e. sets or Δ^0_2 sets. Can all degrees of such sets be degrees of categoricity of computable structures? A positive answer would presumably relativize to all degrees $\mathbf{0}^{(m)}$, following the methods of Section 5.

Moreover, there is no obvious reason why all categorically definable arithmetical degrees must lie in the intervals $[\mathbf{0}^{(n)}, \mathbf{0}^{(n+1)}]$. A similar situation arose in the paper [9] by Harizanov, Miller, and Morozov, but the result there was finally generalized to degrees outside these intervals by the use of Π^0_1-function singletons. (If $\mathcal{T} \subseteq \omega^{<\omega}$ is a computable subtree with a unique infinite path f, then the degree of f is called a Π^0_1-function singleton.) Possibly the techniques used there might extend to categoricity spectra as well, but the situation is somewhat different. [9] concerns the *automorphism spectrum* of a computable structure $\mathcal{A}$, defined as the set of Turing degrees of all nontrivial automorphisms of $\mathcal{A}$. Thus it is specifically a property of a single presentation of a structure, and indeed [9] gives examples of isomorphic computable structures with distinct automorphism spectra. In contrast, the categoricity spectra of computable structures $\mathcal{A}$ and $\mathcal{B}$ are clearly equal whenever $\mathcal{A} \cong \mathcal{B}$; this is a property of the isomorphism type of $\mathcal{A}$, not of the presentation.

Next, there is the tantalizing possibility suggested by the single known non-arithmetical degree of categoricity $\mathbf{0}^{(\omega)}$ from Theorem 6.1. Can the construction there be extended to higher hyperarithmetical degrees? Or is there a different proof for such degrees? Or could the result actually be false for some degree $\mathbf{0}^{(\alpha)}$ with $\omega < \alpha < \omega_1^{CK}$? The results in [7], which lift many known arithmetical results to hyperarithmetical degrees, might be useful here.

Finally, in Definition 6.2 we defined the *strong degree of categoricity*, without addressing the question of whether there is any other kind of degree of categoricity. That is, could a computable structure $\mathcal{A}$ have a degree of categoricity $\boldsymbol{d}$ such that for every computable $\mathcal{B}$ and $\mathcal{C}$ isomorphic to $\mathcal{A}$, there exists an isomorphism f from $\mathcal{B}$ onto $\mathcal{C}$ with $\boldsymbol{d} \not\leq_T f$? By Theorem 6.3, this would be the only hope for finding a nonhyperarithmetical degree of categoricity.

References

[1] P. Cholak, S.S. Goncharov, B. Khoussainov, & R.A. Shore; Computably categorical structures and expansions by constants, *Journal of Symbolic Logic* **64** (1999), 13–37.

[2] A. Fröhlich & J.C. Shepherdson; Effective procedures in field theory, *Phil. Trans. Royal Soc. London, Series A* **248** (1956) 950, 407–432.

[3] S.S. Goncharov; Problem of the number of non-self-equivalent constructivizations, *Algebra and Logic* **19** (1980), 401–414.

[4] S.S. Goncharov; Nonequivalent constructivizations, *Proc. Math. Inst. Sib. Branch Acad. Sci.* (Novosibirsk: Nauka, 1982).

[5] S.S. Goncharov; Autostable models and algorithmic dimensions, *Handbook of Recursive Mathematics*, vol. 1 (Amsterdam: Elsevier, 1998), 261–287.

[6] S.S. Goncharov & B. Khoussainov; Complexity of categorical theories with computable models, *Algebra and Logic* **43** 6 (2004), 365–373.

[7] S. Goncharov, V. Harizanov, J. Knight, C. McCoy, R. Miller, & R. Solomon; Enumerations in computable structure theory, *Annals of Pure and Applied Logic* **136** 3 (2005), 219–246.

[8] V.S. Harizanov; Pure computable model theory, *Handbook of Recursive Mathematics*, vol. 1 (Amsterdam: Elsevier, 1998), 3–114.

[9] V.S. Harizanov, R. Miller, & A. Morozov; Automorphism spectra of computable structures, to appear.

[10] J. Harrison; Doctoral dissertation, Stanford University (1967).

[11] D.R. Hirschfeldt, B. Khoussainov, R.A. Shore, & A.M. Slinko; Degree spectra and computable dimensions in algebraic structures, *Annals of Pure and Applied Logic* **115** (2002), 71–113.

[12] N. Jacobson; *Basic Algebra I*, Second Edition (New York: W.H. Freeman & Co., 1985).

[13] G. Kreisel, J. Shoenfield, & H. Wang; Number theoretic concepts and recursive well-orderings, *Arch. Math. Logik Grundlagenforsch* **5** (1960), 42–64.

[14] D. Marker; Non-Σ_n-axiomatizable almost strongly minimal theories, *Journal of Symbolic Logic* **54** (1989), 921–927.

[15] R. Miller; $\boldsymbol{d}$-Computable categoricity for algebraic fields, *The Journal of Symbolic Logic* **74** 4 (2009), 1325–1351.

[16] Y.N. Moschovakis; *Descriptive Set Theory*, vol. 100 of *Studies in Logic and the Foundations of Mathematics* (Amsterdam: North-Holland Pub. Co., 1980).

[17] L.J. Richter; Degrees of structures, *Journal of Symbolic Logic* **46** (1981), 723–731.

[18] R.I. Soare; *Recursively Enumerable Sets and Degrees* (New York: Springer-Verlag, 1987).

Kurt Gödel Research Center for Mathematical Logic
University of Vienna
Währinger Strasse 25
A-1090 Wien, Austria
E-mail: efokina@logic.univie.ac.at

N.G. Chebotarev Research Institute of Mechanics and Mathematics
Kazan State University
Universitetskaya St. 17
Kazan 420008, Russia
E-mail: Iskander.Kalimullin@ksu.ru

Department of Mathematics
Queens College – C.U.N.Y.
65-30 Kissena Blvd.
Flushing, New York 11367 U.S.A.
Ph.D. Programs in Computer Science and Mathematics
The Graduate Center of C.U.N.Y.
365 Fifth Avenue
New York, New York 10016 U.S.A.
E-mail: Russell.Miller@qc.cuny.edu