COMPUTABILITY OF FRAÏSSÉ LIMITS

BARBARA F. CSIMA, VALENTINA S. HARIZANOV, RUSSELL MILLER,
AND ANTONIO MONTALBÁN

Abstract. Fraïssé studied countable structures $\mathcal{S}$ through analysis of the *age* of $\mathcal{S}$, i.e., the set of all finitely generated substructures of $\mathcal{S}$. We investigate the effectiveness of his analysis, considering effectively presented lists of finitely generated structures and asking when such a list is the age of a computable structure. We focus particularly on the *Fraïssé limit*. We also show that degree spectra of relations on a sufficiently nice Fraïssé limit are always upward closed unless the relation is definable by a quantifier-free formula. We give some sufficient or necessary conditions for a Fraïssé limit to be spectrally universal. As an application, we prove that the computable atomless Boolean algebra is spectrally universal.

Contents

1. Introduction

Computable model theory studies the algorithmic complexity of countable structures, of their isomorphisms, and of relations on such structures. Since algorithmic properties often depend on data presentation, in computable model theory classically isomorphic structures can have different computability-theoretic properties. One of the main goals of computable model theory is to obtain computability-theoretic versions of various classical model-theoretic notions and results. For example, it is natural to look at the algorithmic analogues of the

The first author was partially supported by Canadian NSERC Discovery Grant 312501.

The second author was partially supported by the NSF grants DMS-0704256 and DMS-0904101.

The third author was partially supported by CUNY Research Foundation grants numbered 60095-34-35, 67182-00-36, 68470-00 37, 69723-00 38, 61467-00 39, and 80209-04-12, and by grant # 13397 from the Templeton Foundation.

The fourth author was partially supported by the NSF grant DMS-0600824.

notion of a structure. These notions have existed at least since the publication of van der Waerden's *Moderne Algebra* [24] in 1930, in which an *explicitly* given field is one in which the elements are uniquely represented by distinguishable symbols with which one can perform the field operations algorithmically. This idea gave rise to the notion of a *computable structure*, a basic concept in computable model theory.

When we say structure, we mean an $\mathcal{L}$-structure, where $\mathcal{L}$ is a fixed computable language. A *computable language* is a countable language for which the set of symbols and their arities are algorithmically presented. Since we consider only countable structures $\mathcal{A}$, we can assume that their domains $\mathrm{dom}(\mathcal{A})$ are initial segments of ω. The *atomic diagram* of a structure $\mathcal{A}$ is the set of all quantifier-free sentences of $\mathcal{L} \cup \{a : a \in \mathrm{dom}(\mathcal{A})\}$. A structure is *computable* if its atomic diagram is computable. For example, there is a computable dense linear ordering without endpoints. There is also a computable random graph. On the other hand, while the standard model of arithmetic is computable, Tennenbaum showed that there is no computable nonstandard model of Peano Arithmetic (see [1]).

Although these definitions may appear static, computability theory studies dynamic processes, and we usually think of computable structures as being built, one element at a time, rather than presented in their entirety all at once. When we study a computable structure $\mathcal{A}$, we can only consider finitely many elements at any given step in our examination of the structure. Each finite subset F of the domain generates a substructure $\mathcal{A}_F \subseteq \mathcal{A}$, and since $\mathcal{A}$ is computable, the generating process is effective: $\mathcal{A}_F$ has a computably enumerable domain, with the same computable functions and relations as $\mathcal{A}$, and thus $\mathcal{A}_F$ is itself computably presentable, just from our knowledge of $\mathcal{A}$ and F. (See the Pullback Lemma 2.2 for details.)

Therefore, it is natural for computable model theory to consider the finitely generated substructures of $\mathcal{A}$. The set of all such substructures was named the *age* of $\mathcal{A}$ and was first studied (in the context of pure model theory) by Fraïssé [7]. Fraïssé studied under which conditions a class of finitely generated structures is the age of some structure. We look at this question in an effective context in Section 2, introducing the notion of a *computable age*. If among all structures with a certain age there is a homogeneous one, it is called the *Fraïssé limit* of the set of finitely generated structures. Fraïssé also studied which conditions are necessary for a class of finitely generated structures to have a Fraïssé limit. We look at this question in an effective context in Section 3. In both of these sections, the interesting cases are when the finitely generated structures might be infinite. When all the structures in the age are finite, all the proofs in [14] are already effective. This will not be the case for results in the following sections. In Section 4, we look at various examples of computable Fraïssé limits and their properties.

It is standard to generalize the definition of computable structure by defining the *Turing degree* of a countable structure with domain ω to be the Turing degree of its atomic diagram. We will denote the Turing degree of $\mathcal{A}$ by $deg(\mathcal{A})$. Hence, $\mathcal{A}$ is computable iff $deg(\mathcal{A}) = \mathbf{0}$, the degree of $\emptyset$. This only refers to the degree of the given presentation of $\mathcal{A}$. However, there could be isomorphic copies of $\mathcal{A}$ with different Turing degrees. The *Turing degree spectrum*, $\mathrm{Spec}(\mathcal{A})$, of a countable structure $\mathcal{A}$ is

$$\mathrm{Spec}(\mathcal{A}) = \{\deg(\mathcal{B}) : \mathcal{B} \cong \mathcal{A}\}.$$

These spectra may have various structural properties within the Turing degrees. For example, Slaman [22] and Wehner [25] independently showed that there is a structure whose spectrum contains precisely the nonzero Turing degrees, and Miller [19] showed that there is a linear ordering the spectrum of which does not contain $\mathbf{0}$, but contains all nonzero Δ^0_2 (that is, limit computable) Turing degrees.

A structure $\mathcal{A}$ is called *automorphically trivial* if its domain contains a finite subset $\{a_0, \ldots, a_{n-1}\}$ such that every permutation f of $\mathrm{dom}(\mathcal{A})$ with $f(a_i) = a_i$ for $i < n$ is an automorphism of $\mathcal{A}$. For example, the complete graph on ω-many vertices is automorphically trivial. On the other hand, a linear ordering is automorphically trivial if and only if it is finite. It is not hard to prove that for an automorphically trivial structure, all isomorphic copies have the same Turing degree. If, in addition, the language of the structure is finite, then that degree is $\mathbf{0}$ (see [12]). The following important theorem is due to Knight [18].

Theorem 1.1 (Knight)**.** *Let $\mathcal{A}$ be a structure that is not automorphically trivial. Then the Turing degree spectrum of $\mathcal{A}$ is upward closed under $\leq_T$. That is, for any two Turing degrees $\mathbf{c} \leq_\mathbf{T} \mathbf{d}$, if $\mathbf{c} \in Spec(\mathcal{A})$, then also $\mathbf{d} \in Spec(\mathcal{A})$.*

In [2], Ash and Nerode considered complexity of an additional relation R on the domain of $\mathcal{A}$, that is, a relation not named in $\mathcal{L}$. More precisely, they investigated syntactic conditions on $\mathcal{A}$ and R under which for every isomorphism f from $\mathcal{A}$ onto a computable model $\mathcal{B}$, $f(R)$ is computable, or computably enumerable (abbreviated by c.e.). This leads to the following general definition.

Definition 1.2. Let $\mathcal{P}$ be a computability theoretic complexity class. An additional relation R on the domain of a computable structure $\mathcal{A}$ is called *intrinsically* $\mathcal{P}$ on $\mathcal{A}$ if the image of R under every isomorphism from $\mathcal{A}$ onto a computable structure belongs to $\mathcal{P}$.

Harizanov defined in [9] the *Turing degree spectrum* of a relation R on a computable structure $\mathcal{A}$, $\mathrm{DgSp}_\mathcal{A}(R)$, as the set of Turing degrees of the images of R under all isomorphisms from $\mathcal{A}$ onto computable structures. It turns out that, unlike spectra of structures, many spectra of relations have upper bounds under Turing reducibility. For example, the adjacency relation on a computable linear order is always $\mathbf{0}'$-computable. However, we show in Section 5 that if $\mathcal{A}$ is a nice (in a sense we will specify) Fraïssé limit, then the degree spectrum $\mathrm{DgSp}_\mathcal{A}(R)$ of any relation R is upward closed in the Turing degrees, unless R can be defined by a quantifier-free formula with parameters in $\mathcal{A}$. In the latter case, $\mathrm{DgSp}_\mathcal{A}(R) = \{\mathbf{0}\}$.

In [12], Harizanov and R. Miller studied connections between these two types of spectrum, defining the notion of spectral universality for computable models of certain theories. Such structures $\mathcal{S}$ exhibit a close connection between the spectra of countable models of the theory and the spectra of the relations on $\mathcal{S}$.

Definition 1.3. A computable structure $\mathcal{S}$ is *spectrally universal* for a theory T if for every automorphically nontrivial countable model $\mathcal{A}$ of T, there is an embedding $f : \mathcal{A} \to \mathcal{S}$ such that $\mathcal{A}$, as a structure, has the same degree spectrum as $f(\mathrm{dom}(\mathcal{A}))$, as a relation on the domain of $\mathcal{S}$.

In particular, Harizanov and Miller studied the computable dense linear ordering and the computable random graph, proving similar results for both.

Theorem 1.4. *([12]) Let $\mathcal{A} = (\omega, \prec)$ be a computable dense linear ordering without endpoints.*
(i) The structure $\mathcal{A}$ is spectrally universal for the theory of linear orderings.
(ii) For any unary relation R on $\mathcal{A}$, the following are equivalent:
(a) The degree spectrum $DgSp_\mathcal{A}(R)$ is upward closed under Turing reducibility;
(b) The relation R is not intrinsically computable;
(c) R cannot be defined by a quantifier-free formula with parameters from $dom(\mathcal{A})$.

(The equivalence of (b) and (c) had already been shown by Moses in [20].)

As a corollary, Harizanov and Miller obtained from Miller's result in [19] that there is a relation R on $\mathcal{A}$ such that $\mathrm{DgSp}_\mathcal{A}(R)$ does not contain $\mathbf{0}$, but contains all nonzero Δ^0_2 degrees.

Theorem 1.5. *([12]) Let $\mathcal{G}$ be a computable random graph.*
(i) The structure $\mathcal{G}$ is spectrally universal for the theory of (symmetric irreflexive) graphs.
(ii) For any unary relation R on $\mathcal{G}$, the following are equivalent:
(a) The degree spectrum $DgSp_{\mathcal{G}}(R)$ is upward closed under Turing reducibility;
(b) The relation R is not intrinsically computable;
(c) R cannot be defined by a quantifier-free formula with parameters from $dom(\mathcal{G})$.

It is natural to ask whether there are spectrally universal structures for other theories. Both the random graph and the countable dense linear ordering are Fraïssé limits, as defined below, for the classes of finite graphs and finite linear orderings, respectively. Harizanov and Miller conjectured that the computable atomless Boolean algebra is spectrally universal for the theory of Boolean algebras. This structure is the Fraïssé limit for the class of finite Boolean algebras. They conjectured further that other computable Fraïssé limits of classes of finite structures might be spectrally universal for their associated theories. In this paper we generalize some of their results to arbitrary Fraïssé limits of classes of finite or finitely generated structures, and provide conditions under which computable Fraïssé limits will or will not satisfy other results from [12].

In Section 6 we provide necessary conditions for a theory to have a spectrally universal model. One is that it has to be locally finite. The other is a restriction on the number of atomic types that are realized by only finitely many different tuples.

In Section 7 we use the result of Section 3.2 to get a sufficient condition for the existence of a spectrally universal model. As an application, we prove the conjecture mentioned above that the computable atomless Boolean algebra is spectrally universal.

1.1. **Classical results about Fraïssé limits and background definitions.** The material in this subsection is from [14, Chapter 6]. Let $\mathcal{D}$ be an $\mathcal{L}$-structure. The *age* of $\mathcal{D}$ is the class of all finitely generated structures that can be embedded in $\mathcal{D}$. More generally, a class of finitely generated structures is called an *age* if it is (up to replacing its members with isomorphic structures) the age of a structure. Fraïssé showed that a (nonempty) finite or countable class $\mathbf{K}$ of finitely generated structures is an age of a finite or a countable structure if and only if $\mathbf{K}$ has the hereditary property and the joint embedding property, as defined below.

Definition 1.6. (see [14]) Let $\mathbf{K}$ be a class of finitely generated structures.

(i) We say that $\mathbf{K}$ has the *hereditary property*, abbreviated by HP, if whenever $\mathcal{A} \in \mathbf{K}$ and $\mathcal{B}$ is a finitely generated substructure of $\mathcal{A}$, then $\mathcal{B}$ is isomorphic to some structure in $\mathbf{K}$.

(ii) We say that $\mathbf{K}$ has the *joint embedding property*, abbreviated by JEP, if for every $\mathcal{A}, \mathcal{B} \in \mathbf{K}$ there is some $\mathcal{C} \in \mathbf{K}$ such that $\mathcal{A}$ and $\mathcal{B}$ embed into $\mathcal{C}$.

Theorem 1.7 (Fraïssé, see [14])**.** *A class of finitely generated structures $\mathbf{K}$ is an age if and only if $\mathbf{K}$ satisfies the HP and JEP properties.*

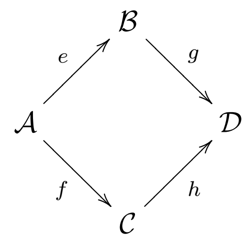
Definition 1.8. A structure $\mathcal{D}$ is *ultrahomogeneous* (the term used in [14, Chapter 6]) if every isomorphism between finitely generated substructures of $\mathcal{D}$ extends to an automorphism of $\mathcal{D}$.

Definition 1.9. Let $\mathbf{K}$ be a class of finitely generated structures. A structure $\mathcal{D}$ is the *Fraïssé limit* of $\mathbf{K}$ if $\mathcal{D}$ is countable, ultrahomogeneous and has age $\mathbf{K}$.

Theorem 1.10 (Fraïssé)**.** *The Fraïssé limit of a class of finitely generated structures is unique up to isomorphism.*

Definition 1.11. A class of finitely generated structures $\mathbf{K}$ satisfies the *amalgamation property*, abbreviated by AP, if whenever $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbf{K}$ and there are embeddings $e : \mathcal{A} \to \mathcal{B}$,

$f : \mathcal{A} \to \mathcal{C}$, then there is a $\mathcal{D} \in \mathbf{K}$ and embeddings $g : \mathcal{B} \to \mathcal{D}$ and $h : \mathcal{C} \to \mathcal{D}$ such that $ge = hf$.



Theorem 1.12. *(Fraïssé, see [14]) A class of finitely generated structures* $\mathbf{K}$ *has a Fraïssé limit if and only if* $\mathbf{K}$ *satisfies HP, JEP and AP.*

Definition 1.13. $\mathrm{Th}_{\mathcal{L}}(\mathbf{K})$ is the class of all sentences ψ of $\mathcal{L}$ such that $\mathcal{A} \models \psi$ for all structures $\mathcal{A}$ in $\mathbf{K}$.

