

# Enumerations in computable structure theory

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# 0 Introduction

Families of sets with special enumeration properties have been used to produce a number of interesting examples in computable structure theory. Selivanov [17] constructed a family of sets that Goncharov [11] used to produce a structure that is computably categorical but not relatively computably categorical. Manasse [14] used Selivanov's family of sets to produce a computable structure with a relation that is intrinsically computably enumerable (c.e.) but not relatively intrinsically c.e. Goncharov [10] constructed families of sets that he then used to produce computable structures with computable dimension  $n$ , for all finite  $n$  [11]. Wehner [20] constructed a family of sets that yields a structure with isomorphic copies in exactly the non-computable Turing degrees. Slaman [18] produced another such structure.

Here, we lift the results of Goncharov, Manasse, and Slaman and Wehner to higher levels. Using Selivanov's construction, in relativized form, we show that for each computable successor ordinal  $\alpha$ , there is a computable structure that is  $\Delta_\alpha^0$  categorical, but not relatively  $\Delta_\alpha^0$  categorical. From this structure, we obtain another computable structure, with a relation that is intrinsically  $\Delta_\alpha^0$ , but not relatively intrinsically  $\Delta_\alpha^0$ . Using the enumeration results of Goncharov, relativized, we show that for each computable successor ordinal  $\alpha$ , and each finite  $n$ , there is a computable structure with exactly  $n$  computable copies, up to  $\Delta_\alpha^0$  isomorphism. Using the enumeration result of Wehner, also relativized, we show that for each computable successor ordinal  $\alpha$ , there is a structure with copies in just the degrees of sets  $X$  such that  $\Delta_\alpha^0(X)$  is not  $\Delta_\alpha^0$ . In particular, for each finite  $n$ , there is a structure with copies in just the non- $low_n$  degrees.

Section 1 has some basic definitions. In Section 2, we state the enumeration results of Selivanov, Goncharov, and Wehner. In Section 3, we say how enumeration properties of a family of sets translate into properties of certain graph structures derived from the family. In Section 4, we prove the basic results of Goncharov, Manasse, and Slaman and Wehner, using the results from Sections 1, 2 and 3. In Section 5, we describe a construction that, for a computable successor ordinal  $\alpha$ , transforms a graph  $\mathcal{G}$  into a structure  $\mathcal{G}^*$  such that  $\mathcal{G}$  has a  $\Delta_\alpha^0$  copy iff  $\mathcal{G}^*$  has a computable copy. We indicate how various special features of  $\mathcal{G}$  translate into features of  $\mathcal{G}^*$ . This construction requires the existence of a pair of structures  $\mathcal{B}_0, \mathcal{B}_1$ , which are uniformly relatively  $\Delta_\alpha^0$  categorical and have nice properties with respect to the standard back-and-forth relations  $\leq_\gamma$  for  $\gamma < \alpha$ . We describe the structures in Section 5, but we delay proving that they have the required properties until Section 7. In Section 6, we use the construction taking  $\mathcal{G}$  to  $\mathcal{G}^*$  to lift the results from Section 4. In Section 8, we state some open problems.

## 1 Background

We consider only computable languages, and structures with universe contained in a computable set of constants. We identify sentences with their Gödel numbers. In measuring complexity of a structure  $\mathcal{A}$ , we identify the structure with its atomic diagram  $D(\mathcal{A})$ . Thus,  $\mathcal{A}$  is a subset of  $\omega$ , and it makes sense to say that  $\mathcal{A}$  is *computable*, or to speak of the *Turing*

degree of  $\mathcal{A}$ . Our main goal in this section is to give definitions, and state some known results. All of the material may be found in [4], with examples and proofs.

## 1.1 Notions related to computable categoricity

Let  $\mathcal{A}$  be a computable structure. We say that  $\mathcal{A}$  is *computably categorical* if for all computable  $\mathcal{B} \cong \mathcal{A}$ , there is a computable isomorphism from  $\mathcal{A}$  onto  $\mathcal{B}$ . Similarly,  $\mathcal{A}$  is  $\Delta_\alpha^0$  *categorical* if for all computable  $\mathcal{B} \cong \mathcal{A}$ , there is a  $\Delta_\alpha^0$  isomorphism. We say that  $\mathcal{A}$  is *relatively computably categorical* if for all  $\mathcal{B} \cong \mathcal{A}$ , there is an isomorphism that is computable relative to  $\mathcal{B}$ , and  $\mathcal{A}$  is *relatively  $\Delta_\alpha^0$  categorical* if for all  $\mathcal{B} \cong \mathcal{A}$ , there is a  $\Delta_\alpha^0(\mathcal{B})$  isomorphism.

There are syntactical conditions that imply  $\Delta_\alpha^0$  categoricity, and are equivalent to relative  $\Delta_\alpha^0$  categoricity. The conditions involve the existence of nice “Scott families”. The notion comes from the proof of Scott’s Isomorphism Theorem, which says that for a countable structure  $\mathcal{A}$ , there is an  $L_{\omega_1\omega}$  sentence whose countable models are just the copies of  $\mathcal{A}$  [16], [13]. Scott derived the “Scott sentence” for  $\mathcal{A}$  from a family of  $L_{\omega_1\omega}$  formulas defining the orbits of tuples in  $\mathcal{A}$ .

A *Scott family* for  $\mathcal{A}$  is a set  $\Phi$  of formulas, with a fixed finite tuple of parameters  $\bar{c}$  in  $\mathcal{A}$ , such that

1. each tuple in  $\mathcal{A}$  satisfies some  $\varphi \in \Phi$ , and
2. if  $\bar{a}, \bar{b}$  are tuples in  $\mathcal{A}$  satisfying the *same* formula  $\varphi \in \Phi$ , then there is an automorphism of  $\mathcal{A}$  taking  $\bar{a}$  to  $\bar{b}$ .

According to this definition, a Scott family for  $\mathcal{A}$  may contain formulas that are not satisfied by any tuple in  $\mathcal{A}$ . If  $\Phi$  has parameters  $\bar{c}$ , and  $\mathcal{A} \models \varphi(\bar{a})$ , where  $\varphi \in \Phi$ , then  $\varphi$  defines the orbit of  $\bar{a}$  in the expanded structure  $(\mathcal{A}, \bar{c})$ . If there are nice isomorphisms from  $\mathcal{A}$  onto its copies, then we expect a nice Scott family. A *formally c.e. Scott family* is a c.e. Scott family consisting of finitary existential formulas. A *formally  $\Sigma_\alpha^0$  Scott family* is a  $\Sigma_\alpha^0$  Scott family consisting of “computable  $\Sigma_\alpha$ ” formulas.

A detailed discussion of computable infinitary formulas is given in [4]. For our purposes here, an intuitive definition, together with one characteristic property, will suffice. Roughly speaking, computable infinitary formulas are  $L_{\omega_1\omega}$  formulas in which the infinite disjunctions and conjunctions are over c.e. sets. There is a useful hierarchy of computable infinitary formulas. A *computable  $\Sigma_0$  or  $\Pi_0$  formula* is a finitary quantifier-free formula. For  $\alpha > 0$ , a *computable  $\Sigma_\alpha$  formula* is a c.e. disjunction of formulas of the form  $\exists \bar{u} \psi$ , where  $\psi$  is computable  $\Pi_\beta$  for some  $\beta < \alpha$ , and a *computable  $\Pi_\alpha$  formula* is a c.e. conjunction of formulas of the form  $\forall \bar{u} \psi$ , where  $\psi$  is computable  $\Sigma_\beta$  for some  $\beta < \alpha$ . (To make this precise, we would assign indices to the formulas, based on Kleene’s system of notations for computable ordinals.) The important property of these formulas is given in the following theorem.

**Theorem 1.1.** *For a structure  $\mathcal{A}$ , the set  $\{\bar{a} : \mathcal{A} \models \varphi(\bar{a})\}$  is  $\Sigma_\alpha^0(\mathcal{A})$  if  $\varphi(\bar{x})$  is computable  $\Sigma_\alpha$ , and  $\Pi_\alpha^0(\mathcal{A})$  if  $\varphi(\bar{x})$  is computable  $\Pi_\alpha$ . Moreover, this holds with all imaginable uniformity, over structures and formulas.*

It is easy to see that if  $\mathcal{A}$  has a formally c.e. Scott family, then it is relatively computably categorical, so it is computably categorical. More generally, if  $\mathcal{A}$  has a formally  $\Sigma_\alpha^0$  Scott family, then we can see, using Theorem 1.1, that it is relatively  $\Delta_\alpha^0$  categorical, so it is  $\Delta_\alpha^0$  categorical. Goncharov [11] showed that, under some added effectiveness conditions (on a single copy), if  $\mathcal{A}$  is computably categorical, then it has a formally c.e. Scott family. Ash [1] showed that, under some effectiveness conditions (on a single copy), if  $\mathcal{A}$  is  $\Delta_\alpha^0$  categorical, then it has a formally  $\Sigma_\alpha^0$  Scott family. For the relative notions, the effectiveness conditions disappear. The following result is from [5] and [8].