This leads to the following question: If the class $\mathbf{K}$ of all finitely generated models of a theory T forms an age with AP and has computable Fraïssé limit $\mathcal{F}$, when is $\mathcal{F}$ spectrally universal for T?

Recall that a class $\mathcal{K}$ of structures is *locally finite* if every finitely generated structure in $\mathcal{K}$ is finite, and a theory T is said to be *locally finite* if the class of its models is locally finite. A class $\mathcal{K}$ is *uniformly locally finite* if there exists a function $g : \omega \to \omega$ such that every structure in $\mathcal{K}$ which is generated by n elements contains at most $g(n)$ elements, and $\mathcal{K}$ is *computably locally finite* if this function g may be chosen to be computable. For example, the theory of Boolean algebras is computably locally finite via the function $g(n) = 2^{(2^n)}$.

2. Computable Ages

Now we turn to the computable model theoretic aspects of the structures in the previous section. First, we introduce some definitions.

Definition 2.1. Let $\mathbf{K}$ be a set of finitely generated structures in the same language.

1. A *representation* of $\mathbf{K}$ is a sequence $\mathbb{K} = \{(\mathcal{A}_i, \vec{a}_i)\}_{i\in\omega}$, where the domain of each $\mathcal{A}_i$ is a subset of ω, each $\mathcal{A}_i$ is generated by the finite tuple $\vec{a}_i$, and $\mathbf{K} = \{\mathcal{A}_i\}_{i\in\omega}$.
2. We say that this representation is *computable* if the sequence $\{\vec{a}_i\}_{i\in\omega}$ of tuples is computable and the functions, relations, and constants on the structures $\mathcal{A}_i$ are uniformly computable. Notice that this implies that each $\mathcal{A}_i$ has a c.e. domain the index of which is computable in i.
3. We say that $\mathbf{K}$ is *computably representable* if $\mathbf{K}$ has a computable representation. We will say $\mathbb{K}$ is a *computable age* if $\mathbb{K}$ is a computable representation of an age.

Note that the structures $\mathcal{A}_i$ in a computable representation $\mathbb{K}$ are not necessarily computable structures, according to our definition, unless their domains are initial segments of ω. However, the structures $\mathcal{A}_i$ have c.e. domains and computable functions and relations. We call such a structure a *c.e. presentation* of its isomorphism type. The following well-known easy lemma justifies this generalization, in an effective and uniform way.

Lemma 2.2 (Pullback Lemma)**.** *Every c.e. presentation of a structure is isomorphic to a computable copy. Moreover, this copy can be found computably from a c.e. index for the domain of the original structure. Furthermore, the isomorphism from the computable copy to the c.e. one is computable and it can be found uniformly.*

The notion of isomorphisms of classes of structures also has a computable counterpart.

Definition 2.3. Two sets of finite structures $\mathbf{J}$ and $\mathbf{K}$ are *isomorphic* if every structure in $\mathbf{J}$ is isomorphic to one in $\mathbf{K}$, and every structure in $\mathbf{K}$ is isomorphic to one in $\mathbf{J}$. (Two structures in $\mathbf{J}$ may be isomorphic to each other, and likewise in $\mathbf{K}$. Therefore, this definition does allow $\mathbf{J}$ and $\mathbf{K}$ to have different numbers of isomorphic copies of a single structure.)

Now, given two representations $\mathbb{K} = \{(\mathcal{A}_i, \vec{a}_i)\}_{i\in\omega}$ and $\mathbb{J} = \{(\mathcal{B}_i, \vec{b}_i)\}_{i\in\omega}$, we say that $\mathbb{K}$ and $\mathbb{J}$ are *computably isomorphic* if the isomorphisms can be found computably uniformly, or, in other words, if there exists a computable sequence $\{(j_i, \vec{d}_i) : i \in \omega\}$ such that $\vec{d}_i \in \mathcal{B}_{j_i}$ and the map $\vec{a}_i \mapsto \vec{d}_i$ extends to an isomorphism $\mathcal{A}_i \to \mathcal{B}_{j_i}$, and there exists another computable sequence $\{(k_i, \vec{e}_i) : i \in \omega\}$ such that $\vec{e}_i \in \mathcal{A}_{k_i}$ and the map $\vec{b}_i \mapsto \vec{e}_i$ extends to an isomorphism $\mathcal{B}_i \to \mathcal{A}_{k_i}$.

Given a computable structure, its age has one natural computable representation among all its computable presentations.

Definition 2.4. Let $\mathcal{D}$ be a computable structure. Let $\{\vec{a}_i : i \in \omega\}$ be an enumeration of all the tuples from $\mathcal{D}$, and let $\mathcal{A}_i$ be the substructure of $\mathcal{D}$ generated by $\vec{a}_i$. Let $\mathbb{K}_{\mathcal{D}} = \{(\mathcal{A}_i, \vec{a}_i)\}_{i\in\omega}$. We call $\mathbb{K}_{\mathcal{D}}$ *the canonical representation of the age of* $\mathcal{D}$. We say that a representation $\mathbb{K}$ is *a canonical representation of the age of* $\mathcal{D}$ if $\mathbb{K}$ is computably isomorphic to $\mathbb{K}_{\mathcal{D}}$. Sometimes, we will write "canonical age" as an abbreviation for "canonical representation of the age".

In this and the next section we study what properties of a representation $\mathbb{K}$ of a set of finite structures guarantee that:

- $\mathbb{K}$ is a canonical age of some computable structure;
- $\mathbb{K}$ is a canonical age of a Fraïssé limit with various properties;
- $\mathbb{K}$ is the age of a Fraïssé limit.

If $\mathbb{K}$ is a representation of $\mathbf{K}$, we will say $\mathbb{K}$ has a certain property (such as HP, JEP, or being an age) if $\mathbf{K}$ does. We are interested in ages where HP and JEP are computably verifiable.

Before continuing, let us say a few words about maps defined on finitely generated structures. Suppose $(\mathcal{A}, \vec{a})$ is a finitely generated structure and $\mathcal{B}$ is any structure. To specify an embedding $f : (\mathcal{A}, \vec{a}) \to \mathcal{B}$, we need only to specify $f(\vec{a})$, the behavior of f on the generators. We write $f : \vec{a} \to \mathcal{B}$ to denote that f is a function mapping each element of $\vec{a}$ into $\mathcal{B}$, with $f(\vec{a})$ denoting the tuple of images of these elements under f. If such an f extends uniquely to all of $\mathcal{A}$ and defines a 1-1 homomorphism from $\mathcal{A}$ into $\mathcal{B}$, then we say that f *extends to an embedding* $\bar{f} : \mathcal{A} \hookrightarrow \mathcal{B}$.

Definition 2.5. Let $\mathbb{K} = \{(\mathcal{A}_i, \vec{a}_i)\}_{i\in\omega}$ be a computable age.

(1) The age $\mathbb{K}$ has the *computable hereditary property*, abbreviated by CHP, if there is a computable function that given an index i (for $\mathcal{A}_i$) and a tuple $\vec{c} \in \mathcal{A}_i$, returns an index j (for $\mathcal{A}_j$), a tuple $\vec{a} \in \mathcal{A}_j$, and a bijection between the finite tuples $\vec{a}$ and $\vec{c}$, which extends to an embedding of $\mathcal{A}_j$ into $\mathcal{A}_i$.

(2) The age $\mathbb{K}$ has the *computable joint embedding property*, abbreviated by CJEP, if there is a computable function that given indices i and j (for $\mathcal{A}_i$ and $\mathcal{A}_j$), returns an index k (for $\mathcal{A}_k$) and tuples $\vec{b}$ and $\vec{c}$ in $\mathcal{A}_k$ such that the maps $\vec{a}_i \mapsto \vec{b}$ and $\vec{a}_j \mapsto \vec{c}$ extend to embeddings $\mathcal{A}_i \hookrightarrow \mathcal{A}_k$ and $\mathcal{A}_j \hookrightarrow \mathcal{A}_k$.

The canonical age of a computable structure always has these properties.

Lemma 2.6. *Let* $\mathcal{D}$ *be a computable structure. Then* $\mathbb{K}_{\mathcal{D}}$ *has CJEP and CHP.*

Proof. For CJEP, given $\mathcal{A}_i$ and $\mathcal{A}_j$ generated by $\vec{a}_i$ and $\vec{a}_j$, look for k such that $\vec{a}_k \in \mathcal{D}$ contains both $\vec{a}_i$ and $\vec{a}_j$. Then, both $\mathcal{A}_i$ and $\mathcal{A}_j$ embed into $\mathcal{A}_k$ via the inclusion maps. For CHP, given $\mathcal{A}_i$ and $\vec{c} \in \mathcal{A}_i$, look for k such that $\vec{a}_k = \vec{c}$. Then $\mathcal{A}_k$ embeds into $\mathcal{A}_i$ via the inclusion map. □

Observation 2.7. If $\mathbb{K}$ contains only finite structures, then CJEP is equivalent to JEP and CHP is equivalent to HP. The reason is that in this case one can just search for the desired structures.

It turns out that we can get CHP almost for free.

Theorem 2.8. *Let* $\mathbf{K}$ *be a computably representable set of finitely generated structures with HP. Then* $\mathbf{K}$ *has a computable representation* $\mathbb{K}$ *that has CHP.*

Proof. Let $\tilde{\mathbb{K}} = \{(\mathcal{B}_i, \vec{b}_i)\}_{i\in\omega}$ be a computable representation of $\mathbf{K}$. For each i, let $\{\vec{a}_{i,e} : e \in \omega\}$ be a computable list of all the tuples of $\mathcal{B}_i$, uniformly in i. Let $\mathcal{A}_{i,e}$ be the substructure of $\mathcal{B}_i$ generated by $\vec{a}_{i,e}$. Let $\mathbb{K} = \{(\mathcal{A}_{i,e}, \vec{a}_{i,e})\}_{i,e\in\omega}$.

Since $\mathbf{K}$ has HP, it is not hard to see that $\mathbb{K}$ is a representation of $\mathbf{K}$. We claim that $\mathbb{K}$ has CHP. Given $\mathcal{A}_{i,j} \in \mathbb{K}$ and a tuple $\vec{c} \in \mathcal{A}_{i,j} \subseteq \mathcal{B}_i$, there is a k such that $\vec{c} = \vec{a}_{i,k}$. Then $\mathcal{A}_{i,k}$ and the identity map $\vec{c} = \vec{a}_{i,k}$ are as needed for CHP. □

If $\mathbf{K}$ is an age, then, by definition, there is a structure $\mathcal{D}$ such that $\mathbf{K}$ is the age of $\mathcal{D}$. To characterize those $\mathbb{K}$ that are ages of computable structures, we need the following known lemma.

Lemma 2.9. *Let* $\mathcal{D}_0 \hookrightarrow \mathcal{D}_1 \hookrightarrow \cdots$ *be a chain of c.e. presentations of structures, with both the presentations and the embeddings* $\delta_i : \mathcal{D}_i \hookrightarrow \mathcal{D}_{i+1}$ *computable uniformly in* i*. We call this a* computable chain of structures*. Then there exists a computable presentation* $\mathcal{C}$ *of the union of the chain over these embeddings, and embeddings* $\theta_i : \mathcal{D}_i \hookrightarrow \mathcal{C}$ *that are computable uniformly in* i*.*

Proof. Essentially, the usual construction of the direct limit $\mathcal{C}$ works effectively. First, using Lemma 2.2, we can assume that each $\mathcal{D}_i$ is actually a computable structure. Given $i < j$, let $\delta_{i,j} = \delta_{j-1} \circ .... \circ \delta_{i+1} \circ \delta_i \colon \mathcal{D}_i \hookrightarrow \mathcal{D}_j$. Consider the set of pairs $E = \{(d, i) : i \in \omega, d \in \mathcal{D}_i\}$. Now, given $(d_0, i_0), (d_1, i_1) \in E$, we let $(d_0, i_0) \equiv (d_1, i_1)$ if $\delta_{i_0,k}(d_0) = d_{i_1,k}(d_1)$ where $k = \max\{i_0, i_1\}$. Note that $\equiv$ is a computable equivalence relation. Let C be $E/\equiv$ and given $d_i \in \mathcal{D}_i$, let $\theta_i(d_i)$ be the equivalence class of (d_i, i). (We encode $C = E/\equiv$ as a subset of ω by taking the least element of each equivalence class. This set is computable and hence isomorphic to an initial segment of ω.) It is not hard to see that $\mathcal{C}$ is the union of the images of the maps $\theta_i : \mathcal{D}_i \hookrightarrow \mathcal{C}$ and that for every $i < j$, $\theta_i = \theta_j \circ \delta_{i,j}$. The functions and relations on $\mathcal{C}$ are then defined in the obvious way, using the maps θ_i, and, clearly, the structure $\mathcal{C}$ is a presentation of the union of the chain, with computable domain. □

Theorem 2.10. *Let* $\mathbb{K} = \{(\mathcal{A}_i, \vec{a}_i)\}_{i\in\omega}$ *be a computable age. Then* $\mathbb{K}$ *is a canonical age of a computable structure if and only if it has CJEP and CHP.*

Proof. ($\Rightarrow$) Lemma 2.6.

($\Leftarrow$) We construct a uniformly computable chain $\mathcal{A}_{d(0)} \hookrightarrow \mathcal{A}_{d(1)} \hookrightarrow \cdots$ of structures in $\mathbb{K}$. Let $d(0) = 0$ and $d(n+1)$ be the joint embedding of $\mathcal{A}_{n+1}$ and $\mathcal{A}_{d(n)}$, given by the CJEP. Then Lemma 2.9 yields a computable structure $\mathcal{D}$ with age $\mathbb{K}$.

We claim that $\mathbb{K}$ is computably isomorphic to $\mathbb{K}_{\mathcal{D}}$. Given $\mathcal{A}_i \in \mathbb{K}$, we know how $\mathcal{A}_i$ embeds into $\mathcal{A}_{d(i)}$, and how $\mathcal{A}_{d(i)}$ embeds into $\mathcal{D}$. So we can find a tuple in $\mathcal{D}$ which generates a structure isomorphic to $\mathcal{A}_i$. Therefore, we can find a structure in $\mathbb{K}_{\mathcal{D}}$ isomorphic to $\mathcal{A}_i$. Conversely, given a tuple $\vec{c} \in \mathcal{D}$, and hence a structure $D_{\vec{c}}$ in $\mathbb{K}_{\mathcal{D}}$, there is an n such that $\vec{c}$ comes from $\mathcal{A}_{d(n)}$. Since $\mathbb{K}$ has CHP, we can find a structure $\mathcal{A}_i \in \mathbb{K}$ which is isomorphic to the substructure of $\mathcal{A}_{d(n)}$ generated by $\vec{c}$. $\mathcal{A}_i$ is then isomorphic to $\mathcal{D}_{\vec{c}}$, as wanted. □

3. Computable Fraïssé limits

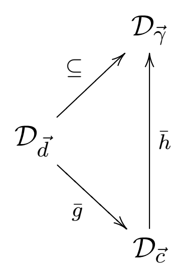
In this section we study what conditions a computable age $\mathbb{K}$ has to satisfy in order to have a computable Fraïssé limit. First we look at some properties of computable Fraïssé limits.