**Theorem 1.2 (Ash-Knight-Manasse-Slaman, Chisholm).** *A computable structure  $\mathcal{A}$  is relatively  $\Delta_\alpha^0$  categorical iff it has a formally  $\Sigma_\alpha^0$  Scott family. In particular,  $\mathcal{A}$  is relatively computably categorical iff it has a formally c.e. Scott family.*

It would be pleasant if computable categoricity and relative computable categoricity were the same—then we could drop the effectiveness conditions from Goncharov’s result. However, Goncharov [11] showed that this is not the case, using an enumeration result of Selivanov [17]. There are examples with further properties. Cholak, Goncharov, Khoussainov, and Shore [9] gave an example of a structure that is computably categorical, but ceases to be after naming a constant. It follows from Theorem 1.2 that such a structure is not relatively computably categorical.

A *rigid* structure is one with no non-trivial automorphisms. If a rigid structure is  $\Delta_\alpha^0$  categorical, then it is also  $\Delta_\alpha^0$  *stable*; i.e., every isomorphism from  $\mathcal{A}$  onto a computable copy is  $\Delta_\alpha^0$ . For a rigid structure  $\mathcal{A}$ , we may replace the notion of a Scott family by that of a *defining family*, where this is a set  $\Phi$  of formulas with just  $x$  free, and with a fixed finite tuple of parameters, such that

1. each element of  $\mathcal{A}$  satisfies some formula  $\varphi(x) \in \Phi$ , and
2. no formula of  $\Phi$  is satisfied by more than one element of  $\mathcal{A}$ .

If  $\mathcal{A}$  is rigid, and the isomorphisms from  $\mathcal{A}$  to its copies are nice, then we expect a nice defining family. A defining family  $\Phi$  is said to be *formally c.e.* if it is a c.e. set of finitary existential formulas, and it is *formally  $\Sigma_\alpha^0$*  if it is a  $\Sigma_\alpha^0$  set of computable  $\Sigma_\alpha$  formulas. For a rigid computable structure  $\mathcal{A}$ , there is a formally c.e. Scott family iff there is a formally c.e. defining family, and there is a formally  $\Sigma_\alpha^0$  Scott family iff there is a formally  $\Sigma_\alpha^0$  defining family. The parameters in the Scott family and the defining family will be the same.

## 1.2 Intrinsically and relatively intrinsically $\Sigma_\alpha^0$ relations

Let  $\mathcal{A}$  be a computable structure, and let  $R$  be a relation on  $\mathcal{A}$ . We say that  $R$  is *intrinsically c.e.* if in all computable  $\mathcal{B} \cong \mathcal{A}$ , the image of  $R$  is c.e., and  $R$  is *intrinsically  $\Sigma_\alpha^0$*  if in all computable  $\mathcal{B} \cong \mathcal{A}$ , the image of  $R$  is  $\Sigma_\alpha^0$ . We say that  $R$  is *relatively intrinsically c.e.* if in all  $\mathcal{B} \cong \mathcal{A}$ , the image of  $R$  is c.e. relative to  $\mathcal{B}$ , and  $R$  is *relatively intrinsically  $\Sigma_\alpha^0$*  if in all  $\mathcal{B} \cong \mathcal{A}$ , the image of  $R$  is  $\Sigma_\alpha^0(\mathcal{B})$ .

If  $R$  is definable in  $\mathcal{A}$  by a computable  $\Sigma_\alpha$  formula, with a finite tuple of parameters, then  $R$  is relatively intrinsically  $\Sigma_\alpha^0$ , so it is intrinsically  $\Sigma_\alpha^0$ . Ash and Nerode [6] showed that under some effectiveness conditions, on a single copy, if  $R$  is intrinsically c.e., then it is defined by a computable  $\Sigma_1$  formula, with a finite tuple of parameters. Barker [7] showed that under some effectiveness conditions, on a single copy, if  $R$  is intrinsically  $\Sigma_\alpha^0$ , then it is defined by a computable  $\Sigma_\alpha$  formula, with a finite tuple of parameters. For the relative notions, the effectiveness conditions are not needed. The following result is in [5] and [8].

**Theorem 1.3 (Ash-Knight-Manasse-Slaman, Chisholm).** *Let  $\mathcal{A}$  be a computable structure. Then a relation  $R$  on  $\mathcal{A}$  is relatively intrinsically  $\Sigma_\alpha^0$  iff it is defined by a computable  $\Sigma_\alpha$  formula, with a finite tuple of parameters. In particular,  $R$  is relatively intrinsically c.e. iff it is defined by a computable  $\Sigma_1$  formula, with parameters.*

It would be pleasant if the intrinsically c.e. and relatively intrinsically c.e. relations were the same. However, Manasse [14] produced an example showing that this is not so. His construction also used the family of sets constructed by Selivanov [17].

### 1.3 Notions related to computable dimension

The *computable dimension* of a structure  $\mathcal{A}$  is the number of computable copies, up to computable isomorphism. Similarly, the  $\Delta_\alpha^0$  *dimension* is the number of computable copies, up to  $\Delta_\alpha^0$  isomorphism. Goncharov [11] showed that there are structures of computable dimension  $n$  for all finite  $n$ . McCoy [15] showed that computable dimension does not relativize.

**Theorem 1.4 (McCoy).** *Suppose  $\mathcal{A}$  is a computable structure. If  $\mathcal{A}$  is not computably categorical, then for all  $n > 1$ , there exist  $\mathcal{B}_1, \dots, \mathcal{B}_n$  isomorphic to  $\mathcal{A}$  such that for  $1 \leq i < j \leq n$ , there is no  $(\bigoplus_{1 \leq k \leq n} \mathcal{B}_k)$ -computable isomorphism from  $\mathcal{B}_i$  onto  $\mathcal{B}_j$ .*

## 2 Basic enumeration results

We begin with some definitions. For  $\mathcal{S} \subseteq P(\omega)$  an *enumeration* is a binary relation  $\nu$  such that

$$\mathcal{S} = \{\nu(i) : i \in \omega\}, \text{ where } \nu(i) = \{x : (i, x) \in \nu\}.$$

When we say that the enumeration is *computable* (c.e., respectively) we mean that the binary relation is computable (c.e., respectively). We note that in some of the literature,  $\nu$  is called *computable* when the binary relation is merely c.e. It is easy to see that a family  $\mathcal{S}$  has a computable enumeration just in case the family

$$\mathcal{S}^+ = \{A \oplus \bar{A} : A \in \mathcal{S}\}$$

has a c.e. enumeration.

An enumeration is *Friedberg* if it is 1 – 1, in the sense that if  $i \neq j$ , then  $\nu(i) \neq \nu(j)$ . Suppose  $\nu, \mu$  are two enumerations of the same family  $\mathcal{S}$ . We write  $\nu \leq \mu$  if there is a

computable function  $f$  such that for all  $i$ ,  $\nu(i) = \mu(f(i))$ ; i.e., we can effectively pass from a  $\nu$ -index to a  $\mu$ -index for the same set. We say that  $\nu$  and  $\mu$  are *computably equivalent* if  $\mu \leq \nu$  and  $\nu \leq \mu$ . Note that if  $\mu$  and  $\nu$  are Friedberg enumerations of the same family  $\mathcal{S}$ , then  $\mu \leq \nu$  implies  $\nu \leq \mu$ .

A family  $\mathcal{S} \subseteq P(\omega)$  is *discrete* if for each  $A \in \mathcal{S}$ , there exists  $\sigma \in 2^{<\omega}$  such that for all  $B \in \mathcal{S}$ ,  $\sigma \subseteq \chi_B$  iff  $B = A$ . The family is *effectively discrete* if there is a c.e. set  $E \subseteq 2^{<\omega}$  such that

- (a) for each  $A \in \mathcal{S}$ , there is  $\sigma \in E$  such that  $\sigma \subseteq \chi_A$ , and
- (b) for all  $\sigma \in E$  and all  $A, B \in \mathcal{S}$ , if  $\sigma \subseteq \chi_A, \chi_B$ , then  $A = B$ .

In [17], Selivanov proved the following.

**Theorem 2.1 (Selivanov).** *There exists a family  $\mathcal{S} \subseteq P(\omega)$  which has a unique computable Friedberg enumeration, up to computable equivalence, and is discrete but not effectively discrete.*

Actually, Selivanov produced a family of functions  $f \in \omega^\omega$  such that the family of sets  $A_f = \{\langle x, f(x) \rangle : x \in \omega\}$ , representing the graphs of the functions, has the properties above. For such a family, any c.e. enumeration is actually computable. Hence, Selivanov's family also has a unique c.e. Friedberg enumeration, up to computable equivalence.

In [10], Goncharov proved the following.

**Theorem 2.2 (Goncharov).** *For each finite  $n \geq 1$ , there is a family of sets with just  $n$  c.e. Friedberg enumerations, up to computable equivalence.*

We do not know whether there are similar examples of families with just  $n$  computable Friedberg enumerations, up to computable equivalence.

In [20], Wehner proved the following.

**Theorem 2.3 (Wehner).**

1. *There is a family  $\mathcal{S} \subseteq P(\omega)$  such that for each noncomputable set  $X$ ,  $\mathcal{S}$  has an enumeration computable in  $X$ , but  $\mathcal{S}$  has no computable enumeration.*
2. *There is a family  $\mathcal{S} \subseteq P(\omega)$  such that for each noncomputable set  $X$ ,  $\mathcal{S}$  has an enumeration c.e. relative to  $X$ , but  $\mathcal{S}$  has no c.e. enumeration.*

The enumeration results of Selivanov, Goncharov, and Wehner all relativize. In the next section, we describe a general method for turning a family of sets with special enumeration properties into a directed graph structure with related properties.

### 3 Turning a family of sets into a graph

Let  $\mathcal{S}$  be a family of sets. For each  $A \in \mathcal{S}$ , a *daisy graph*  $\mathcal{G}_A$  consists of one *index* point  $a$  at the center, with  $a \rightarrow a$ , and, for each  $n \in A$ , a *petal* of the form

$$a \rightarrow a_0 \rightarrow \cdots \rightarrow a_n \rightarrow a .$$

The petals are disjoint except for the index point, which is common to all. Let  $\mathcal{G}(\mathcal{S})$  be the union of a disjoint family of daisy graphs  $\mathcal{G}_A$ , one for each  $A \in \mathcal{S}$ .

We put the important facts about this construction into the following technical lemmas.

**Lemma 3.1.** *For any family  $\mathcal{S} \subseteq P(\omega)$ , both  $\mathcal{G}(\mathcal{S})$  and  $\mathcal{G}(\mathcal{S}^+)$  are rigid graphs.*

This is clear.

**Lemma 3.2.** *If  $\mathcal{S}$  has a unique c.e. Friedberg enumeration, up to computable equivalence, then  $\mathcal{G}(\mathcal{S})$  is computably categorical. Similarly, if  $\mathcal{S}$  has a unique computable Friedberg enumeration, up to computable equivalence, then  $\mathcal{G}(\mathcal{S}^+)$  is computably categorical.*

*Proof.* If  $\nu$  is a c.e. Friedberg enumeration of  $\mathcal{S}$ , then we can use  $\nu$  to produce a computable copy of  $\mathcal{G}(\mathcal{S})$ , with a computable function taking  $i$  to the index point for the daisy graph  $\mathcal{G}_{\nu(i)}$ . If  $\mathcal{H}$  is a computable copy of  $\mathcal{G}(\mathcal{S})$ , then  $\mathcal{H}$  yields a c.e. Friedberg enumeration  $\mu$  of  $\mathcal{S}$ , as follows. First, there is a computable function taking the  $i^{\text{th}}$  index point of  $\mathcal{H}$  to  $i$ . We can easily recognize index points in  $\mathcal{H}$ —they are the points  $a$  such that  $\mathcal{H} \models a \rightarrow a$ . If  $a$  is the  $i^{\text{th}}$  index point, then we let  $\mu(i)$  be the set coded in the daisy graph with  $a$  as its center. If  $\mathcal{S}$  has a unique computable Friedberg enumeration, up to computable equivalence, then for any two computable copies of  $\mathcal{G}(\mathcal{S})$ , we effectively match up the index points, and we can then effectively match up the remaining points in the daisies to obtain a computable isomorphism.

If  $\mathcal{S}$  has a unique computable Friedberg enumeration, up to computable equivalence, then  $\mathcal{S}^+$  has a unique computable c.e. enumeration, up to computable equivalence, and we have seen that  $\mathcal{G}(\mathcal{S}^+)$  is computably categorical. □

**Lemma 3.3.** *If  $\mathcal{S}$  has just  $n$  c.e. Friedberg enumerations, up to computable equivalence, then  $\mathcal{G}(\mathcal{S})$  has computable dimension  $n$ . Similarly, if  $\mathcal{S}$  has just  $n$  computable Friedberg enumerations, up to computable equivalence, then  $\mathcal{G}(\mathcal{S}^+)$  has computable dimension  $n$ .*

*Proof.* Suppose that  $\mathcal{S}$  has just  $n$  c.e. Friedberg enumerations, up to computable equivalence. Let  $\nu_1, \dots, \nu_n$  be computably inequivalent c.e. enumerations of  $\mathcal{S}$ . For each  $k$ , let  $\mathcal{H}_k$  be a computable copy of  $\mathcal{G}(\mathcal{S})$  with a computable function taking each  $i$  to the index point for  $\nu_k(i)$  in  $\mathcal{H}_k$ . For  $k \neq m$ , the fact that  $\nu_k$  and  $\nu_m$  are not computably equivalent means that there is no computable isomorphism from  $\mathcal{H}_k$  to  $\mathcal{H}_m$ . Therefore,  $\mathcal{G}(\mathcal{S})$  has computable dimension at least  $n$ . Suppose  $\mathcal{H}$  is a computable copy of  $\mathcal{G}(\mathcal{S})$ , and let  $\nu$  be a computable enumeration of  $\mathcal{S}$  with a computable function taking  $i$  to the index point for the daisy graph of type  $\mathcal{G}_{\nu(i)}$  in  $\mathcal{H}$ . For some  $k$ ,  $\nu$  is computably equivalent to  $\nu_k$ , and then we have a computable isomorphism from  $\mathcal{H}$  onto  $\mathcal{H}_k$ . Therefore,  $\mathcal{G}(\mathcal{S})$  has computable dimension at most  $n$ .

If  $\mathcal{S}$  has just  $n$  computable Friedberg enumerations, up to computable equivalence, then  $\mathcal{S}^+$  has just  $n$  c.e. enumerations, up to computable equivalence, and we have seen that  $\mathcal{G}(\mathcal{S}^+)$  has computable dimension  $n$ . □

**Lemma 3.4.** *If  $\mathcal{S}$  is discrete, then every element of  $\mathcal{G}(\mathcal{S}^+)$  has a finitary existential definition with no parameters.*

*Proof.* First, let  $a$  be the index point for the daisy graph of type  $\mathcal{G}_{A \oplus \bar{A}}$ , where  $A \in \mathcal{S}$ . Because  $\mathcal{S}$  is discrete, we can fix a finite binary string  $\alpha$  such that  $\alpha \subseteq \chi_A$  and for any  $B \neq A$  from  $\mathcal{S}$ ,  $\alpha \not\subseteq \chi_B$ . The string  $\alpha$  corresponds to a particular collection of odd and even length cycles in the daisy graph with index point  $a$ . From this, we get an existential formula defining  $a$  in  $\mathcal{G}(\mathcal{S}^+)$ . If  $c$  is some other element of the daisy graph containing  $a$ , then for some  $n$  and  $k$ ,  $c$  is the  $k^{\text{th}}$  element of a cycle of length  $n$  which starts and ends with  $a$ . Using this, we get an existential definition for  $c$ . □

**Lemma 3.5.** *Suppose  $\mathcal{S}$  has a computable Friedberg enumeration, and is discrete but not effectively discrete. Then  $\mathcal{G}(\mathcal{S}^+)$  does not have a formally c.e. defining family.*

*Proof.* We may apply the proof of Lemma 3.2 and suppose that  $\mathcal{G}(\mathcal{S}^+)$  is computable. Suppose there is a formally c.e. defining family  $\Phi$ , hoping for a contradiction. By Lemma 3.4, each element of  $\mathcal{G}(\mathcal{S}^+)$  has a finitary existential definition, and we may assume that there are no parameters in the formulas of  $\Phi$ . We consider the c.e. set  $D$  consisting of pairs  $(\varphi, a)$  such that  $\varphi \in \Phi$ , and  $a$  is an index point satisfying  $\varphi(x)$  in  $\mathcal{G}(\mathcal{S}^+)$ . For each such pair, the formula  $\varphi(x)$  describes the way  $a$  sits in a finite subgraph of  $\mathcal{G}(\mathcal{S}^+)$ , where the finite subgraph includes part of the daisy with index point  $a$ , and possibly parts of some other daisies, with index points  $b_1, \dots, b_n$ .