3.1. Computable properties of Fraïssé limits.

We say that a computable structure $\mathcal{F}$ is the Fraïssé limit of $\mathbf{K}$ if $\mathcal{F}$ is homogeneous and has age $\mathbf{K}$.

Definition 3.1. Given a computable structure $\mathcal{D}$, and a tuple $\vec{d} \in \mathcal{D}$, we use $\mathcal{D}_{\vec{d}}$ to denote the substructure of $\mathcal{D}$ generated by $\vec{d}$. We say that $\mathcal{D}$ is *computably homogeneous* if there exists a computable function which, given a tuple $\vec{d} \in \mathcal{D}$, a map $g \colon \vec{d} \to \mathcal{D}$, and $x \in \mathcal{D}$, returns a tuple $\vec{\gamma} \in \mathcal{D}$ and $y \in \mathcal{D}_{\vec{\gamma}}$ such that:

- $\mathcal{D}_{\vec{d}} \subseteq \mathcal{D}_{\vec{\gamma}}$; and
- if $\vec{c} = \langle g(\vec{d}), x\rangle$, and g extends to an embedding $\mathcal{D}_{\vec{d}} \hookrightarrow \mathcal{D}$, then the function $h : \vec{c} \to \mathcal{D}_{\vec{\gamma}}$ with $h(x) = y$ and $h(g(z)) = z$ for $z \in \vec{d}$ extends to an embedding $\mathcal{D}_{\vec{c}} \hookrightarrow \mathcal{D}_{\vec{\gamma}}$.



Even if g does not extend to an embedding, we still require that $\mathcal{D}_{\vec{d}} \subseteq \mathcal{D}_{\vec{\gamma}}$.

If $\mathcal{D}_{\vec{c}}$ is an extension of the image $g(\mathcal{D}_{\vec{d}})$ by an arbitrary finite number of generators, we can still effectively compute some $\vec{\gamma}$ such that $\mathcal{D}_{\vec{\gamma}}$ extends $\mathcal{D}_{\vec{d}}$ and embeds $\mathcal{D}_{\vec{c}}$ so that the diagram commutes, simply by iterating this process over the generators of $\mathcal{D}_{\vec{c}}$ one by one.

Proposition 3.2. *Let $\mathcal{F}_0$ and $\mathcal{F}_1$ be two computably homogeneous computable structures the canonical ages of which are computably isomorphic. Then $\mathcal{F}_0$ and $\mathcal{F}_1$ are computably isomorphic.*

Proof. The classical proof that any two ω-homogeneous structures with the same age are isomorphic actually produces a computable isomorphism if the two structures are computably presented and are computably homogeneous. See Lemmas 6.1.3 and 6.1.4 in [14]. □

We immediately obtain the following corollary. We say that a computably presentable age $\mathbf{K}$ is *computably categorical* if any two computable presentations of it are computably isomorphic. For example, this is always the case when all the structures in $\mathbf{K}$ are finite.

Corollary 3.3. *If $\mathbf{K}$ is computably categorical and all computable Fraïssé limits of $\mathbf{K}$ are computably homogeneous, then the Fraïssé limit of $\mathbf{K}$ is computably categorical.* □

Corollary 3.4. *If $\mathbf{K}$ is locally finite and has a computable Fraïssé limit $\mathcal{F}$, then $\mathcal{F}$ is computably categorical.*

Proof. Since $\mathbf{K}$ is locally finite, all its computable representations, including the canonical age of $\mathcal{F}$, are computably isomorphic. Since $\mathcal{F}$ is locally finite and homogeneous, it is computably homogeneous. Now, by the corollary above, $\mathcal{F}$ is computably categorical. □

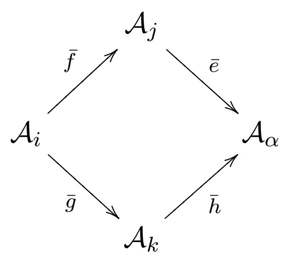
Essentially by the same proof we get the following corollary.

Corollary 3.5. *If $\mathcal{F}$ is computably homogeneous, then every isomorphism between finitely generated substructures of $\mathcal{F}$ extends to a computable automorphism of $\mathcal{F}$.*

Observation 3.6. If $\mathcal{F}$ is a locally finite computable structure and is homogeneous, then it is computably homogeneous. The reason is that we can find $\vec{\gamma}$ and y as in Definition 3.1 just by searching.

3.2. **Existence of computable Fraïssé limits.** Next, we turn from homogeneity to amalgamation. Let $\mathbb{K} = \{(\mathcal{A}_i, \vec{a}_i)\}_{i\in\omega}$ be a computably presented age. As above, we write $f : \vec{a}_i \to \mathcal{A}_j$ to denote that f maps the generators of $\mathcal{A}_i$ into $\mathcal{A}_j$, and we say that f *extends to an embedding* $\bar{f} : \mathcal{A}_i \hookrightarrow \mathcal{A}_j$ if the extension of f to $\mathcal{A}_i$ is uniquely defined and is an injective homomorphism.

Definition 3.7. $\mathbb{K}$ has the *computable amalgamation property*, or *CAP*, if there exists a computable function such that given indices i, j, k for structures $\mathcal{A}_i, \mathcal{A}_j, \mathcal{A}_k$ and maps $f\colon \vec{a}_i \to \mathcal{A}_j$ and $g\colon \vec{a}_i \to \mathcal{A}_k$, returns an index α of a structure $\mathcal{A}_\alpha$ and maps $e\colon \vec{a}_j \to \mathcal{A}_\alpha$ and $h\colon \vec{a}_k \to \mathcal{A}_\alpha$ such that if f and g extend to embeddings $\bar{f}$ and $\bar{g}$ as in the diagram below, then e and h extend to embeddings such that the diagram commutes.

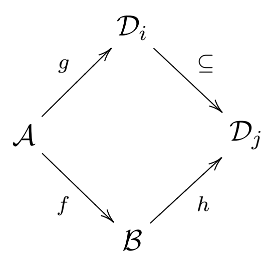


Even if f and g do not extend to embeddings, we ask for e to extend to an embedding $\mathcal{A}_j \hookrightarrow \mathcal{A}_\alpha$.

For locally finite ages $\mathbb{K}$ in finite signatures, CAP is equivalent to AP.

Now we turn to the existence of computable Fraïssé limits. The following lemma gives a way of constructing Fraïssé limits. Its proof can be found in [14].

Lemma 3.8 (How to build a Fraïssé limit)**.** *Let* $\mathbf{K}$ *be an age with AP. Let* $\mathcal{D} = \cup_{i\in\omega}\mathcal{D}_i$*, where* $\{\mathcal{D}_i\}_{i\in\omega}$ *is a chain of structures in* $\mathbf{K}$ *with the property that whenever* $f : \mathcal{A} \to \mathcal{B}$ *is an embedding of structures in* $\mathbf{K}$*, and there is an embedding* $g : \mathcal{A} \to \mathcal{D}_i$ *for some* $i \in \omega$*, then there is some* $j > i$ *and an embedding* $h : \mathcal{B} \to \mathcal{D}_j$ *which extends* f*. Then* $\mathcal{D}$ *is a Fraïssé limit of* $\mathbf{K}$*.*

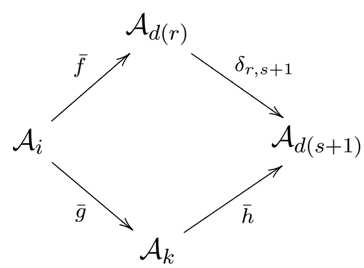


Theorem 3.9. *Let $\mathbb{K}$ be a computable age with CHP and CJEP. Then $\mathbb{K}$ is a canonical age of a computably homogeneous Fraïssé limit if and only if $\mathbb{K}$ has CAP.*

Proof. First, assume that $\mathbb{K}$ has CAP (and CJEP and CHP). To build a computable copy $\mathcal{F}$ of its Fraïssé limit, we build a uniformly computable chain $\mathcal{A}_{d(0)} \hookrightarrow \mathcal{A}_{d(1)} \hookrightarrow \cdots$, with elementary embeddings $\delta_q : \mathcal{A}_{d(q)} \hookrightarrow \mathcal{A}_{d(q+1)}$ also uniformly computable, and apply Lemma 2.9 to get a computable structure $\mathcal{F}$. For $i < j$, we use $\delta_{i,j}$ to denote $\delta_{j-1} \circ \cdots \circ \delta_i \colon \mathcal{A}_i \hookrightarrow \mathcal{A}_j$.

By Lemma 3.8, for $\mathcal{F}$ to be a Fraïssé limit it suffices to meet for each $\langle i, r, k, f, g\rangle$ the following requirement.

$R_{\langle i,r,k,f,g\rangle}$: If $f : \vec{a}_i \to \mathcal{A}_{d(r)}$ and $g : \vec{a}_i \to \mathcal{A}_k$ extend to embeddings $\bar{f}$ and $\bar{g}$, then there exist $s > r$ and an embedding $\bar{h}$ such that the following diagram commutes.

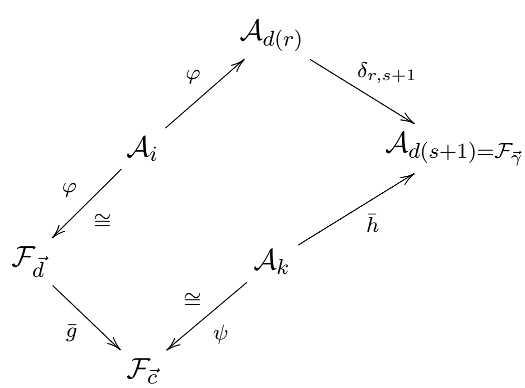


Let $d(0) = 0$, and define $d(s + 1)$ inductively for all s as follows.

Suppose $s = \langle i, r, k, f, g\rangle$, where $i, r, k \in \omega$, $\mathcal{A}_i$ is generated by an n-tuple $\vec{a}_i$ (for some n), and $\mathcal{A}_k$ is generated by an $(n + 1)$-tuple $\vec{a}_k$, with $f : \vec{a}_i \to \mathcal{A}_{d(r)}$ and $g : \vec{a}_i \to \mathcal{A}_k$ from some uniform enumeration of all such maps. We may assume that $r \leq s$. Applying CAP to $\delta_{r,s} \circ f\colon \vec{a}_i \to \mathcal{A}_{d(s)}$ and $g\colon \vec{a}_i \to \mathcal{A}_k$, we get α, e, h such that e extends to an embedding $\mathcal{A}_{d(s)} \hookrightarrow \mathcal{A}_\alpha$. Moreover, if $\delta_{r,s} \circ f$ and g actually extend to embeddings, then $\bar{e} \circ \delta_{r,s} \circ \bar{f} = \bar{h} \circ \bar{g}$. In any case, we let $\mathcal{A}_{d(s+1)} = \mathcal{A}_\alpha$ and $\delta_s = \bar{e}$.

It is not hard to see that our construction satisfies the requirements $R_{\langle i,r,k,f,g\rangle}$.

To see that $\mathcal{F}$ is computably homogeneous, consider $g, \vec{d}, x$ and $\vec{c}$ as in Definition 3.1. We need to compute $\vec{\gamma}$ and y. (The structure now is $\mathcal{F}$, and we write $\mathcal{F}_{\vec{d}}$ for the substructure generated by the tuple $\vec{d}$.) First, find an r such that all coordinates of $\vec{d}$ and $\vec{c}$ lie in $\mathcal{A}_{d(r)}$, and use CHP on $\mathbb{K}$ to find i and k and isomorphisms $\varphi : \mathcal{A}_i \to \mathcal{F}_{\vec{d}}$ and $\psi : \mathcal{A}_k \to \mathcal{F}_{\vec{c}}$, with $\varphi(\vec{a}_i) = \vec{d}$ and $\psi(\vec{a}_k) = \vec{c}$. Set $s = \langle i, r, k, \varphi \restriction \vec{a}_i, \psi^{-1} \circ g \circ \varphi\rangle$. Then at stage $s + 1$ we defined $d(s + 1)$ and δ_s so that the diagram below commutes.



Set $\vec{\gamma} = \vec{a}_{d(s+1)}$. (More precisely, $\vec{\gamma}$ should be the image of $\vec{a}_{d(s+1)}$ in $\mathcal{F}$, as built from the chain $\mathcal{A}_{d(0)} \hookrightarrow \mathcal{A}_{d(1)} \hookrightarrow \cdots$.) Since $s > r$, we clearly have $\mathcal{F}_{\vec{d}} \subseteq \mathcal{A}_{d(r)} \subseteq \mathcal{A}_{d(s+1)} = \mathcal{F}_{\vec{\gamma}}$. Moreover, if g does extend to an embedding $\bar{g} : \mathcal{F}_{\vec{d}} \hookrightarrow \mathcal{F}_{\vec{c}}$, then since the tuple $\langle g(\varphi(\vec{a}_i)), x\rangle$ is precisely $\psi(\vec{a}_j)$, the CAP shows that the embedding h demanded by Definition 3.1 really does exist. Thus $\mathcal{F}$ is computably homogeneous.

For the converse, let $\mathcal{F}$ be a computably homogeneous computable copy of the Fraïssé limit of $\mathbb{K}$. We can assume that $\mathbb{K} = \mathbb{K}_{\mathcal{F}}$ since CAP is preserved under computable isomorphisms of sets of structures. So $\mathbb{K}$ has CHP and CJEP, and indeed each $\mathcal{A}_p$ in $\mathbb{K}$ is just $\mathcal{F}_{\vec{d}}$. To show that $\mathbb{K}$ has CAP, consider i, j, k, f and g as in Definition 3.7. We may assume that $\vec{a}_i$ is an n-tuple and $\vec{a}_k = \langle b_1, \ldots, b_{n+1}\rangle$ is an $(n+1)$-tuple, and $g(\vec{a}_i) = \langle b_1, \ldots, b_n\rangle$.

Let $\vec{d} = f(\vec{a}_i) \in \mathcal{A}_j$. Let $k = g \circ f^{-1} \colon \vec{d} \to \langle b_1, \ldots, b_n\rangle$. Let $x = b_{n+1}$ and $\vec{c} = \vec{d} * x$.