By enlarging the finite subgraph, we may suppose that any petal represented in it is completely included, and there are enough petals to give information distinguishing among the sets  $A, B_i$  that correspond to the indices  $a, b_i$ . That is, for each distinct pair of sets  $X, Y \in \{A, B_1, \dots, B_n\}$ , there must be some number  $k$  such that  $k \in X$  and  $k \notin Y$ , or  $k \in Y$  and  $k \notin X$ . These differences are recorded in the graph by the existence of an appropriate even length cycle in one daisy graph and an odd length cycle in the other daisy graph. Furthermore, we can find such differences effectively by searching. From the lengths of the petals, we see that  $\alpha \subseteq \chi_A$ , and  $\beta_i \subseteq \chi_{B_i}$ , where  $\alpha, \beta_i$  are distinct sequences of the same finite length. Note that if  $c$  is an index point corresponding to some  $C \in \mathcal{S}$ , where  $\alpha \subseteq \chi_C$ , then  $c$  also satisfies the formula  $\varphi(x)$ , so  $C = A$ . We have a c.e. set  $E$  of finite sequences  $\alpha$  obtained effectively in this way from the pairs  $(\varphi, a)$  in  $D$ . Therefore,  $\mathcal{S}$  is effectively discrete, a contradiction. □

For  $\mathcal{S} \subseteq P(\omega)$ , we may also form a graph structure  $\mathcal{G}^\infty(\mathcal{S})$  made up of infinitely many copies of  $\mathcal{G}_A$  for each  $A \in \mathcal{S}$ . The structure  $\mathcal{G}^\infty(\mathcal{S})$  is not rigid. Copies of  $\mathcal{G}^\infty(\mathcal{S})$  correspond to *arbitrary* enumerations of  $\mathcal{S}$ —not just to Friedberg enumerations.



**Lemma 3.6.** *Let  $\mathcal{S} \subseteq P(\omega)$ . For any set  $X$ , there is an enumeration of  $\mathcal{S}$  c.e. in  $X$  iff there is a copy of  $\mathcal{G}^\infty(\mathcal{S})$  computable in  $X$ . Similarly, there is an enumeration of  $\mathcal{S}$  computable in  $X$  iff there is a copy of  $\mathcal{G}^\infty(\mathcal{S}^+)$  computable in  $X$ .*

*Proof.* Clearly, if there is a copy of  $\mathcal{G}^\infty(\mathcal{S})$  computable in  $X$ , then there is an enumeration of  $\mathcal{S}$  c.e. in  $X$ ; in fact, we get an enumeration in which each element of  $\mathcal{S}$  has infinitely many indices. Now, suppose  $\nu$  is an enumeration of  $\mathcal{S}$  c.e. in  $X$ . We can define another enumeration  $\mu$  by  $\mu(\langle i, j \rangle) = \nu(i)$ . This  $\mu$  enumerates every set in  $\mathcal{S}$  infinitely many times, and it is also c.e. in  $X$ . From  $\mu$ , we get a copy of  $\mathcal{G}^\infty(\mathcal{S})$  computable in  $X$ .

If there is a copy of  $\mathcal{G}^\infty(\mathcal{S}^+)$  computable in  $X$ , then there is an enumeration of  $\mathcal{S}$  computable in  $X$ ; in fact, there is an enumeration in which each element of  $\mathcal{S}$  has infinitely many indices. If there is an enumeration of  $\mathcal{S}$  computable in  $X$ , then there is an enumeration of  $\mathcal{S}^+$  c.e. in  $X$ . We get a copy of  $\mathcal{G}^\infty(\mathcal{S}^+)$  computable in  $X$ , as above. □

## 4 Results of Goncharov, Manasse, Slaman, and Wehner

In this section, we review the basic results that we plan to lift. Here is the result of Goncharov [11].

**Theorem 4.1 (Goncharov).** *There is a rigid graph structure  $\mathcal{G}$  that is computably categorical but not relatively computably categorical.*

*Proof.* We take the family  $\mathcal{S}$  from Selivanov's Theorem (Theorem 2.1). By Lemma 3.1, the structure  $\mathcal{G}(\mathcal{S}^+)$  is rigid. By Lemma 3.2, it is computably categorical. By Lemma 3.5, it has no formally c.e. defining family. Therefore, by Theorem 1.2, it is not relatively computably categorical. □

As a corollary of Theorem 4.1, we obtain Manasse's result on intrinsically c.e. and relatively intrinsically c.e. relations.

**Theorem 4.2 (Manasse).** *There is a computable structure  $\mathcal{A}$  with a relation  $R$  that is intrinsically c.e. but not relatively intrinsically c.e.*

*Proof.* The *cardinal sum* of disjoint structures  $\mathcal{B}_0, \mathcal{B}_1$ , in the same relational language, is formed by taking the disjoint union of the structures and adding predicates  $P_0$  and  $P_1$  which hold of the elements of  $\mathcal{B}_0$  and  $\mathcal{B}_1$ , respectively. Let  $\mathcal{A}$  be the cardinal sum of two disjoint computable copies of the graph structure  $\mathcal{G}$  from Theorem 4.1, and let  $R$  be the unique isomorphism. The fact that  $\mathcal{G}$  is computably categorical implies that  $R$  is intrinsically c.e. Suppose  $R$  is relatively intrinsically c.e., hoping for a contradiction. For any copy  $\mathcal{H}$  of  $\mathcal{G}$ , we take the disjoint union of the universes, and form a copy of  $\mathcal{A}$ . There is an isomorphism from  $\mathcal{G}$  onto  $\mathcal{H}$ , computable in  $\mathcal{H}$ . Then  $\mathcal{G}$  is relatively computably categorical, a contradiction. □

Next, we obtain Goncharov's result on computable dimension.

**Theorem 4.3 (Goncharov).** *For each finite  $n$ , there is a rigid graph structure  $\mathcal{G}$  with computable dimension  $n$ .*

*Proof.* By Goncharov's Enumeration Theorem (Theorem 2.2), there is a family of sets  $\mathcal{S}$  with just  $n$  c.e. Friedberg enumerations, up to computable equivalence. By Lemma 3.1,  $\mathcal{G}(\mathcal{S})$  is a rigid graph. By Lemma 3.3, it has computable dimension  $n$ . □

Here is the result of Slaman and Wehner on degrees of structures.

**Theorem 4.4 (Slaman, Wehner).** *There is a structure  $\mathcal{A}$  with copies in just the non-computable degrees.*

*Proof.* By Theorem 2.3, there is a family of sets  $\mathcal{S}$  with enumerations c.e. in all non-computable sets, but no c.e. enumeration. By Lemma 3.6,  $\mathcal{G}^\infty(\mathcal{S})$  has copies computable in  $X$ , for all non-computable sets  $X$ , but no computable copy. (We could equally well take a family  $\mathcal{S}$  with enumerations computable in all non-computable sets, but no computable enumeration, and form  $\mathcal{G}^\infty(\mathcal{S}^+)$ .) □

## 5 Coding a $\Delta_\alpha^0$ structure in a computable one

To lift the basic results of Goncharov and Manasse, we first relativize them, producing a directed graph  $\mathcal{G}$  that is  $\Delta_\alpha^0$ . We then pass to a computable structure  $\mathcal{G}^*$ , using a pair of structures to code the arrow relation (from the graph). For a graph  $\mathcal{G}$ , and a pair of structures  $\mathcal{B}_0, \mathcal{B}_1$  for the same relational language, let  $\mathcal{G}^* = (G \cup U, G, U, Q, \dots)$ , where

1.  $G$  is the universe of  $\mathcal{G}$ ,
2.  $G$  and  $U$  are disjoint,
3.  $Q$  is a ternary relation assigning to each pair  $a, b \in G$  an infinite set  $U_{(a,b)}$ , where  $x \in U_{(a,b)}$  iff  $Qabx$ ,
4. the sets  $U_{(a,b)}$  form a partition of  $U$ ,
5. each of the other relations of  $\mathcal{G}^*$  (in  $\dots$ ) corresponds to some symbol in the language of  $\mathcal{B}_0, \mathcal{B}_1$ , and is the union of its restrictions to the sets  $U_{(a,b)}$ ,
6. for each pair  $a, b \in G$ , if  $\mathcal{U}_{(a,b)}$  is the structure  $(U_{(a,b)}, \dots)$ , then

$$\mathcal{U}_{(a,b)} \cong \begin{cases} \mathcal{B}_0, & \text{if } \mathcal{G} \models a \rightarrow b, \\ \mathcal{B}_1, & \text{otherwise.} \end{cases}$$

We give conditions on the pair of structures  $\mathcal{B}_i$  ( $i = 0, 1$ ) under which a  $\Delta_\alpha^0$  graph structure  $\mathcal{G}$  gives rise to a computable structure  $\mathcal{G}^*$ . We need some definitions. The *standard back-and-forth relations*  $\leq_\beta$  on the set of pairs  $\{(i, \bar{b}) : \bar{b} \in \mathcal{B}_i\}$ , are defined inductively as follows:

- (i)  $(i, \bar{b}) \leq_1 (j, \bar{c})$  if the existential formulas true of  $\bar{c}$  in  $\mathcal{B}_j$  are true of  $\bar{b}$  in  $\mathcal{B}_i$ ,
- (ii) if  $\beta > 1$ ,  $(i, \bar{b}) \leq_\beta (j, \bar{c})$  if for all  $\bar{c}'$  in  $\mathcal{B}_j$ , and all  $\gamma$  such that  $1 \leq \gamma < \beta$ , there exists  $\bar{b}'$  in  $\mathcal{B}_i$  such that  $(j, \bar{c}, \bar{c}') \leq_\gamma (i, \bar{b}, \bar{b}')$ .

**Remark:** By a result of Karp [12],  $(i, \bar{b}) \leq_\beta (j, \bar{c})$  iff all  $\Pi_\beta$  formulas of  $L_{\omega_1\omega}$  true of  $\bar{b}$  in  $\mathcal{B}_i$  are true of  $\bar{c}$  in  $\mathcal{B}_j$  (not just the computable  $\Pi_\beta$  formulas).

A pair of structures  $\{\mathcal{B}_0, \mathcal{B}_1\}$  is  $\alpha$ -friendly if the structures are computable and the standard back-and-forth relations  $\leq_\beta$  for  $\beta < \alpha$  are c.e., uniformly in  $\beta$ . (To make this precise, we fix a notation  $a$  for  $\alpha$  in  $O$  and identify each ordinal  $\beta < \alpha$  with its unique notation  $b <_O a$ .)

**Lemma 5.1.** *Let  $\alpha$  be a computable successor ordinal. Let  $\mathcal{B}_0, \mathcal{B}_1$  be such that*

1. *the pair  $\{\mathcal{B}_0, \mathcal{B}_1\}$  is  $\alpha$ -friendly,*
2.  *$\mathcal{B}_0, \mathcal{B}_1$  satisfy the same  $\Pi_\beta$  sentences (of  $L_{\omega_1\omega}$ ) for  $\beta < \alpha$ ,*
3. *each  $\mathcal{B}_i$  ( $i = 0, 1$ ) satisfies some computable  $\Pi_\alpha$  sentence that is not true in the other.*

*Then for any  $\Delta_\alpha^0$  set  $S$ , there is a uniformly computable sequence  $(\mathcal{C}_n)_{n \in \omega}$  such that*

$$\mathcal{C}_n \cong \begin{cases} \mathcal{B}_0, & \text{if } n \in S, \\ \mathcal{B}_1, & \text{otherwise.} \end{cases}$$

Lemma 5.1 is related to results in [3], where a  $\Pi_\alpha^0$  set (as opposed to a  $\Delta_\alpha^0$  one) is coded in a computable sequence of structures. The proof of Lemma 5.1 uses the same machinery; namely, Ash's  $\alpha$ -systems. The reader who is not familiar with this machinery will find a thorough discussion in [4].

*Proof.* Suppose  $\alpha = \beta + 1$ . We give a uniform effective procedure for constructing  $\mathcal{C}_n$ . Let  $C$  be an infinite computable set of constants, for the universe. We have a  $\Delta_\beta^0$  function  $g_n : \omega \rightarrow \{0, 1\}$  which is eventually constant, with limit value 1 if  $n \in S$ , and 0 otherwise. We want

$$\mathcal{C}_n \cong \begin{cases} \mathcal{B}_0, & \text{if } \lim_k g_n(k) = 1, \\ \mathcal{B}_1, & \text{if } \lim_k g_n(k) = 0. \end{cases}$$

For simplicity, we suppose that  $\mathcal{B}_0, \mathcal{B}_1$  are structures for a finite relational language. Also, for convenience, we suppose that they have disjoint, computable universes.

To put ourselves in a position to apply Ash's metatheorem, we begin by defining a  $\beta$ -system  $(L, U, P, \hat{\ell}, E, (\leq_\gamma)_{\gamma < \beta})$ . Let  $L$  be the set of all finite partial 1 – 1 functions from  $C$  to  $\mathcal{B}_0$ , or  $\mathcal{B}_1$ . Let  $U = \{0, 1\}$ . Let  $\hat{\ell} = \emptyset$ . For  $\ell \in L$ , let  $E(\ell)$  be the set of atomic sentences and negations of atomic sentences that  $\ell$  makes true in  $\mathcal{B}_0$ , or  $\mathcal{B}_1$ . Let  $P$  be the set of finite alternating sequences  $\ell_0 u_1 \ell_1 u_2 \ell_2 \dots$  (ending with an element of  $U$  or  $L$ ) such that

1.  $\ell_0 = \hat{\ell}$ ,
2.  $u_k \in U$ , and  $\ell_k \in L$ ,

3.  $\text{dom}(\ell_k)$  includes the first  $k$  elements of  $C$ ,
4. if  $u_k = 0$ , then  $\text{ran}(\ell_k) \subseteq \mathcal{B}_0$ , if  $u_k = 1$ , then  $\text{ran}(\ell_k) \subseteq \mathcal{B}_1$ , and in either case,  $\text{ran}(\ell_k)$  includes the first  $k$  elements of the structure,
5. if  $u_k = u_{k+1}$ , then  $\ell_k \subseteq \ell_{k+1}$ .

For  $\ell, \ell' \in L$ , we let  $\ell \leq_0 \ell'$  if  $E(\ell) \subseteq E(\ell')$ , and for  $0 < \gamma < \beta$ , we let  $\ell \leq_\gamma \ell'$  if  $\text{dom}(\ell) \subseteq \text{dom}(\ell')$ , and for any extension  $\mu'$  of  $\ell'$  and any  $\delta < \gamma$ , there is an extension  $\mu$  of  $\ell$ , such that  $\mu' \leq_\delta \mu$ . These are the standard back-and-forth relations.

We have defined the  $\beta$ -system. We can show that Ash's four conditions are satisfied.

1.  $\ell \subseteq_0 \ell'$  implies  $E(\ell) \subseteq E(\ell')$ ,
2.  $\ell \leq_\gamma \ell'$  implies  $\ell \leq_\delta \ell'$  if  $\gamma > \delta$ ,
3.  $\leq_\gamma$  is transitive and reflexive,
4. if  $\sigma \ell^0 u \in P$ ,  $\ell^0 \leq_{\gamma_0} \ell^1 \leq_{\gamma_1} \dots \leq_{\gamma_{k-1}} \ell^k$ , and  $\beta > \gamma_0 > \gamma_1 > \dots > \gamma_{k-1} > \gamma_k$ , then there exists  $\ell^*$  such that  $\sigma \ell^0 u \ell^* \in P$  and for  $i = 0, 1, \dots, k$ , we have  $\ell^i \leq_{\gamma_i} \ell^*$ .

The first three conditions are obvious. For Condition 4, the important thing is that for any  $\gamma < \beta$  and any extension  $\mu$ , there is  $\mu'$ , with range in the opposite structure, such that  $\mu \leq_\gamma \mu'$ .

Next, we define a  $\Delta_\beta^0$  instruction function  $g_n^*$  (related to the function  $g_n$ ), such that if  $\sigma \in P$ , where  $\sigma$  has length  $2k + 1$ , then  $g_n^*(\sigma) = \lim_k g_n(k)$ . A *run* of  $(P, g_n^*)$  is an infinite path  $\pi = \ell_0 u_1 \ell_1 \dots$  through  $P$  in which the terms from  $U$  are given by  $g_n^*$ . For the run  $\pi$ ,  $F^{-1} = \cup_k \ell_k$  is a 1-1 function from  $C$  onto the desired structure  $\mathcal{B}_i$ , with inverse  $F$ . If  $\mathcal{C}_n$  is the structure induced by  $F$  on  $C$ , then

$$E(\pi) = \cup_k E(\ell_k) = D(\mathcal{C}_n) .$$

By Ash's metatheorem, there is a run  $\pi$  such that  $E(\pi)$  is c.e. Then the resulting  $\mathcal{C}_n$  is computable. Moreover, the uniformity in the metatheorem means that given  $n$ , we can effectively find a computable index for  $\mathcal{C}_n$ .  $\square$

We need pairs of structures  $\mathcal{B}_i$  satisfying the hypotheses of Lemma 5.1. In addition, each  $\mathcal{B}_i$  will be *uniformly relatively*  $\Delta_\alpha^0$  *categorical*; i.e., given an  $X$ -computable index for  $\mathcal{C} \cong \mathcal{B}_i$ , we can find a  $\Delta_\alpha^0(X)$  index for an isomorphism from  $\mathcal{B}_i$  onto  $\mathcal{C}$ . By the comments following Theorem 1.1, to show that a structure  $\mathcal{B}$  is uniformly relatively  $\Delta_\alpha^0$  categorical, it is enough to show that it has a formally  $\Sigma_\alpha^0$  Scott family  $\Phi$  with no parameters. We introduce some notation to describe certain structures. If  $\mathcal{C}_1, \mathcal{C}_2$  are structures for the same relational language, we write  $\mathcal{C}_1 | \mathcal{C}_2$  for the cardinal sum, where this includes unary predicates for the two universes.