Now the computable homogeneity of $\mathcal{F}$ applied to k, $\vec{d}$ and x gives us $\vec{\gamma}$ and y as in Definition 3.1. Let $\mathcal{F}_\alpha$ be a finitely generated substructure of $\mathcal{F}$ that contains both $\mathcal{F}_{\vec{\gamma}}$ and $\mathcal{A}_j$. Note that $\mathcal{F}_\alpha \in \mathbb{K}$. Let $e \colon \mathcal{A}_j \to \mathcal{F}_\alpha$ be the inclusion map and $h \colon \vec{b} \to \mathcal{F}_{\vec{\gamma}}$ be as in the definition of homogeneity. Note that $\mathcal{F}_\alpha$, e and h are as required for CAP. □

Now we still wish to characterize those computable ages which have computable Fraïssé limits. However we no longer require the Fraïssé limit to be computably homogeneous and we also do not require the age to be a canonical age of the Fraïssé limit. It turns out that a much weaker condition on the age is necessary. This analysis will be useful later when we study spectrally universal structures.

Definition 3.10. We say that a computable age $\mathbb{K}$ has the *computable extension property*, if there is a partial computable function which, given $i \in \omega$ and a quantifier-free formula $\theta(\vec{a}, \vec{x})$ with $\vec{a} \in \mathcal{A}_i$, outputs a structure $(\mathcal{A}_j, \vec{a}_j)$, an embedding $f : \mathcal{A}_i \hookrightarrow \mathcal{A}_j$, and a tuple $\vec{b}$ such that $\mathcal{A}_j \models \theta(f(\vec{a}), \vec{b})$ if such j and $\vec{b}$ exist, and does not halt otherwise. When such $(\mathcal{A}_j, \vec{a}_j)$ and $\vec{b}$ exist we say that $\theta(\vec{a}, \vec{x})$ is *consistent* with $\mathcal{A}_i$ in $\mathbb{K}$.

Remark 3.11. If all structures in $\mathbb{K}$ are finite, then any computable representation of $\mathbb{K}$ has the computable extension property.

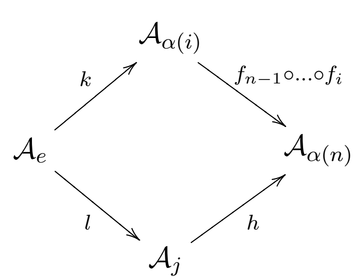
Theorem 3.12. *Let* $\mathbf{K}$ *be an age that satisfies AP. Then* $\mathbf{K}$ *has a computable Fraïssé limit if and only if* $\mathbf{K}$ *has a computable representation which has the computable extension property.*

Proof. First assume that $\mathbf{K}$ has a computable Fraïssé limit $\mathcal{D}$. Let $\mathbb{K}_{\mathcal{D}}$ be the canonical representation of the age of $\mathcal{D}$. We claim that it has the computable extension property. Given $(\mathcal{A}_i, \vec{a}_i) \in \mathbb{K}_{\mathcal{D}}$ and $\theta(\vec{a}, \vec{x})$, if $\vec{a} \in \mathcal{A}_i$, just search for $\vec{b} \in \mathcal{D}$ such that $\mathcal{D} \models \theta(\vec{a}, \vec{b})$. Recall that $\mathcal{A}_i \subseteq \mathcal{D}$. If such a $\vec{b}$ is ever found, return the substructure of $\mathcal{D}$ generated by $\vec{a}_i$ and $\vec{b}$.

For the other direction, let $\mathbb{K} = \{(\mathcal{A}_i, \vec{a}_i)\}$ be a computable representation of $\mathbf{K}$ which has the computable extension property. We will use a finite injury priority construction, and construct a Fraïssé limit $\mathcal{D}$ of $\mathbf{K}$ by finite approximations.

Similarly to Theorem 3.9, we will ensure that $\mathcal{D}$ is a Fraïssé limit by building a chain $\mathcal{A}_{\alpha(0)} \hookrightarrow \mathcal{A}_{\alpha(1)} \hookrightarrow \ldots \hookrightarrow \mathcal{D}$ with embeddings $f_i \colon \mathcal{A}_{\alpha(i)} \hookrightarrow \mathcal{A}_{\alpha(i+1)}$, and $h_i \colon \mathcal{A}_{\alpha(i)} \hookrightarrow \mathcal{D}$ such that $h_i = h_{i+1} \circ f_i$ and $\mathcal{D} = \cup_{i\in\omega} h_i[\mathcal{A}_{\alpha(i)}]$. However, this chain of structures will not be computable, so we cannot use Lemma 2.9 to build $\mathcal{D}$. Instead, we have to build $\mathcal{D}$ separately. We will build $\mathcal{D}$ by finite approximations, satisfying the following requirements.

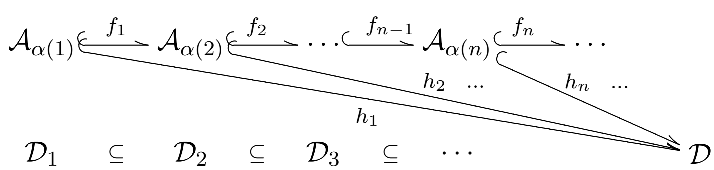
$R_n = R_{\langle e,i,j,k,l\rangle}$: If $l : \mathcal{A}_e \to \mathcal{A}_j$ and $k : \mathcal{A}_e \to \mathcal{A}_{\alpha(i)}$ are embeddings, then there exists an embedding h such that the following diagram commutes. (Here $l : \mathcal{A}_e \to \mathcal{A}_j$ is the l-th map of the form $\vec{a}_e \to \mathcal{A}_j$ in some effective listing of such maps, which may or may not extend to embeddings.)



In Theorem 3.9, using CAP, we were able to immediately meet requirement R_n at stage $n+1$, where $\mathcal{A}_{\alpha(n)}$ was just the amalgamator of $\mathcal{A}_j$ and $\mathcal{A}_{\alpha(n-1)}$, and we knew that $\mathcal{A}_{\alpha(i)} \hookrightarrow \mathcal{A}_{\alpha(n-1)}$. Now, at any stage $s+1$, for each requirement R_n we can only *guess* at an amalgamator of $\mathcal{A}_{\alpha(i)}$ and $\mathcal{A}_j$, and we will build $\mathcal{D}_{s+1}$ one step towards this amalgamator. We will have to be careful about our guesses for the different requirements, to try to have them be consistent.

At stage s of the construction, we will define a number n_s, a sequence $\mathcal{A}_{\alpha(1,s)}$, $\mathcal{A}_{\alpha(2,s)}$,..., $\mathcal{A}_{\alpha(n_s,s)}$, and for each $n < n_s$, a map $f_{n,s}\colon \vec{a}_{\alpha(n,s)} \to \vec{a}_{\alpha(n+1,s)}$, such that at stage s they seem to be a chain of embeddings (i.e., if we look at the elements of $\mathcal{A}_{\alpha(n,s)}$ which are less than s, the extension of $f_{n,s}$ preserves the structure). We might discover later that $f_{n,s}$ does not extend to an embedding $\mathcal{A}_{n,s} \hookrightarrow \mathcal{A}_{n+1,s}$. We will show that for each n, $\alpha(n) = \lim_s \alpha(n,s)$ exists and that $\lim_s n_s = \infty$. At stage s we also define a finite set D_s and assign a truth value to each of the first s atomic formulas $\varphi_i(\vec{d})$ for some enumeration of all the atomic formulas, where $\vec{d}$ are elements of D_s. We also define a partial onto function $h_{n_s,s}\colon \mathcal{A}_{n_s,s} \to D_s$; the values of the atomic values on D_s come from the values of these formulas evaluated on $h^{-1}_{n_s,s}(D_s) \subseteq \mathcal{A}_{n_s,s}$. Let $\theta_s(D_s)$ be the conjunction of all these formulas. We have to make sure that if $t < s$, then $D_t \subseteq D_s$ and $\theta_s(D_s) \Rightarrow \theta_t(D_t)$. This will ensure that $\mathcal{D} = \bigcup_s \mathcal{D}_s$ has a computable atomic diagram, and hence is a computable structure. Here we use $\mathcal{D}_s$ to denote the finite structure defined by the atomic diagram $\theta_s(D_s)$.

Suppose that $\mathcal{A}_i \cap D_s = \vec{b}$. We say that $\mathcal{A}_i$ is *consistent with* $\mathcal{D}_s$ if $\mathcal{A}_i$ is consistent with $\theta_s(\vec{a}, \vec{x})$ as in Definition 3.10. At each step s we always make sure that for every $n \leq n_s$, $\mathcal{A}_{\alpha(n,s)}$ is consistent with $\mathcal{D}_s$, even if $\alpha(n,s)$ is going to change later.



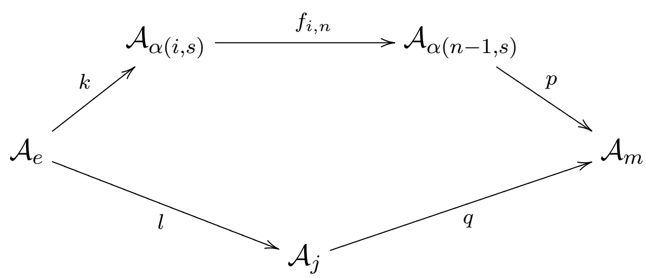
We usually think of the maps f_i and h_i as inclusions. So, the elements of $\mathcal{A}_{\alpha(i)}$ are naturally in $\mathcal{A}_{\alpha(j)}$ for $i < j$, and in $\mathcal{D}$. We will abuse notation and use this convention.

Let us now turn our attention to a requirement R_n with $n = \langle e,i,j,k,l\rangle$. Note that if some $l : \mathcal{A}_e \to \mathcal{A}_j$ or $k : \mathcal{A}_e \to \mathcal{A}_{\alpha(i)}$ is not an embedding, we will find out after a finite amount of time. At any stage s such that we know that $l : \mathcal{A}_e \to \mathcal{A}_j$ or that $k : \mathcal{A}_e \to \mathcal{A}_{\alpha(i,s)}$ is not an embedding, we consider requirement $R_{\langle e,j,k,i\rangle}$ *inactive*. If $\alpha(i,s) = \alpha(i)$ and $k : \mathcal{A}_e \to \mathcal{A}_{\alpha(i,s)}$ is not an embedding, or if $l : \mathcal{A}_e \to \mathcal{A}_j$ is not an embedding, then $R_{\langle e,i,j,k,l\rangle}$ will never be active again, and it will be satisfied.

Our goal is that for each n, if R_n is active infinitely often, then $\alpha(n) = \lim_s \alpha(n,s)$ will exist, and $\mathcal{A}_{\alpha(n)}$ will meet requirement R_n.

As we said before, R_n will guess the amalgamator. It will do it by looking at all the possible amalgamators, one by one. R_n will make a guess and believe it until it sees it is not correct, and only then will it move to the next guess. To keep track of these guesses, we will use an auxiliary function $\beta(n,s)$. For each $n = \langle e,i,j,k,l\rangle$ that we consider at stage $s+1$, we will have $\beta(n,s+1) = \langle m,p,q\rangle$ *guess* that $p : \mathcal{A}_{\alpha(n-1,s)} \to \mathcal{A}_m$ and $q : \mathcal{A}_j \to \mathcal{A}_m$ are embeddings such that the following diagram commutes.

(1)



We will have that $\mathcal{A}_{\alpha(n,s+1)}$ is an extension of $\mathcal{A}_m$ consistent with $\mathcal{D}_s$. So at stage $s+1$ we will *guess* that $\mathcal{A}_{\alpha(0,s+1)} \hookrightarrow \ldots \hookrightarrow \mathcal{A}_{\alpha(n,s+1)}$, we will *know* that each $\mathcal{A}_{\alpha(i,s)}$ is consistent with $\mathcal{D}_s$, and we will extend $\mathcal{D}_{s+1}$ to a substructure of $\mathcal{A}_{\alpha(n,s+1)}$. We do this since we *guess* that each $\mathcal{A}_{\alpha(i,s+1)}$ will meet requirement R_i.

Note that if we assume that $\mathcal{A}_{\alpha(i)} = \lim_s \mathcal{A}_{\alpha(i,s)}$ exists for all $i < n$, then requirement R_n either becomes inactive forever, or else, since $\mathbb{K}$ has AP, there is an amalgamator of $\mathcal{A}_{\alpha(n-1)}$ and $\mathcal{A}_j$, which we will find after finitely many wrong guesses.

We say that requirement R_n *requires attention* at stage $s+1$ if R_n is active and either $\beta(n,s)$ is undefined, or $\beta(n,s) = \langle m,p,q \rangle$ is defined, but we have just found out that $\mathcal{A}_m$ does not amalgamate $\mathcal{A}_{\alpha(n-1,s)}$ and $\mathcal{A}_j$ via p and q.

Construction:

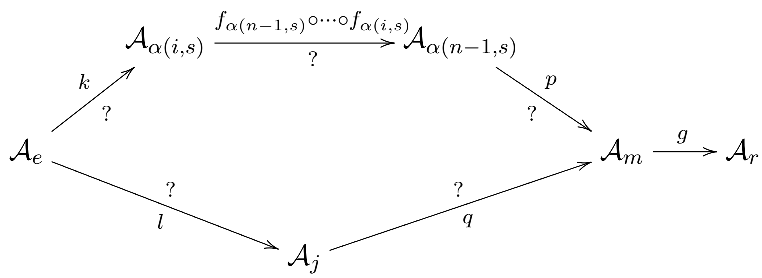
Since the age $\mathbb{K} = \{(\mathcal{A}_i, \vec{a}_i)\}_{i\in\omega}$ is given in such a way that the generators and the functions and relations are uniformly computable, we view the finitely generated structures as being revealed to us stage by stage, by applying at each step every function (there are finitely many) to what we have observed in the domain so far, and then looking at the relations on this new part of the domain.

Stage 0*:* Let $\alpha(0,0) = 0$, $\beta(0,0) = 0$ and $\mathcal{D}_0$ consist of $\vec{a}_0$ in $\mathcal{A}_0$, and $\theta_0(D_0)$ be the empty conjunction that is always true.