**Proposition 5.2.** *For each computable successor ordinal  $\alpha \geq 2$ , there exist  $\mathcal{B}_0, \mathcal{B}_1$  such that*

1. the pair  $\{\mathcal{B}_0, \mathcal{B}_1\}$  is  $\alpha$ -friendly,

2.  $\mathcal{B}_0, \mathcal{B}_1$  satisfy the same  $\Pi_\beta$  sentences (of  $L_{\omega_1\omega}$ ) for  $\beta < \alpha$ ,
3. each  $\mathcal{B}_i$  ( $i = 0, 1$ ) satisfies some computable  $\Pi_\alpha$  sentence that is not true in the other,
4. each  $\mathcal{B}_i$  is uniformly relatively  $\Delta_\alpha^0$  categorical.

We note that if  $\alpha$  is a limit ordinal, then structures that satisfy the same  $\Pi_\beta$  formulas for all  $\beta < \alpha$  also satisfy the same  $\Pi_\alpha$  formulas. Therefore, there is no possibility of extending Proposition 5.2 to limit ordinals. Since proving Proposition 5.2 immediately would disrupt the flow of the argument, we present the structures that satisfy the proposition, but we delay proving that these structures have the required properties until Section 7. The structures we use are all either linear orders or cardinal sums of linear orders. We use  $\omega$  to denote the order type of the natural numbers, and  $\mathbb{Z}$  to denote the order type of the integers. For any ordering  $\xi$ , we write  $\xi^*$  for the reverse ordering. For any ordinal  $\delta > 0$ , we define

$$\xi_\delta = \sum_{\gamma < \delta} \mathbb{Z}^\gamma \cdot \omega.$$

We treat  $\mathbb{Z}^0$  as a single point, so  $\xi_\delta$  is

$$\xi_\delta = \omega + \mathbb{Z} \cdot \omega + \mathbb{Z}^2 \cdot \omega + \cdots + \mathbb{Z}^\gamma \cdot \omega + \cdots .$$

Now, we describe the pairs of structures corresponding to successor ordinals  $\alpha \geq 2$ . For  $\alpha = 2$ , we let  $\mathcal{B}_0, \mathcal{B}_1$  be orderings of type  $\omega$  and  $\omega^*$ . For a successor ordinal  $\alpha > 2$ , we can write  $\alpha$  as either  $\gamma + 2n + 1$  or  $\gamma + 2n + 2$ , where  $n \in \omega$  and  $\gamma$  is either a limit ordinal or zero. Notice that if  $\gamma = 0$ , then  $n \geq 1$ . If we let  $\beta = \gamma + n$ , then  $2\beta + 1 = 2\gamma + 2n + 1 = \gamma + 2n + 1$ . Similarly,  $2\beta + 2 = \gamma + 2n + 2$ . So, to consider all successor ordinals  $\alpha > 2$ , it suffices to look at  $2\beta + 1$  and  $2\beta + 2$  for all  $\beta \geq 1$ .

For  $2\beta + 1$ , we use the cardinal sums  $\xi_\beta | (\xi_\beta + \mathbb{Z}^\beta)$  and  $(\xi_\beta + \mathbb{Z}^\beta) | \xi_\beta$ . For example, for  $\alpha = 3$ , when  $\beta = 1$ , we use  $\omega | (\omega + \mathbb{Z})$  and  $(\omega + \mathbb{Z}) | \omega$ , and for  $\alpha = 5$ , when  $\beta = 2$ , we use

$$(\omega + \mathbb{Z} \cdot \omega) | (\omega + \mathbb{Z} \cdot \omega + \mathbb{Z}^2) \text{ and } (\omega + \mathbb{Z} \cdot \omega + \mathbb{Z}^2) | (\omega + \mathbb{Z} \cdot \omega).$$

To meet the conditions of Proposition 5.2, it suffices to prove the following lemma.

**Lemma 5.3.** *For all  $\beta \geq 1$ ,*

1.  $\xi_\beta | (\xi_\beta + \mathbb{Z}^\beta) \equiv_{2\beta} (\xi_\beta + \mathbb{Z}^\beta) | \xi_\beta$ ,
2. each of  $\xi_\beta | (\xi_\beta + \mathbb{Z}^\beta)$  and  $(\xi_\beta + \mathbb{Z}^\beta) | \xi_\beta$  satisfies a computable  $\Pi_{2\beta+1}$  sentence not true in the other,
3. the pair  $\{\xi_\beta | (\xi_\beta + \mathbb{Z}^\beta), (\xi_\beta + \mathbb{Z}^\beta) | \xi_\beta\}$  is  $(2\beta + 1)$ -friendly,
4.  $\xi_\beta | (\xi_\beta + \mathbb{Z}^\beta), (\xi_\beta + \mathbb{Z}^\beta) | \xi_\beta$  are uniformly relatively  $\Delta_{2\beta+1}^0$  categorical.

For  $2\beta + 2$ , we use the orders  $\mathbb{Z}^\beta \cdot \omega$  and  $\mathbb{Z}^\beta \cdot \omega^*$ . For example, for  $\alpha = 4$ , when  $\beta = 1$ , we use  $\mathbb{Z} \cdot \omega$  and  $\mathbb{Z} \cdot \omega^*$ , and for  $\alpha = 6$ , when  $\beta = 2$ , we use  $\mathbb{Z}^2 \cdot \omega$  and  $\mathbb{Z}^2 \cdot \omega^*$ . To meet the conditions of Proposition 5.2, it suffices to prove the following lemma.

**Lemma 5.4.** *For all  $\beta \geq 1$ ,*

1.  $\mathbb{Z}^\beta \cdot \omega \equiv_{2\beta+1} \mathbb{Z}^\beta \cdot \omega^*$ ,
2. *each of  $\mathbb{Z}^\beta \cdot \omega$  and  $\mathbb{Z}^\beta \cdot \omega^*$  satisfies a computable  $\Pi_{2\beta+2}$  sentence not true in the other,*
3. *the pair  $\{\mathbb{Z}^\beta \cdot \omega, \mathbb{Z}^\beta \cdot \omega^*\}$  is  $(2\beta + 2)$ -friendly,*
4.  $\mathbb{Z}^\beta \cdot \omega, \mathbb{Z}^\beta \cdot \omega^*$  *are uniformly relatively  $\Delta_{2\beta+2}^0$  categorical.*

We now continue with the general argument, delaying the proofs of Lemmas 5.3 and 5.4 until Section 7.

**Lemma 5.5.** *Let  $\alpha$  be a computable successor ordinal, and let  $\mathcal{B}_0, \mathcal{B}_1$  be as in Proposition 5.2. Suppose  $\mathcal{G}$  is a graph structure, and  $\mathcal{G}^*$  is constructed from  $\mathcal{G}, \mathcal{B}_i$  in the way that was described at the beginning of this section. Then  $\mathcal{G}$  has a  $\Delta_\alpha^0$  copy iff  $\mathcal{G}^*$  has a computable copy. More generally, for any  $X \subseteq \omega$ ,  $\mathcal{G}$  has a  $\Delta_\alpha^0(X)$  copy iff  $\mathcal{G}^*$  has an  $X$ -computable copy. In addition,*

- (a) *if  $\mathcal{G}$  has just one  $\Delta_\alpha^0$  copy, up to  $\Delta_\alpha^0$  isomorphism, then  $\mathcal{G}^*$  is  $\Delta_\alpha^0$  categorical,*
- (b) *if  $\mathcal{G}$  has just  $n$   $\Delta_\alpha^0$  copies, up to  $\Delta_\alpha^0$  isomorphism, then  $\mathcal{G}^*$  has  $\Delta_\alpha^0$  dimension  $n$ ,*
- (c) *if  $\mathcal{G}$  has no  $\Sigma_\alpha^0$  Scott family consisting of finitary existential formulas, then  $\mathcal{G}^*$  has no formally  $\Sigma_\alpha^0$  Scott family.*

*Proof.* For (a), suppose that  $\mathcal{G}^*$  is computable, let  $\mathcal{H}^*$  be a computable copy of  $\mathcal{G}^*$ , and let  $\mathcal{H}$  be the image of  $\mathcal{G}$  under the isomorphism. From the computable  $\Pi_\alpha$  sentences distinguishing the structures  $\mathcal{B}_i$ , we get computable  $\Pi_\alpha$  and  $\Sigma_\alpha$  definitions of the relation  $\rightarrow$  on  $G$ . Now,  $\mathcal{H}$  is  $\Delta_\alpha^0$  (the universe is computable and  $\rightarrow$  is  $\Delta_\alpha^0$ ). Therefore, there is a  $\Delta_\alpha^0$  isomorphism  $f$  from  $\mathcal{G}$  onto  $\mathcal{H}$ . For each pair  $(a, b)$  in  $\mathcal{G}$ , we can effectively determine a computable index for the structure  $\mathcal{U}_{(f(a), f(b))}$  corresponding to  $\mathcal{U}_{(a, b)}$ , and using  $\Delta_\alpha^0$ , we can determine whether it is a copy of  $\mathcal{B}_1$  or  $\mathcal{B}_2$ . Since  $\mathcal{B}_i$  is uniformly  $\Delta_\alpha^0$  categorical, we can effectively find a  $\Delta_\alpha^0$  index for an isomorphism  $f_{(a, b)}$  from  $\mathcal{U}_{(a, b)}$  onto  $\mathcal{U}_{(f(a), f(b))}$ . Then the union of  $f$  and the functions  $f_{(a, b)}$  is a  $\Delta_\alpha^0$  isomorphism from  $\mathcal{G}^*$  onto  $\mathcal{H}^*$ .