Stage $s+1$*:* Let n be least such that R_n requires attention at stage $s+1$. Let $\langle m,p,q \rangle$ be least such that at stage $s+1$, $p : \mathcal{A}_{\alpha(n-1,s)} \to \mathcal{A}_m$ and $q : \mathcal{A}_j \to \mathcal{A}_m$ appear to be embeddings such that (2) holds. Let $\vec{a}$ be that portion of $\mathcal{D}_s$ that sits naturally inside of $\mathcal{A}_{\alpha(n-1,s)}$. By induction hypothesis, $\theta_s(\vec{a},\vec{x})$ is consistent with $\mathcal{A}_{\alpha(n-1,s)}$. It follows from the fact that $\mathbb{K}$ has AP that if $p : \mathcal{A}_{\alpha(n-1,s)} \to \mathcal{A}_m$ is really an embedding then $\theta(p(\vec{a}),\vec{x})$ is consistent with $\mathcal{A}_m$. Now if $\theta(p(\vec{a}),\vec{x})$ is consistent with $\mathcal{A}_m$, then the computable extension property will give us $g : \mathcal{A}_m \to \mathcal{A}_r$, and a tuple $\vec{b}$ such that $\mathcal{A}_r \models \theta(g(p(\vec{a})),\vec{b})$. So, for $\langle m,p,q \rangle$, we run the computable extension property with $\mathcal{A}_m$ and $\theta(p(\vec{a}),\vec{x})$, and simultaneously continue to check whether $p : \mathcal{A}_{\alpha(n-1,s)} \to \mathcal{A}_m$ is an embedding, until either we are given $\mathcal{A}_r$, or we find out that $p : \mathcal{A}_{\alpha(n-1,s)} \to \mathcal{A}_m$ is not an embedding. In the latter case we increment $\langle m,p,q \rangle$ by 1, and try again. Since $\mathbb{K}$ has AP, this process must halt, and we will be left with some $\mathcal{A}_m$ that *might* be an amalgamator of $\mathcal{A}_{\alpha(n-1,s)}$ and $\mathcal{A}_j$, and some $\mathcal{A}_r$, together with an embedding $g : \mathcal{A}_m \to \mathcal{A}_r$, and a tuple $\vec{b}$ such that $\mathcal{A}_r \models \theta(g(p(\vec{a})),\vec{b})$. In the following diagram the "?"

indicates that we do not know whether the map is an embedding.

(2)

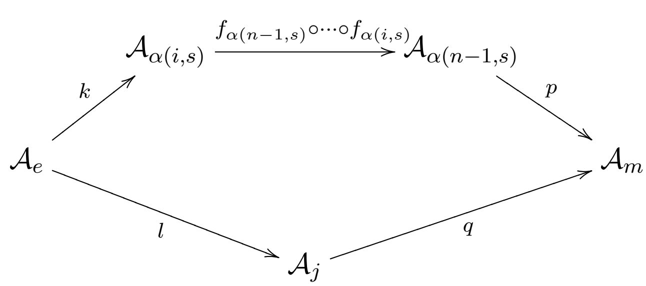


Set $g(p(\mathcal{D}_s))$ to be the image of $\mathcal{D}_s$ in $\mathcal{A}_r$. Let $\mathcal{C}$ be the (partial) substructure of $\mathcal{A}_r$ resulting from applying all the functions and constants with index less than s once to $g(p(\mathcal{D}_s))$ and also adding the atomic diagram of $\vec{a}_r$. We would like to let $\mathcal{D}_{s+1} = \mathcal{C}$. However, we must first make sure that such a definition would be consistent with all $\mathcal{A}_{\alpha(l,s)}$ for $l < n$. Note that if it is true that $\mathcal{A}_{\alpha(0,s)} \hookrightarrow \mathcal{A}_{\alpha(1,s)} \hookrightarrow \dots \hookrightarrow \mathcal{A}_{\alpha(n-1,s)} \hookrightarrow \mathcal{A}_r$, then $\mathcal{C}$ would be consistent with all of them, since it is a substructure of $\mathcal{A}_r$. For each $0 < l < n$, in turn, use the computable extension property to check whether $\mathcal{C}$ is consistent with $\mathcal{A}_{\alpha(l,s)}$ (that is, let θ be the diagram of $\mathcal{C}$ in $\mathcal{A}_r$, let $\vec{a} = \mathcal{A}_{\alpha(l,s)} \cap h^{-1}_{l,s}[\mathcal{D}_s]$, and run the computable extension property on $\mathcal{A}_{\alpha(l,s)}$ and $\theta(\vec{a}, \vec{x})$; if this halts then $\mathcal{C}$ is consistent with $\mathcal{A}_{\alpha(l,s)}$), and at the same time search to see if we find out that $f_{\alpha(l-1,s)} : \mathcal{A}_{\alpha(l-1,s)} \to \mathcal{A}_{\alpha(l,s)}$ is not an embedding. If $\mathcal{C}$ is actually consistent with all $\mathcal{A}_{\alpha(l,s)}$, then all the functions given by the computable extension property will halt and we will know it. In this case, we define $\mathcal{D}_{s+1} = \mathcal{C}$, $\alpha(l, s+1) = \alpha(l, s)$ for $l < n$, $\beta(l, s+1) = \beta(l, s)$ for $l < n$, $\beta(n, s+1) = \langle m, p, q\rangle$, $\mathcal{A}_{\alpha(n,s+1)} = \mathcal{A}_r$, and $f_{\alpha(n-1,s+1)} = p \circ g$. Otherwise, we will find that for some $l < n$, $f_{\alpha(l-1,s)} : \mathcal{A}_{\alpha(l-1,s)} \to \mathcal{A}_{\alpha(l,s)}$ is not an embedding. In this case, we let $\mathcal{D}_{s+1} = \mathcal{D}_s$, $\alpha(k, s+1) = \alpha(k, s)$ for $k < l$, $\beta(k, s+1) = \beta(k, s)$ for $k < l$, and leave $\alpha(k, s+1)$ and $\beta(k, s+1)$ undefined for all $k \geq l$.

Lemma 3.13. *For all n, $\alpha(n) = \lim_s \alpha(n, s)$ and $\beta(n) = \lim_s \beta(n, s)$ exist, R_n requires attention at most finitely often, and either the hypotheses in R_n fail, or $\mathcal{A}_{\alpha(n)}$ meets requirement R_n.*

Proof. We proceed by induction on n. The statement clearly holds for $n = 0$. Suppose it holds for all $n' < n$. Let $n = \langle e, i, j, k, l\rangle$. By induction hypothesis, $\alpha(i)$ exists. So if the hypotheses for R_n fail, then we will find out after a finite amount of time, and so there will be a point after which for all s, $\alpha(n, s) = \alpha(n-1, s)$, $\beta(n, s)$ is undefined, and R_n does not require attention at stage s. Suppose the hypotheses for R_n hold. Then since $\mathbb{K}$ has AP, there is some $\langle m, p, q\rangle$ such that the following diagram commutes.

(3)



Let s' be least such that $\alpha(n', s) = \alpha(n')$ for all $n' < n$ and all $s \geq s'$. Let $\langle m, p, q\rangle$ be least such that (3) holds. Then for each $\langle m', p', q'\rangle < \langle m, p, q\rangle$ there is some stage $t > s'$ at which we know that (3) does not hold for $\langle m', p', q'\rangle$ in place of $\langle m, p, q\rangle$. Hence there is a least stage t' such that for all $s \geq t'$, $\beta(n, s) = \langle m, p, q\rangle$. Thus $\mathcal{A}_{\alpha(n,t')}$ will be an extension of $\mathcal{A}_m$ that is consistent with $\mathcal{D}_{t'}$. Since $\mathcal{A}_{\alpha(0)} \preccurlyeq ... \preccurlyeq \mathcal{A}_{\alpha(n-1)} \preccurlyeq \mathcal{A}_{\alpha(n,t')}$, we will have $\alpha(n,s) = \alpha(n,t')$ for all $s \geq t'$. At stage t' we will ensure that all generators of $\mathcal{A}_{\alpha(n)}$ are included in $\mathcal{D}_{t'}$. At any stage $s > t'$ where we extend $\mathcal{D}_s$, we build it to be a substructure of an extension of $\mathcal{A}_{\alpha(n)}$, with one more application of functions than at the previous stage. Hence we will have $\mathcal{A}_{\alpha(n)} \preccurlyeq \mathcal{D}$. This proves the lemma. □

Note that $\mathcal{D}$ is actually a structure since we have closed it under all the functions (operations), one at a time. We have that $\mathcal{D} = \cup_{i\in\omega} h_i[\mathcal{A}_{\alpha(i)}]$ because all the generators of the $\mathcal{A}_{\alpha(n)}$ are eventually included in some $\mathcal{D}_s$. □

4. Examples

This section is devoted to examples of Fraïssé limits for various classes **K** of finitely generated structures. We will focus on several properties of the Fraïssé limits: computable representability, ω-categoricity, computable categoricity, and spectral universality. The classes we consider are linear orderings, graphs, Boolean algebras, p-groups, algebraic extensions of $\mathbb{Z}_p$, $\mathbb{Z}$-modules, and various types of classes with a successor function. Some of these classes contain precisely the finitely generated models of a particular first-order theory T that may or may not be c.e. or finitely axiomatizable; for others there is no such theory.

Table 1 summarizes much of the information we get in this section, showing the classes **K** for which we have computable Fraïssé limits, along with their properties.

$\mathbf{K}_\omega$	T	$\mathcal{F}$	**K** ULF	Th($\mathcal{F}$) ω-cat.	**K** LF	$\mathcal{F}$ c.c.	$\mathcal{F}$ s.u.
linear orderings	Yes	dense LO	Yes	Yes	Yes	Yes	Yes
graphs	Yes	random graph	Yes	Yes	Yes	Yes	Yes
Boolean algs.	Yes	atomless BA	Yes	Yes	Yes	Yes	Yes
successor S	Yes	$(\cup_{n>0}\mathbb{Z}_n)^\omega \cup \mathbb{Z}^\omega$	No	No	No	No	No
S, loops only	No	$(\cup_{n>0}\mathbb{Z}_n)^\omega$	No	No	Yes	Yes	Yes
S, no loops	Yes	$\mathbb{Z}^\omega$	No	No	No	No	No
$\mathbb{Z}$-modules	Yes	$(\mathbb{Q}/\mathbb{Z} \sqcup \mathbb{Q})^{<\omega}$	No	No	No	No	No
torsion $\mathbb{Z}$-mod.	No	$(\mathbb{Q}/\mathbb{Z})^{<\omega}$	No	No	Yes	Yes	Yes
torsion-free $\mathbb{Z}$-mod.	Yes	$\mathbb{Q}^{<\omega}$	No	No	No	No	No
p-groups	No	$(\mathbb{Z}_{p^\infty})^{<\omega}$	No	No	Yes	Yes	Yes
algebraic exts. of $\mathbb{Z}_p$	No	$\overline{\mathbb{Z}_p}$	No	No	Yes	Yes	No

Table 1. In each row, $\mathbf{K}_\omega$ is a class of countable models and $\mathcal{F}$ is the Fraïssé limit of the subclass **K** containing the finitely generated structures in $\mathbf{K}_\omega$. The column labeled "T" tells us whether $\mathbf{K}_\omega$ is the class of countable models of a first-order theory. LF stands for locally finite, ULF stands for uniformly LF, ω-cat. for ω-categorical, c.c. for computably categorical and s.u. for spectrally universal for $\mathbf{K}_\omega$.

Two well-known examples are presented in [12]: the countable dense linear ordering and the random graph. These are similar in many ways. Each is the Fraïssé limit for the class **K** of finite models of a finitely $\forall$-axiomatizable theory (the theories of linear orderings and graphs, of course) in a relational language. Hence both of these classes $\mathbf{K}$ are uniformly locally finite. Both Fraïssé limits are computably categorical, and their theories are both ω-categorical. Both are spectrally universal, as proven in [12].

The countable atomless Boolean algebra is the Fraïssé limit of the class of finite Boolean algebras, and satisfies all the same conditions as the previous two, except that the language of Boolean algebras is not relational. However, n elements generate a Boolean algebra of size at most 2^{2^n}, so the theory is still uniformly locally finite, and it will be shown in Section 7.1 to be spectrally universal.

Now consider a language with a single unary function symbol S, and with an axiom saying that S is one-to-one, so that we may think of S as a successor function. If we adjoin more axioms to say that there are no finite loops (i.e., $S^n(x) \neq x$, for each $n > 0$), then the resulting theory is c.e. but not finitely axiomatizable, and the class $\mathbf{K}_\infty$ of finitely generated models of this theory contains precisely the finite unions of ω-chains. This class of structures has a Fraïssé limit $\mathcal{F}_\infty$, which is the disjoint union of countably many $\mathbb{Z}$-chains, which we denote by $\mathbb{Z}^\omega$, and so $\mathcal{F}_\infty$ is computably representable. However, this theory is not locally finite, let alone uniformly so, and, moreover, $\mathcal{F}_\infty$ is not spectrally universal by Lemma 6.3. It is also not computably categorical, because the relation "x and y are in the same $\mathbb{Z}$-chain" can be computable or noncomputable in different computable copies of $\mathbb{Z}^\omega$. Also, the theory of $\mathcal{F}_\infty$ is not ω-categorical, since any finite union of $\mathbb{Z}$ chains is elementarily equivalent to $\mathcal{F}_\infty$.

We can also form the class $\mathbf{K_0}$ of models of a successor function in which every element lies on a (finite) loop. By compactness this is not the class of finitely-generated models of any set of sentences in this language, but it is locally finite (although not uniformly) and does have a Fraïssé limit $\mathcal{F}_0$, consisting of countably many loops of size n for each $n > 0$. We often write $\mathcal{F}_0$ as $(\sqcup_{n>0}\mathbb{Z}_n)^\omega$. The theory $\mathrm{Th}(\mathcal{F}_0)$ is not ω-categorical, but $\mathcal{F}_0$ is computably representable, computably categorical, and spectrally universal for the class of countable models with no $\mathbb{Z}$-chains.

Finally, in the language of successor we can allow both ω-chains and finite loops. This gives a finitely axiomatizable theory, with a non-locally-finite class $\mathbf{K}$ of finitely generated models, the Fraïssé limit of which, $\mathcal{F}$, is the disjoint union of $\mathcal{F}_0$ and $\mathcal{F}_\infty$ above. As with $\mathcal{F}_\infty$, $\mathcal{F}$ is neither computably categorical nor spectrally universal, and its theory is not ω-categorical.

Abelian groups act very much in the same way as structures in the language of successor. Again the basic theory is finitely axiomatizable, with a non-locally-finite class $\mathbf{K}$ of finitely generated models, and there is one subclass containing the torsion-free abelian groups and another one containing the torsion abelian groups, which behave just as the corresponding subclasses did in the language of successor. The Fraïssé limit of the torsion abelian groups is the additive group $(\mathbb{Q}/\mathbb{Z})^{<\omega}$, the algebraic direct product of countably many copies of $(\mathbb{Q}/\mathbb{Z})$. The torsion-free abelian groups have Fraïssé limit $\mathbb{Q}^{<\omega}$, and the Fraïssé limit of the class of all abelian groups is the disjoint union of these two. (For us, the *algebraic direct product* of countably many abelian groups contains those countable sequences in which cofinitely many elements are the identity element 0.) These Fraïssé limits have exactly the same properties named above for $\mathcal{F}_0$, $\mathcal{F}_\infty$ and $\mathcal{F}$, respectively.