For (b), let  $\mathcal{H}_i^*$ ,  $i < n$ , be computable copies of  $\mathcal{G}^*$ , and let  $\mathcal{H}_i$  be the  $\Delta_\alpha^0$  copy of  $\mathcal{G}$  definable in  $\mathcal{H}^*$ . If  $f$  is a  $\Delta_\alpha^0$  isomorphism from  $\mathcal{H}_i^*$  onto  $\mathcal{H}_j^*$ , then by restricting  $f$ , we get a  $\Delta_\alpha^0$  isomorphism from  $\mathcal{H}_i$  onto  $\mathcal{H}_j$ . If  $g$  is a  $\Delta_\alpha^0$  isomorphism from  $\mathcal{H}_i$  onto  $\mathcal{H}_j$ , then we can extend  $g$ , as in part (a), to a  $\Delta_\alpha^0$  isomorphism from  $\mathcal{H}_i^*$  onto  $\mathcal{H}_j^*$ . From this, (b) is clear.

The proof of (c) requires some model-theoretic effort. The main idea is that from the point of view of  $\Delta_\alpha^0$ , there is no extra structure on  $G$  in  $\mathcal{G}^*$  beyond the coded graph structure.

**Claim 1:** Let  $\bar{a}, \bar{b}$  and  $\bar{a}', \bar{b}'$  be two pairs of tuples from  $\mathcal{G}^*$  with the following properties:  $|\bar{a}| = |\bar{a}'|$ ;  $|\bar{b}| = |\bar{b}'|$ ;  $\bar{a}$  and  $\bar{a}'$  are in  $G$ ; each  $d$  from  $\bar{b}$  is in  $U_{(a_1, a_2)}$  for some pair  $a_1, a_2 \in \bar{a}$  and each  $d'$  in  $\bar{b}'$  is similarly connected to some pair of elements from  $\bar{a}'$ ; if two elements of  $\bar{a}$  or  $\bar{b}$  are equal, then so are the corresponding elements in  $\bar{a}'$  or  $\bar{b}'$  and *vice versa*; and if  $d \in \bar{b}$

is in  $U_{(a_1, a_2)}$ , where  $a_1, a_2 \in \bar{a}$ , then  $d' \in U_{(a'_1, a'_2)}$  for the corresponding  $d' \in \bar{b}'$  and  $a'_1, a'_2 \in \bar{a}'$  and *vice versa*. Fix any  $\beta < \alpha$ . Suppose that for all  $a_1, a_2 \in \bar{a}$ , if  $\bar{d}$  is the part of  $\bar{b}$  in  $U_{(a_1, a_2)}$  and  $\bar{d}'$  is the corresponding part of  $\bar{b}'$  in  $U_{(a'_1, a'_2)}$ , we have  $(U_{(a_1, a_2)}, \bar{d}) \leq_\beta (U_{(a'_1, a'_2)}, \bar{d}')$ . Then  $(\mathcal{G}^*, \bar{a}, \bar{b}) \leq_\beta (\mathcal{G}^*, \bar{a}', \bar{b}')$ .

Proof of Claim 1: The proof proceeds by induction on  $\beta$ . The base case, where  $\beta = 1$ , is easy to check—we use the fact that  $\mathcal{B}_i \leq_1 \mathcal{B}_j$ . Assuming that the claim holds for all  $\gamma < \beta$ , we prove it for  $\beta$ . Let  $\bar{v}'$  be a tuple in  $\mathcal{G}^*$  which breaks into  $\bar{v}'_1 \in G$  and  $\bar{v}'_2 \in U$ . It suffices to prove that for any given  $\gamma < \beta$ , there is a tuple  $\bar{v}$  which breaks into  $\bar{v}_1 \in G$  and  $\bar{v}_2 \in U$  such that  $(\mathcal{G}^*, \bar{a}', \bar{b}', \bar{v}') \leq_\gamma (\mathcal{G}^*, \bar{a}, \bar{b}, \bar{v})$ . We assume, without loss of generality, that the tuples  $\bar{a}'$  and  $\bar{v}'_1$  are disjoint and the tuples  $\bar{b}'$  and  $\bar{v}'_2$  are disjoint. Furthermore, we assume that for any  $y \in \bar{v}'_2$ , there are  $x_1, x_2$  from  $\bar{a}', \bar{v}'_1$  such that  $y \in U_{(x_1, x_2)}$ . We can achieve this property by slightly expanding our tuples.

Let  $\bar{v}_1$  be a tuple of elements of  $G$ , disjoint from  $\bar{a}$ , and such that  $|\bar{v}_1| = |\bar{v}'_1|$ . Consider an arbitrary pair of distinct elements  $r', s'$  from  $\bar{a}', \bar{v}'_1$ . Let  $\bar{d}'$  be the tuple of elements from  $\bar{b}'$  which are in  $U_{(r', s')}$  (this tuple is empty unless  $r', s'$  are both from  $\bar{a}'$ ) and let  $\bar{w}'$  be the tuple of elements from  $\bar{v}'_2$  which are in  $U_{(r', s')}$ . Let  $r, s$  be the elements of  $\bar{a}, \bar{v}_1$  which correspond to  $r'$  and  $s'$ , and let  $\bar{d}$  be the tuple of elements from  $\bar{b}$  which are in  $U_{(r, s)}$  (this tuple is empty unless  $r, s$  are both from  $\bar{a}$ ). If  $r', s'$  are both from  $\bar{a}'$ , then  $(U_{(r, s)}, \bar{d}) \leq_\beta (U_{(r', s')}, \bar{d}')$  by the hypothesis of the claim. Otherwise,  $\bar{d}$  and  $\bar{d}'$  are empty and, by the properties of our coding structures (since  $\beta < \alpha$ ), we have  $U_{(r, s)} \leq_\beta U_{(r', s')}$ . In either case, there is a tuple of elements  $\bar{w}$  from  $U_{(r, s)}$  such that  $(U_{(r', s')}, \bar{d}', \bar{w}') \leq_\gamma (U_{(r, s)}, \bar{d}, \bar{w})$ . We declare the tuple  $\bar{w}$  to be the part of  $\bar{v}_2$  that corresponds to  $\bar{w}'$ , as a part of  $\bar{v}'_2$ .

We repeat this process for each pair  $r', s'$  from  $\bar{a}', \bar{v}'_1$  to build  $\bar{v}_2$ . Notice that once  $\bar{v}_2$  is completed, we have satisfied the hypotheses of this claim with the sequences  $\bar{a}', \bar{v}'_1, \bar{b}', \bar{v}'_2$  and  $\bar{a}, \bar{v}_1, \bar{b}, \bar{v}_2$  and the ordinal  $\gamma$ . Then, by the induction hypothesis,  $(\mathcal{G}^*, \bar{a}', \bar{b}', \bar{v}') \leq_\gamma (\mathcal{G}^*, \bar{a}, \bar{b}, \bar{v})$ , as required. This completes the proof of Claim 1.