A p-group is an abelian group in which every element has order a power of the prime p. The Fraïssé limit of the class of finitely generated p-groups is often written as $(\mathbb{Z}_{p^\infty})^{<\omega}$, the algebraic direct product of countably many copies of the Prüfer p-group. This Fraïssé limit is quite similar to $\mathcal{F}_0$ above.

Algebraic fields of characteristic p are discussed in Section 6.2, since they are a natural example of a locally finite class of models the Fraïssé limit of which is not spectrally universal. For us a field is *algebraic* if it is an algebraic extension of its prime field. We must require our fields to be algebraic extensions of $\mathbb{Z}_p$ in order for the class of finitely generated models to be locally finite. In this case the Fraïssé limit is the algebraic closure of $\mathbb{Z}_p$, which is computably categorical but not spectrally universal, and its theory is not ω-categorical. If we allowed transcendental extensions, then the Fraïssé limit would be the algebraic closure of the function field $\mathbb{Z}_p(X_0, X_1, \ldots)$, which is still not computably categorical, nor spectrally universal for the class of countable fields of characteristic p, and whose theory is not ω-categorical.

Table 1 suggests several implications, some already known and some which we now prove.

Proposition 4.1. *Let* $\mathbf{K}$ *be the age of a Fraïssé limit* $\mathcal{F}$*, in a finite language. Then the following hold.*

1. *The age* $\mathbf{K}$ *is uniformly locally finite iff* $Th(\mathcal{F})$ *is* ω*-categorical and has quantifier elimination.*
2. *If* $\mathcal{F}$ *is computable and spectrally universal for a class* $\mathbf{K}_\omega$ *such that* $\mathbf{K}$ *is the set of all finitely generated structures in* $\mathbf{K}_\omega$*, then* $\mathbf{K}$ *is locally finite.*
3. *If* $\mathbf{K}$ *is locally finite, then* $\mathcal{F}$ *is relatively computably categorical.*

Proof. Part (1) appears in [14, Corollary 6.4.2]. Part (2) is a restatement of Lemma 6.3. For (3), suppose that $\tilde{\mathcal{F}}$ is a structure with domain ω isomorphic to $\mathcal{F}$. By ultrahomogeneity, every finite partial automorphism $f : \mathcal{A} \to \tilde{\mathcal{A}}$ (with $\mathcal{A} \subset \mathcal{F}$ and $\tilde{\mathcal{A}} \subset \tilde{\mathcal{F}}$) extends to one more element $a \in \mathcal{F}$. But $\mathcal{A}$ and a together generate a finite substructure $\mathcal{B} \subset \mathcal{F}$, by local finiteness, and we may identify all elements of $\mathcal{B}$ and search through $\tilde{\mathcal{F}}$ (using an $\tilde{\mathcal{F}}$-oracle) until we find a corresponding substructure $\tilde{\mathcal{B}}$ extending $\tilde{\mathcal{A}}$, and then extend f to map all of $\mathcal{B}$ onto $\tilde{\mathcal{B}}$. The same works for any $\tilde{a} \in \tilde{\mathcal{F}}$, so by going back and forth, we may build an isomorphism, computable in $\tilde{\mathcal{F}}$, from $\mathcal{F}$ onto $\tilde{\mathcal{F}}$. □

Counterexamples to the converse of (3) exist, although none appear in Table 1. For instance, let $\mathcal{F}$ be any of the computably categorical Fraïssé limits from the table (all of which are also relatively computably categorical), and let $\mathcal{F}'$ be the disjoint union of $\mathcal{F}$ with a $\mathbb{Z}$-chain, with the language augmented by one new unary relation symbol R, which holds of all elements on the $\mathbb{Z}$-chain, and one new function symbol S which names the successor of an element on the $\mathbb{Z}$-chain and is the identity function on $\overline{R}$. Relative computable categoricity allows us to build an isomorphism from the complement $\overline{R}$ in $\mathcal{F}$ onto $\overline{R}$ in any copy of $\mathcal{F}$, and pairing up the elements of the $\mathbb{Z}$-chains in two copies of $\mathcal{F}$ is trivial using R and S, so there exists an isomorphism computable in the degrees of the two structures. Thus, $\mathcal{F}'$ is relatively computably categorical. However, $\mathcal{F}'$ is the Fraïssé limit of the class $\mathbf{K}'$ of disjoint unions (defined by R), where the R-part is an ω-chain under S, and the $\overline{R}$-part is an element of the class $\mathbf{K}$ corresponding to $\mathcal{F}$. This class $\mathbf{K}'$ is not locally finite, so we have the desired counterexample to the converse of (3).

Next, we ask how a Fraïssé limit could fail to be computably representable. Theorems 3.9 and 3.12 provide useful conditions, but the simplest examples arise from producing unusual theories related to those already described in this section.

For instance, in the language of the successor function S, let T be the theory whose axioms are:

$$S(x) = S(y) \Rightarrow x = y$$

And for each $n \in \emptyset'$, the axiom

$$(\forall x) S^n(x) \neq x.$$

Thus if $\mathcal{A}$ is a model of this theory, then each member of $\mathcal{A}$ belongs either to a $\mathbb{Z}$-chain or to a loop, but there can be no loops of size n with $n \in \emptyset'$. Here $\emptyset'$ represents the jump of the empty set, which is c.e., so this T is c.e. Clearly, the finitely generated models of T satisify HP, JEP, and AP, so T has a Fraïssé limit $\mathcal{F}$, consisting of countably many $\mathbb{Z}$-chains and countably many loops of length n for each $n > 1$ such that $n \notin \emptyset'$. However, from any copy of

$\mathcal{F}$ one could enumerate the complement of $\emptyset'$, so the least degree of any representation of $\mathcal{F}$ is $\mathbf{0}'$. Indeed, the age of $\mathcal{F}$ is not computably representable, for the same reason. One could do the same with any other noncomputable c.e. set in place of $\emptyset'$, of course, and get this result for any Σ_1 degree above $\mathbf{0}$.

5. Upward closure of degree spectra of relations

Knight proved that the degree spectra of any non-trivial structures is upward closed (see Theorem 1.1). This result does not hold for degree spectra of relations. For example, if one considers the successor relation on a computable linear ordering, this relation is always co-c.e., and therefore its degree spectrum is not upward closed in the Turing degrees. However, we show that in the case where the structure is a sufficiently nice Fraïssé limit, the degree spectrum of any "non-trivial" relation on the structure is upward closed.

Theorem 5.1. *Let $\mathcal{F}$ be a computable structure over a finite language $\mathcal{L}$, which is computably locally finite, homogeneous, and the existential diagram of which is computable. Let U be a unary relation on $\mathcal{F}$. Then the following are equivalent.*

1. *The degree spectrum of U on $\mathcal{F}$ is not upward closed under Turing reducibility.*
2. *The relation U is definable by a quantifier-free formula with parameters in $\mathcal{F}$.*
3. *The relation U is intrinsically computable.*

Corollary 5.2. *Let $\mathbf{K}$ be a class of finite structures over a finite language $\mathcal{L}$ such that $Th_{\mathcal{L}}(\mathbf{K})$ is computably axiomatizable and locally finite, and with computable Fraïssé limit $\mathcal{F}$. Then $\mathcal{F}$ is as in Theorem 5.1.*

Proof. Since $\mathbf{K}$ is an age, $\mathbf{K}$ contains all finite models of $\mathrm{Th}_{\mathcal{L}}(\mathbf{K})$. So $\mathbf{K}$ is computably locally finite by Lemma 6.2, and hence so is $\mathcal{F}$. The structure $\mathcal{F}$ is homogeneous because it is a Fraïssé limit. Finally, given an existential formula about $\mathcal{F}$, we search to either find that the formula holds in $\mathcal{F}$, or else we wait for $\mathrm{Th}_{\mathcal{L}}(\mathbf{K})$ to prove that this existential formula does not hold in any finitely generated substructure of $\mathcal{F}$. So, the existential theory of $\mathcal{F}$ is decidable. □

Proof of Theorem 5.1. (2) $\Rightarrow$ (3) $\Rightarrow$ (1) is immediate, so we show that if (2) fails, then $\mathrm{DgSp}_{\mathcal{F}}(U)$ is upward closed. Suppose U is a unary relation on $\mathcal{F}$, which is not quantifier-free definable with parameters. Suppose also that $(\mathcal{F}, S) \cong (\mathcal{F}, U)$, where $S \in \mathbf{d}$. Let $\mathbf{c} > \mathbf{d}$, and let $C \in \mathbf{c}$. We will build an automorphism $g : \mathcal{F} \to \mathcal{F}$ such that $C \leq_T g(S)$ and $g \leq_T C$, which will ensure that $g(S) \equiv_T C$, as desired.

We build the automorphism by stages. At stage s, we will define g_s on a finite domain D_s and range R_s. The R_s will be uniformly computable in s, while the D_s will only be $C-$uniformly computable. At each stage s, we have to make sure that g_s extends to an isomorphism from the finite structure generated by D_s to the one generated by R_s. By homogeneity, this implies that g_s extends to an automorphism of $\mathcal{F}$. So, for each s, R_s and D_s must have the same quantifier-free type. When we refer to "types" in this proof, we mean "quantifier-free type" (this abuse of terminology is justified because $\mathcal{F}$ has quantifier elimination [14, Theorem 6.4.1]).

Construction:

Stage 0*:* Let $g_0 = D_0 = R_0 = \emptyset$.

Stage $s + 1 = 2e + 1$*:* Say $R_s = \{y_1, \ldots, y_k\}$. For an arbitrary x, since $\mathcal{F}$ is uniformly locally finite and the language is finite, there are only finitely many candidates for the atomic diagram of $\{y_1, y_2, \ldots, y_k, x\}$. This is because there is a bound $g(k + 1)$ on the size of the structure generated by $\{y_1, y_2, \ldots, y_k, x\}$, and since the language is finite, there are finitely many possible ways for the functions, relations and constants of $\mathcal{L}$ to be defined on those at most $g(k + 1)$ many elements. Moreover, since $\mathcal{F}$ is computably locally finite, we can list these candidates, and since the existential theory of $\mathcal{F}$ is decidable, we know which of these candidates actually appear in $\mathcal{F}$. Hence we can partition the domain of $\mathcal{F}$ into finitely many computable pieces $Z_1, \ldots, Z_n$, where x and z are in the same Z_i exactly if they have the same 1-type over $\{y_1, \ldots, y_k\}$. Note that n can be found computably, and that each Z_i is quantifier-free definable over R_s. Now $D_s = \{g_s^{-1}(y_1), \ldots, g_s^{-1}(y_k)\}$. Let $B_1, \ldots, B_n$ partition the domain of $\mathcal{F}$ into distinct 1-types over D_s. Note that the B_i are uniformly C-computable.

For $1 \leq i \leq n$, let $x_{i,1}$ be the least member of Z_i not in R_s. Choose $x_{i,2}, ..., x_{i,k_i}$ from Z_i such that $\{x_{i,l}\}_{2 \leq l \leq k_i}$ represent all possible 1-types over $R_s \cup \{x_{i,1}\}$. As was the case for n, each k_i can be found computably. Let $R_{s+1} = R_s \cup \{x_{i,j} \mid 1 \leq i \leq n, 1 \leq j \leq k_i\}$. We now use C to define the extension of the domain D_{s+1} and the map g_{s+1} such that $D_{s+1} \supset D_s$ and $g_{s+1} \supset g_s$.

We claim that for some $l \leq n$, both $B_l \cap S \neq \emptyset$ and $B_l \cap \overline{S} \neq \emptyset$. If there were no such l, then for each B_i, either $B_i \subseteq \overline{S}$ or $B_i \subseteq S$, so that S could be expressed as the union

$$S = \cup_{B_i \subseteq S} B_i,$$

since the sets $B_1, \ldots, B_n$ form a partition of the domain of $\mathcal{F}$. But each B_i is definable without quantifiers over the finite parameter set D_s, so S would then be definable over D_s by the disjunction of the definitions of those sets B_i with $B_i \subseteq S$. By assumption S is not so definable, and hence we may fix the least l with both $B_l \cap S \neq \emptyset$ and $B_l \cap \overline{S} \neq \emptyset$. Let $a \in B_l \cap S$ and $b \in B_l \cap \overline{S}$ (or we could let $a \in B_l \cap \overline{S}$ and $b \in B_l \cap S$, depending on the coding). Since $a \in B_l$, mapping a to $x_{l,1}$ would certainly extend the partial isomorphism g_s. Since among $x_{l,2}, ..., x_{l,k_l}$ there are representatives of all possible 1-types over $R_s \cup \{x_{l,1}\}$, there must be some $x_{l,m}$ such that mapping b to $x_{l,m}$ would extend the partial isomorphism. Since $\mathcal{F}$ is homogeneous, there exists a further extension of this partial isomorphism to all of R_{s+1}. Such a partial isomorphism would have the property that for each $j < l$, the preimages of $x_{j,r}$ are all on the "same side of S" (that is, for each $j < l$, either $(\forall r \leq k_j)[x_{j,r} \in S]$ or $(\forall r \leq k_j)[x_{j,r} \in \overline{S}]$.)

To code C into $g(S)$, we will search for a configuration that allows us to define g as follows. There is some l as described above such that for $j < l$, we can have $g^{-1}(x_{j,r})$ all on the "same side of S". If $e \in C$, we can have $g^{-1}(x_{l,1}) \in B_l \cap S$ and $g^{-1}(x_{l,m}) \in B_l \cap \overline{S}$ for some m. If $e \notin C$ we can have $g^{-1}(x_{l,1}) \in B_l \cap \overline{S}$ and $g^{-1}(x_{l,m}) \in B_l \cap S$ for some m.

So to define g_{s+1}, use the C oracle to examine all possibilities $a_{1,1}, ..., a_{n,k_n}$ until we find one that codes what we want, and define g_{s+1} accordingly.

Stage $s + 1 = 2e + 2$: As before, partition the domain of $\mathcal{F}$ into finitely many computable pieces $Z_1, ..., Z_n$, where x and z are in the same Z_i exactly if they have the same 1-type over $\{y_1, ..., y_k\}$. For $1 \leq i \leq n$, let x_i be the least member of Z_i not in R_s. Let $R_{s+1} = R_s \cup \{x_i \mid 1 \leq i \leq n\}$.

Use C to compute the least a not in D_s. Then $\text{type}(a, D_s) = \text{type}(x_i, R_s)$ for some i, and this is C-computable. We will map $g(a) = x_i$. Since $\mathcal{F}$ is homogeneous, there exist $b_j \notin D_s$ such that extending g_s by mapping $a \to x_i$ and $b_j \to x_j, j \neq i$, is an isomorphism. So we C-computably search for such b_j, defining D_{s+1} and g_{s+1} accordingly.

Verification:

The construction certainly gives $g \leq_T C$, and that the R_s are uniformly computable in s (without a C-oracle). At odd stages we ensure that g is onto, and at even stages we ensure that it is total. At each stage s, g_s is a partial isomorphism, so $g : \mathcal{F} \to \mathcal{F}$ is an automorphism.

It remains to show that $C \leq_T g(S)$. To check whether $e \in C$, compute $R_{2e+1} - R_{2e} = \{x_{i,j} \mid 1 \leq i \leq n, 1 \leq j \leq k_i\}$. Find the least l such that there exists m with $(x_{l,1} \in g(S) \wedge x_{l,m} \notin g(S)) \vee (x_{l,1} \notin g(S) \wedge x_{l,m} \in g(S))$. Such an l exists by construction. Then $e \in C$ iff $x_{l,1} \in g(S)$. □

6. Necessary conditions for spectral universality

In this section we examine when the computable Fraïssé limit $\mathcal{F}$ of a class $\mathbf{K}$ will be spectrally universal, under the assumption that such an $\mathcal{F}$ exists. We state two conditions which are necessary in order to have spectral universality. These conditions make it easy to disprove spectral universality for some Fraïssé limits, as we did in Section 4.

In [12], spectral universality was defined only for theories. Here we extend the definition to a more general case.

Definition 6.1. Let $\mathbf{K}_\omega$ be a class of countable structures. We say that a computable structure $\mathcal{S} \in \mathbf{K}_\omega$ is *spectrally universal for* $\mathbf{K}_\omega$ if for every nontrivial $\mathcal{A} \in \mathbf{K}_\omega$ there exists an embedding $f : \mathcal{A} \hookrightarrow \mathcal{S}$ which preserves the spectrum (i.e., the spectrum $\mathrm{Spec}(\mathcal{A})$ of the structure $\mathcal{A}$ is equal to the spectrum $\mathrm{DgSp}_{\mathcal{S}}(f(\mathcal{A}))$ of the image of $\mathcal{A}$ as a relation on $\mathcal{S}$).

Trivial structures are excluded from consideration in Definition 6.1 because by Theorem 1.1, their spectra are always singletons, hence not very interesting. Notice that if, in Definition 6.1, $\mathbf{K}_\omega$ is the set of countable models of a theory T, then this definition is equivalent to the original definition in [12] of a spectrally universal structure for T.

6.1. Local finiteness.

The first lemma shows how in some cases "computably locally finite" comes for free. Then we show that local finiteness is a necessary condition for spectral universality.

Lemma 6.2. *If T is a locally finite theory in a finite language, then the class $\mathbf{K}$ of finite models of T is uniformly locally finite. If, in addition, T is computably axiomatizable, then T is computably locally finite.*

Proof. For any $m, n > 0$, we have a formula $\theta_n^m(x_1, \dots, x_n)$ which says that $x_1, \dots, x_n$ generate a structure containing at least m distinct elements. (θ_n^m just lists all terms built by applying at most m function symbols to $x_1, \dots, x_n$, and the constant symbols, and states that at least m of these terms are distinct. It is important here that the language be finite, of course.) If T were not uniformly locally finite, then for some fixed n and all m, the sentence

$$(\forall x_1) \cdots (\forall x_n) \neg\theta_n^m(x_1, \dots, x_n)$$

would not be provable from T. Then a compactness argument on new constants $c_1, \dots, c_n$ shows that $T \cup \{\theta_n^m(\vec{c}) : m \in \omega\}$ is consistent, so it is possible for an infinite model of T to be generated by n elements, which contradicts the local finiteness of T. Thus T must be uniformly locally finite.

If, in addition, T is computably axiomatizable, then we can compute $g(n)$, a bound on the maximum number of elements in a structure generated by n elements, as follows. Since the theory T is computably axiomatizable, the set of formulas that are consequences of T is c.e. To compute $g(n)$, we just enumerate the consequences of T until we find an m such that $T \models (\forall x_1 \cdots \forall x_n) \neg\theta_n^{m+1}(x_1, \dots, x_n)$. Since T is uniformly locally finite, we will eventually find an m with this property, and we let $g(n) = m$. □

Lemma 6.3. *If $\mathbf{K}_\omega$ is not a locally finite class, then there is no spectrally universal structure for $\mathbf{K}_\omega$.*

Proof. We fix any computable structure $\mathcal{S}$, and show that $\mathcal{S}$ is not spectrally universal for $\mathbf{K}_\omega$. Let $\mathcal{A}$ be a structure in $\mathbf{K}_\omega$, which is infinite but generated by a finite set $\{a_1, \dots, a_n\}$. Then the image $f(\mathcal{A})$ of $\mathcal{A}$ in $\mathcal{S}$ under any embedding f is the closure in $\mathcal{S}$ of the set $\{f(a_1), \dots, f(a_n)\}$ under the functions named in the language. Hence $f(\mathcal{A})$ is intrinsically c.e. However, $\mathcal{A}$ itself must have an upward closed spectrum, by Theorem 1.1. ($\mathcal{A}$ cannot be automorphically trivial: it is infinite, yet every automorphism of $\mathcal{A}$ is determined by its values on the generating set.) So f did not preserve the spectrum of $\mathcal{A}$. □

6.2. **Finite realizability.** For the purpose of this section, all types will be complete atomic types, i.e., maximally consistent sets of atomic formulas and negated atomic formulas in a fixed finite set of variables (usually $x_1, \ldots, x_n$, sometimes written $\vec{x}$), and we will usually omit the word "atomic." Having proven Lemma 6.3, we will only need to deal with locally finite theories and classes, in finite languages. In this situation an atomic type is always generated by a finite subset of its formulas, and the finite subset can be recognized once it appears, so we may think of atomic types as finite objects. If we allow parameters from a structure in our types, we will so specify. The list of parameters will also normally be finite, usually $a_1, \ldots, a_n$ or $\vec{a}$.

A more interesting example where spectral universality fails is provided by the class $\mathbf{K_p}$ of all fields of a fixed characteristic $p > 0$ which are algebraic over their prime field $\mathbb{Z}_p$. The class $\mathbf{K_p}$ is a locally finite class, but it is not uniformly locally finite, as a single primitive n-th root of unity generates a structure in $\mathbf{K_p}$ of power $\geq n$. The elements of $\mathbf{K_p}$ may be characterized as the finite algebraic field extensions of $\mathbb{Z}_p$. It is not possible to express algebraicity of extensions of $\mathbb{Z}_p$ by any set of sentences in the language of fields, so this $\mathbf{K_p}$ is not the class of countable models of any first-order theory (as was already clear from Lemma 6.3). Nevertheless, the class of algebraic field extensions of $\mathbb{Z}_p$ is well-known and of definite interest. The Fraïssé limit $\mathcal{F}_p$ for the subclass of finite extensions of $\mathbb{Z}_p$ is computably presentable: it is the algebraic closure of the field $\mathbb{Z}_p$.

It is not difficult to see that $\mathcal{F}_p$ fails to be spectrally universal for $\mathbf{K_p}$. Let $\mathcal{A}$ be the computable field of characteristic p which we get by starting with $\mathbb{Z}_p$ and adjoining a primitive p_n-th root of unity whenever any $n > p$ enters the halting set $\emptyset'$. (Here p_n represents the n-th prime number, so $p_n > p$ when $n > p$.) Clearly, $\mathcal{A}$ is an algebraic extension of $\mathbb{Z}_p$. However, any isomorphic image R of $\mathcal{A}$ within $\mathcal{F}_p$ would allow us to compute $\emptyset'$, since for $n > p$ we would have $n \in \emptyset'$ iff any (hence all) of the $(p_n - 1)$-many primitive p_n-th roots of unity in $\mathcal{F}$ belongs to R. Thus, no image of $\mathcal{A}$ in a computable copy of $\mathcal{F}_p$ can have the degree $\mathbf{0}$ in its spectrum, yet $\mathcal{A}$ itself is computably presentable. Therefore, $\mathcal{F}_p$ is not spectrally universal for $\mathbf{K_p}$, and the same would hold for any other field into which all elements of $\mathbf{K_p}$ embed, since such a field must realize all possible roots of unity.

The shortcoming of this Fraïssé limit $\mathcal{F}_p$ has to do with its having types (in this case, the types Γ_n of a primitive p_n-th root of unity, for all n) which are realized only finitely often. By using p_n instead of n, we ensured that we could realize or omit any subset of $\{\Gamma_n : n > 1\}$ without forcing any other Γ_m either to be realized or to be omitted. Thus, we may say that these types are independently realizable. However, Theorem 6.5 will provide a simpler (and more general) proof that $\mathcal{F}_p$ is not spectrally universal, without using any notion of independence.

Definition 6.4. An atomic n-type Δ (without parameters) is *finitely realizable in* $\mathbf{K}$ if there exists an $m > 0$ in ω such that:

- no $\mathcal{A} \in \mathbf{K}$ contains more than m distinct n-tuples realizing Δ; and
- some $\mathcal{A} \in \mathbf{K}$ contains m distinct n-tuples realizing Δ.

This m must be unique, so we also call Δ *m-realizable in* $\mathbf{K}$. If $\mathbf{K}$ is the set of finitely generated models (or equivalently, countable models) of a theory T, then we say that Δ is a finitely realizable type under T.

Theorem 6.5. *Suppose $\mathbf{K}_\omega$ is a class of countable structures, closed under taking substructures, and $\mathbf{K} \subseteq \mathbf{K}_\omega$ is the subclass of finitely generated structures in $\mathbf{K}_\omega$. Assume that there* *are infinitely many atomic types (without parameters) which are finitely realizable in* $\mathbf{K}$*. Then no computable structure* $\mathcal{S}$ *in* $\mathbf{K}_\omega$ *can be spectrally universal for* $\mathbf{K}_\omega$*.*

Proof. Suppose $\mathcal{S}$ were spectrally universal for $\mathbf{K}_\omega$. By its universality, $\mathcal{S}$ must realize every m-realizable atomic type Γ at least m times. Also, for any such Γ, if $\mathcal{S}$ realized Γ more than m times, then $\mathcal{S}$ would have a finitely generated substructure $\mathcal{B}$ that realized Γ more than m times, and, by hypothesis, this $\mathcal{B}$ would also be in $\mathbf{K}_\omega$, contradicting m-realizability of Γ. Hence Γ must be realized exactly m times in $\mathcal{S}$.

Let $\mathcal{A}$ be the substructure of $\mathcal{S}$ generated by all elements of tuples in $\mathcal{S}$ which realize finitely realizable types. Then the only substructure of $\mathcal{S}$ isomorphic to $\mathcal{A}$ is $\mathcal{A}$ itself, so every automorphism of $\mathcal{S}$ must map $\mathcal{A}$ onto itself. By assumption, $\mathcal{A} \in \mathbf{K}_\omega$, so by spectral universality we have $\mathrm{Spec}(\mathcal{A}) = \mathrm{DgSp}_{\mathcal{S}}(\mathcal{A}) = \{\deg(\mathcal{A})\}$, since the image of $\mathcal{A}$ in $\mathcal{S}$ can only be $\mathcal{A}$ itself. Theorem 1.1 then implies that $\mathcal{A}$ is trivial, hence finite. (We may assume $\mathbf{K}_\omega$ to be locally finite, by Lemma 6.3, so if $\mathcal{A}$ were infinite, it would have infinitely many generators, which would have to realize infinitely many distinct types, contradicting triviality.) Hence T has only finitely many finitely realizable types. □

Thus, we could have shown that the algebraic closure $\mathcal{F}_p$ of $\mathbb{Z}_p$ was not spectrally universal for algebraic extensions of $\mathbb{Z}_p$, just by considering embeddings of $\mathcal{F}_p$ into itself.

7. A sufficient condition for spectral universality

In this section we give a sufficient condition for certain Fraïssé limits to be spectrally universal for countable structures of the same age. We will observe that this condition holds in the already known examples of spectrally universal structures. We will also show that the condition is true for the class of finite Boolean algebras, establishing that the countable atomless Boolean algebra is spectrally universal for countable Boolean algebras. This answers a question from [12].

Let $\mathbf{K}$ be an age with AP, let $\mathcal{F}$ be the Fraïssé limit of $\mathbf{K}$, and let $\mathcal{A}$ be a countable model with age included in $\mathbf{K}$. To show that $\mathcal{F}$ is spectrally universal, we will need to find a unary relation R in $\mathcal{F}$, which is isomorphic to $\mathcal{A}$ and has the same spectrum within $\mathcal{F}$ as the spectrum of $\mathcal{A}$. Before stating the condition we want to impose on $\mathcal{A}$ to ensure that such a relation exists, we need the following definition.

Definition 7.1. Let $\mathcal{L}^{\mathcal{A}}$ be the language $\mathcal{L}$ augmented with constant symbols $\mathtt{c}^a$, one for each $a \in A$, and with a unary relation $\mathtt{A}$. Now we consider the class of structures finitely generated over $\mathcal{A}$. Define $\mathbf{F}^{\mathcal{A}}$ to be the set of $\mathcal{L}^{\mathcal{A}}$-structures $\mathcal{B}$ such that:

- the $\mathcal{L}$-reduct of $\mathcal{B}$ is a model of T;
- for $x \in B$, we have that $\mathcal{B} \models \mathtt{A}(x)$ if and only if $\mathcal{B} \models x = \mathtt{c}^a$ for some $a \in A$;
- the map $a \mapsto c^a \colon \mathcal{A} \to \mathcal{B}$ is an $\mathcal{L}$-embedding;
- $\mathcal{B}$ is finitely generated, so, as $\mathcal{L}$-structures, $\mathcal{B}$ is finitely generated over $\mathcal{A}$.

Theorem 7.2. *Let* $\mathbf{K}$ *be a class of finite structures over a finite language* $\mathcal{L}$*, with a computable Fraïssé limit* $\mathcal{F}$*, such that* $Th_{\mathcal{L}}(\mathbf{K})$ *is computably axiomatizable and locally finite. Suppose that for every countable model* $\mathcal{A}$ *with age* $\mathbf{K}$*, there exists an age* $\mathbf{K}^{\mathcal{A}} \subseteq \mathbf{F}^{\mathcal{A}}$ *of* $\mathcal{L}^{\mathcal{A}}$*-structures such that:*

1. $\mathbf{K}^{\mathcal{A}}$ *has HP, JEP and AP;*
2. *if* $\mathcal{C} \subseteq \mathcal{D}$ *are finite models of* T *and* $\mathcal{C} \subseteq \mathcal{A}$*, then there is* $\mathcal{B}$ *in* $\mathbf{K}^{\mathcal{A}}$ *that is an exact amalgamation of* $\mathcal{D}$ *and* $\mathcal{A}$ *over* $\mathcal{C}$ *(i.e., the intersection of the images of* $\mathcal{A}$ *and* $\mathcal{D}$ *in* $\mathcal{B}$ *equals the image of* $\mathcal{C}$*);*
3. *for any presentation* $\tilde{\mathcal{A}}$ *of* $\mathcal{A}$*, there is an* $\tilde{\mathcal{A}}$*-computable representation of* $\mathbf{K}^{\mathcal{A}}$ *with the* $\tilde{\mathcal{A}}$*-computable extension property.*

Then $\mathcal{F}$ is spectrally universal for countable models with age $\mathbf{K}$*.*

Proof. Consider $\mathcal{A}$, a countable model with age $\mathbf{K}$, and let $\mathbf{K}^{\mathcal{A}}$ be given by the hypothesis of the theorem. Since $\mathbf{K}^{\mathcal{A}}$ is an age with AP, $\mathbf{K}^{\mathcal{A}}$ has a Fraïssé limit; call it $\mathcal{F}^{\mathcal{A}}$. We observe that the $\mathcal{L}$-reduct of $\mathcal{F}^{\mathcal{A}}$ is the Fraïssé limit of $\mathbf{K}$, and hence isomorphic to $\mathcal{F}$. The reason is that both are weakly homogeneous and both have age $\mathbf{K}$ (see [14, Lemma 6.1.4]). Let R be the unary relation on $\mathcal{F}$ that corresponds to the interpretation of $\mathtt{A}$ in $F^{\mathcal{A}}$ under this isomorphism. Clearly R and $\mathcal{A}$ are isomorphic. We claim that they have the same spectrum. By Theorem 5.1, it is enough to show is that for every presentation $\tilde{\mathcal{A}}$ of $\mathcal{A}$ there is a relation $\tilde{R}$ in $\mathcal{F}$ that is computable from $\tilde{\mathcal{A}}$ and automorphic to R. We know that $\mathbf{K}^{\mathcal{A}}$ has an $\tilde{\mathcal{A}}$-computable representation which has the $\tilde{\mathcal{A}}$-computable extension property. Then, by Theorem 3.12, $\mathbf{K}^{\tilde{\mathcal{A}}}$ has a Fraïssé limit $\mathcal{F}^{\tilde{\mathcal{A}}}$ computable in $\tilde{\mathcal{A}}$. By the countable categoricity of Fraïssé limits, we have that $\mathcal{F}^{\tilde{\mathcal{A}}}$ is isomorphic to $\mathcal{F}^{\mathcal{A}}$. By the computable categoricity of Fraïssé limits for locally finite $\mathbf{K}$ (Corollary 3.4), we have that the $\mathcal{L}$-reduct of $\mathcal{F}^{\tilde{\mathcal{A}}}$ and $\mathcal{F}$ are isomorphic via an $\tilde{\mathcal{A}}$-computable isomorphism. Let $\tilde{R}$ be the relation in $\mathcal{F}$ induced by the interpretation of $\mathtt{A}$ in $\mathcal{F}^{\tilde{\mathcal{A}}}$, under this isomorphism. Note that $\tilde{R}$ and R are automorphic as unary relations on $\mathcal{F}$ and that $\tilde{R} \leq_T \tilde{\mathcal{A}}$, as desired. □

Remark 7.3. The spectral universality of the dense linear ordering and the random graph were not originally proved using this method [12]. However, an age $\mathbf{K}^{\mathcal{A}}$ as the one in Theorem 7.2 was implicitly used in the original proofs. For the case of linear orderings, $\mathbf{K}^{\mathcal{A}}$ consisted of the $\mathcal{L}^{\mathcal{A}}$ structures $\mathcal{B}$ such that for every $x \in B - A$, either there is a least element of $\mathcal{A}$ greater than it, or there is a greatest element of $\mathcal{A}$ less than it. For the case of graphs, $\mathbf{K}^{\mathcal{A}}$ consisted of the $\mathcal{L}^{\mathcal{A}}$ structures $\mathcal{B}$ such that for every $x \in B - A$, x is adjacent to only finitely many elements of A.

7.1. **The countable atomless Boolean algebra.** Now we will use Theorem 7.2 to show that the countable atomless Boolean algebra is spectrally universal for the class of countable Boolean algebras.

Definition 7.4. Given a Boolean algebra $\mathcal{C}$, and $c \in \mathcal{C}$ we let $\mathcal{C}^c$ be the unique Boolean algebra $\mathcal{D}$ such that:

(1) $\mathcal{D}$ extends $\mathcal{C}$;
(2) $\mathcal{D}$ is generated by $\mathcal{C}$ and a new element d; and
(3) c is the least element of $\mathcal{C}$ greater than d, and 0 is the greatest element of $\mathcal{C}$ less than d.

We need to show that $\mathcal{C}^c$ is well defined.

Lemma 7.5. *Given a Boolean algebra $\mathcal{C}$, $c \in C$, a Boolean algebra satisfying the conditions in Definition 7.4 exists and is unique.*

Proof. Let us start by proving the existence. Define $\mathcal{C}^c = \mathcal{C} \oplus \mathcal{C}_{\leq c}$. Let $f\colon \mathcal{C} \to \mathcal{D}$ be defined by $f(x) = (x, x \wedge c)$. Is not hard to see that f is an embedding, so we view $\mathcal{C}^c$ as an extension of $\mathcal{C}$. Let $d = (0, c) \in \mathcal{C}^c$. Observe that d generates $\mathcal{C}^c$ over $\mathcal{C}$ because $(x, y) = (f(x) \wedge \neg d) \vee (f(y) \wedge d)$ for every $(x, y) \in \mathcal{C}^c$. Note that the least element of $f[\mathcal{C}]$ greater than d is $(c, c) = f(c)$, and the greatest element less than d is $(0, 0) = f(0)$.

Let us now prove the uniqueness. Let $\mathcal{D}$ be an extension of $\mathcal{C}$ satisfying the conditions of the definition of $\mathcal{C}^c$. Given $(x, y) \in \mathcal{C}^c$, let $g(x, y) = (x \wedge \neg d) \vee (y \wedge d) \in \mathcal{D}$. It is not hard to see that g is a homomorphism of Boolean algebras. It is onto because both $\mathcal{C}$ and d are in the image of g: $\mathcal{C} = g[f[\mathcal{C}]]$ and $d = g(0, c)$. To show that g is one-to-one, suppose $g(x, y) = 0$. Then, $x - d = 0$ and hence $x \leq d$, which implies $x = 0$. Also, $y \wedge d = 0$, so $c - y \geq d$, and hence $c - y = c$ and $y = 0$. □

Definition 7.6. Let $\mathcal{A}$ be a Boolean algebra. We say that $\mathcal{D}$ is a *simple* extension of $\mathcal{A}$ if it is of the form $(...(\mathcal{A}^{c_0})^{c_1}...)^{c_k}$, where for each $i \leq k$, $c_i \in (...(\mathcal{A}^{c_0})^{c_1}...)^{c_{i-1}}$. Let $\mathbf{K}^{\mathcal{A}}$ be the age of simple extensions of $\mathcal{A}$, viewed as $\mathcal{L}^{\mathcal{A}}$-structures.

Lemma 7.7. *The age $\mathbf{K}^{\mathcal{A}}$ satisfies the conditions of Theorem 7.2.*

Proof. Let us start by proving condition (2). Let $\mathcal{C} \subseteq \mathcal{D}$ be finite Boolean algebras, and suppose $\mathcal{C} \subseteq \mathcal{A}$. Suppose first that $\mathcal{D}$ is generated by only one element d over $\mathcal{C}$, and moreover, that d is below some atom c of $\mathcal{C}$. Then, $\mathcal{A}^c$ is an exact amalgamation of $\mathcal{A}$ and $\mathcal{D}$. The general case follows by induction.

Condition (1) is also not hard to prove. For HP, consider $(x, y) \in \mathcal{A}^c$. Then the subalgebra generated by (x, y) over $\mathcal{A} \subseteq \mathcal{A}^c$ is the same as the one generated by $(x - y, y - x)$, because $(x \wedge y, x \wedge y) \in \mathcal{A}$. Notice that $(0, 0)$ is the greatest element of $\mathcal{A}$ less than $(x - y, y - x)$, and $(x \triangle y, x \triangle y)$ is the least element of $\mathcal{A}$ greater than $(x - y, y - x)$. So, by Lemma 7.5, the sub-Boolean algebra of $\mathcal{A}^c$ generated by (x, y) over $\mathcal{A}$ is isomorphic to $\mathcal{A}^{x \triangle y}$. For JEP and AP use the fact that $(...((...(\mathcal{A}^{a_0})...)^{a_k})^{b_0}...)^{b_l}$ is an amalgamation of $(...(\mathcal{A}^{a_0})...)^{a_k}$ and $(...(\mathcal{A}^{b_0})...)^{b_l}$.

For condition (3), $\mathbf{K}^{\mathcal{A}}$ clearly has an $\mathcal{A}$-computable representation, $\mathbb{K}^{\mathcal{A}}$. To prove the $\mathcal{A}$-computable extension property, note that given $\mathcal{D} \in \mathbb{K}^{\mathcal{A}}$ and a formula $\varphi(c, x)$ consistent with $\mathcal{D}$, we can just search for a simple extension of $\mathcal{D}$ satisfying $\exists x \varphi(c, x)$. □

As a corollary we obtain the following theorem.

Theorem 7.8. *The countable atomless Boolean algebra is spectrally universal.*

References

[1] C.J. Ash & J.F. Knight; *Computable Structures and the Hyperarithmetical Hierarchy* (Amsterdam: Elsevier, 2000).

[2] C.J. Ash & A. Nerode; Intrinsically recursive relations, in: J.N. Crossley, editor, *Aspects of Effective Algebra* (U.D.A. Book Co., Steel's Creek, Australia, 1981), 26–41.

[3] J. Chisholm; The complexity of intrinsically r.e. subsets of existentially decidable models, *Journal of Symbolic Logic* **55** (1990), 1213–1232.

[4] R.G. Downey, S.S. Goncharov, & D.R. Hirschfeldt; Degree spectra of relations on Boolean algebras, *Algebra and Logic* **42** (2003), 105–111.

[5] R.G. Downey & C.G. Jockusch, Jr.; Every low Boolean algebra is isomorphic to a recursive one, *Proceedings of the American Mathematical Society* **122** (1994), 871–880.

[6] R.G. Downey & J.F. Knight; Orderings with α-th jump degree $\mathbf{0}^{\alpha}$, *Proceedings of the American Mathematical Society* **114** (1992), 545–552.

[7] R. Fraïssé; *Theory of Relations* (Amsterdam, New York: North-Holland, 1986).

[8] S.S. Goncharov, V.S. Harizanov, J.F. Knight, C. McCoy, R.G. Miller, & R. Solomon; Enumerations in computable structure theory, *Annals of Pure and Applied Logic* **136** (2005), 219–246.

[9] V.S. Harizanov; Uncountable degree spectra, *Annals of Pure and Applied Logic* **54** (1991), 255–263.

[10] V.S. Harizanov; Pure computable model theory, in: Yu.L. Ershov, S.S. Goncharov, A. Nerode & J.B. Remmel, editors, *Handbook of Recursive Mathematics*, vol. 1 (Amsterdam: Elsevier, 1998), 3–114.

[11] V.S. Harizanov; Relations on computable structures, *Contemporary Mathematics* (University of Belgrade, 2000), 65–81.

[12] V.S. Harizanov & R.G. Miller; Spectra of structures and relations, *The Journal of Symbolic Logic* **72** (2007), 324–348.

[13] D.R. Hirschfeldt, B. Khoussainov, R.A. Shore, & A.M. Slinko; Degree spectra and computable dimensions in algebraic structures, *Annals of Pure and Applied Logic* **115** (2002), 71–113.

[14] W. Hodges; *A Shorter Model Theory* (Cambridge: Cambridge University Press, 1997).

[15] C.G. Jockusch, Jr. & R.I. Soare; Degrees of orderings not isomorphic to recursive linear orderings, *Annals of Pure and Applied Logic* **52** (1991), 39–64.

[16] B. Khoussainov & R.A. Shore; Computable isomorphisms, degree spectra of relations, and Scott families, *Annals of Pure and Applied Logic* **93** (1998), 153–193.

[17] B. Khoussainov & R.A. Shore; Effective Model Theory: The number of models and their complexity, in: S.B. Cooper & J.K. Truss, editors, *Models and Computability: Invited Papers from Logic Colloquium '97*, LMSLNS **259** (Cambridge: Cambridge University Press, 1999), 193–240.

[18] J.F. Knight; Degrees coded in jumps of orderings, *Journal of Symbolic Logic* **51** (1986), 1034–1042.

[19] R.G. Miller; The Δ_2^0-spectrum of a linear order, *Journal of Symbolic Logic* **66** (2001), 470–486.

[20] M.F. Moses; Relations intrinsically recursive in linear orders, *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik* **32** (1986), 467–472.

[21] L.J. Richter; Degrees of structures, *Journal of Symbolic Logic* **46** (1981), 723–731.

[22] T. Slaman; Relative to any nonrecursive set, *Proceedings of the American Mathematical Society* **126** (1998), 2117–2122.

[23] R.I. Soare; *Recursively Enumerable Sets and Degrees* (New York: Springer-Verlag, 1987).

[24] B.L. van der Waerden; *Moderne Algebra* (Berlin: Springer, 1930). English translation *Algebra*, trans. F. Blum & J.R. Schulenberger (New York: Springer-Verlag, 1991).

[25] S. Wehner; Enumerations, countable structures, and Turing degrees, *Proceedings of the American Mathematical Society* **126** (1998), 2131–2139.

Department of Pure Mathematics, University of Waterloo, Waterloo, ON, Canada
E-mail address: csima@math.uwaterloo.ca
URL: www.math.uwaterloo.ca/∼csima

Department of Mathematics, The George Washington University, Washington, DC 20052, USA
E-mail address: harizanv@gwu.edu

Department of Mathematics, Queens College – CUNY, 65-30 Kissena Blvd., Flushing, NY 11367, USA, *and*, Ph.D. Programs in Mathematics and Computer Science, CUNY Graduate Center, 365 Fifth Avenue, New York, NY 10016, USA
E-mail address: Russell.Miller@qc.cuny.edu
URL: qcpages.qc.cuny.edu/∼rmiller

Department of Mathematics, University of Chicago, Chicago, IL, USA
E-mail address: antonio@math.uchicago.edu
URL: www.math.uchicago.edu/∼antonio