Now, suppose  $\Phi^*$  is a formally  $\Sigma_\alpha^0$  Scott family for  $\mathcal{G}^*$ . Let  $\bar{c}$  denote the set of parameters in these formulas and assume  $\bar{c}$  is split into  $\bar{c}_1 \in G$  and  $\bar{c}_2 \in U$ . We can assume that for each  $y$  from  $\bar{c}_2$ , there are  $x_1, x_2$  from  $\bar{c}_1$  such that  $y \in U_{(x_1, x_2)}$ . To arrive at a contradiction, we produce a  $\Sigma_\alpha^0$  Scott family of finitary existential formulas for  $\mathcal{G}$ . We use  $\Delta_\alpha^0$  as an oracle and give an effective list of this Scott family of finitary existential formulas. For any tuple  $\bar{a} \in G$ , using  $\Delta_\alpha^0$ , we can find a computable  $\Sigma_\alpha$  formula  $\varphi(\bar{c}, \bar{x}) \in \Phi^*$  such that  $\mathcal{G} \models \varphi(\bar{c}, \bar{a})$ . We may suppose that  $\varphi(\bar{c}, \bar{x})$  has the form  $(\exists \bar{u}) \psi(\bar{c}, \bar{x}, \bar{u})$ , where  $\psi$  is computable  $\Pi_\beta$  for some  $\beta < \alpha$ .

Using the oracle  $\Delta_\alpha^0$ , we can find  $\bar{b}$  such that  $\mathcal{G} \models \psi(\bar{c}, \bar{a}, \bar{b})$ . Say  $\bar{b} = \bar{b}_1, \bar{b}_2$ , where  $\bar{b}_1 \in G$  and  $\bar{b}_2 \in U$ . Expanding the tuples, if necessary, we may assume that for each  $y$  from  $\bar{b}_2$ , the “parents” of  $y$  are in  $\bar{c}_1, \bar{a}, \bar{b}_1$ ; i.e.,  $y \in U_{(x_1, x_2)}$  for  $x_1, x_2$  in  $\bar{c}_1, \bar{a}, \bar{b}_1$ . Using  $\Delta_\alpha^0$ , we can determine, for each pair of points  $(a_1, a_2)$  from  $\bar{a}, \bar{b}_1, \bar{c}_1$ , whether there is an arrow from  $a_1$  to  $a_2$  in  $\mathcal{G}$ . Let  $\delta(\bar{c}_1, \bar{x}, \bar{u}_1)$  be a finitary formula (in the language of  $\mathcal{G}$ ) that describes the graph structure on  $\bar{c}_1, \bar{a}, \bar{b}_1$ . Notice that  $\bar{a}$  (as a tuple in  $\mathcal{G}$ ) satisfies the finitary existential formula  $\exists \bar{u}_1 \delta(\bar{c}_1, \bar{x}, \bar{u}_1)$ .

**Claim 2:** If  $\mathcal{G} \models (\exists \bar{u}_1) \delta(\bar{c}_1, \bar{a}', \bar{u}_1)$ , then there is an automorphism of  $\mathcal{G}$  that fixes  $\bar{c}_1$  and takes  $\bar{a}$  to  $\bar{a}'$ .

Proof of Claim 2: Assume that  $\mathcal{G} \models \delta(\bar{c}_1, \bar{a}', \bar{b}'_1)$ . Suppose there is a tuple  $\bar{b}'_2$  from  $U$  such that for  $\bar{b} = \bar{b}'_1, \bar{b}'_2$  we have  $(\mathcal{G}^*, \bar{c}, \bar{a}, \bar{b}) \leq_\beta (\mathcal{G}^*, \bar{c}, \bar{a}', \bar{b}')$ . Then,  $\mathcal{G}^* \models \psi(\bar{c}, \bar{a}', \bar{b}')$  and so  $\mathcal{G}^* \models \varphi(\bar{c}, \bar{a}')$ . Therefore, there is an automorphism of  $\mathcal{G}^*$  that fixes  $\bar{c}$  and takes  $\bar{a}$  to  $\bar{a}'$ . However, any automorphism of  $\mathcal{G}^*$  induces an automorphism of  $\mathcal{G}$ , so  $\bar{a}$  and  $\bar{a}'$  are automorphic in  $\mathcal{G}$ , as required.

It remains to show that there is an appropriate tuple  $\bar{b}'_2$ . We choose  $\bar{b}'_2$  so that we can apply Claim 1 to the sequences:  $\bar{c}_1, \bar{c}_2, \bar{a}, \bar{b}_1, \bar{b}_2$  and  $\bar{c}_1, \bar{c}_2, \bar{a}', \bar{b}'_1, \bar{b}'_2$ . Since  $\bar{c}_1, \bar{a}, \bar{b}_1$  and  $\bar{c}_1, \bar{a}', \bar{b}'_1$  both satisfy  $\delta$ , equality and the graph relation  $\rightarrow$  are preserved between these two tuples. Let  $u, v$  be a pair of distinct elements in  $\bar{c}_1, \bar{a}, \bar{b}_1$ , and let  $u', v'$  be the corresponding pair in  $\bar{c}_1, \bar{a}', \bar{b}'_1$ . Since  $\mathcal{G} \models u \rightarrow v$  iff  $\mathcal{G} \models u' \rightarrow v'$ , we have  $U_{(u,v)} \cong U_{(u',v')}$ . If  $\bar{d}$  is the part of  $\bar{b}_2$  in  $U_{(u,v)}$ , then we can choose  $\bar{d}'$  in  $U_{(u',v')}$  such that  $(U_{(u,v)}, \bar{d}) \leq_\beta (U_{(u',v')}, \bar{d}')$ . Let  $\bar{b}'_2$  be the result of combining the chosen tuples  $\bar{d}'$  in the appropriate way. Now, we can apply Claim 1 to get  $(\mathcal{G}^*, \bar{c}, \bar{a}, \bar{b}) \leq_\beta (\mathcal{G}^*, \bar{c}, \bar{a}', \bar{b}')$ , as required. This completes the proof of Claim 2.

Now, we let  $\Phi$  consist of the formulas  $(\exists \bar{u}) \delta(\bar{c}, \bar{x}, \bar{u})$ , obtained as above. This is a  $\Sigma_\alpha^0$  Scott family for  $\mathcal{G}$ , consisting of existential formulas. This contradiction completes the proof of (c).  $\square$

## 6 Lifting the basic results

Here is our lifting of the result of Goncharov on structures that are computably categorical but not relatively computably categorical.

**Theorem 6.1.** *For each computable successor ordinal  $\alpha$ , there is a structure that is  $\Delta_\alpha^0$  categorical but not relatively  $\Delta_\alpha^0$  categorical.*

*Proof.* We relativize Theorems 2.1 and 4.1 to  $\Delta_\alpha^0$ , getting a rigid  $\Delta_\alpha^0$  graph structure  $\mathcal{G}$  such that

- (1)  $\mathcal{G}$  has just one  $\Delta_\alpha^0$  copy, up to  $\Delta_\alpha^0$  isomorphism,
- (2)  $\mathcal{G}$  has no  $\Sigma_\alpha^0$  Scott family consisting of finitary existential formulas.

Next, we apply Lemma 5.5 to pass from  $\mathcal{G}$  to a computable structure  $\mathcal{G}^*$  that is  $\Delta_\alpha^0$  categorical, with no formally  $\Sigma_\alpha^0$  Scott family. By Theorem 1.2, it follows that  $\mathcal{G}^*$  is not relatively  $\Delta_\alpha^0$  categorical.  $\square$

Here is our lifting of the result of Manasse on relations that are intrinsically c.e. but not relatively intrinsically c.e.

**Theorem 6.2.** *For each computable successor ordinal  $\alpha$ , there is a computable structure with a relation that is intrinsically  $\Sigma_\alpha^0$  but not relatively intrinsically  $\Sigma_\alpha^0$ .*



*Proof.* Let  $\mathcal{G}$  and  $\mathcal{G}^*$  be as in the proof of Theorem 6.1. Let  $\mathcal{A}$  be the cardinal sum of two copies of  $\mathcal{G}^*$  and let  $R$  be the unique isomorphism between the associated copies of  $\mathcal{G}$ . Suppose  $\mathcal{B}$  is a computable copy of  $\mathcal{A}$ , say  $\mathcal{B}$  is the cardinal sum of  $\mathcal{H}_1^*$  and  $\mathcal{H}_2^*$ , and let  $\mathcal{H}_i$  be the copy of  $\mathcal{G}$  associated with  $\mathcal{H}_i^*$ . The structures  $\mathcal{H}_i$  are  $\Delta_\alpha^0$ , and the image of  $R$ —the unique isomorphism from  $\mathcal{H}_1$  onto  $\mathcal{H}_2$ —is  $\Sigma_\alpha^0$ . Therefore,  $R$  is intrinsically  $\Sigma_\alpha^0$ .

We must show that  $R$  is not relatively intrinsically  $\Sigma_\alpha^0$  on  $\mathcal{G}^*$ . Supposing that it is, we arrive at a contradiction by showing that  $\mathcal{G}^*$  is relatively  $\Delta_\alpha^0$  categorical. Let  $\mathcal{H}^*$  be a copy of  $\mathcal{G}^*$ , and let  $\mathcal{H}$  be the associated copy of  $\mathcal{G}$ . We may suppose that  $\mathcal{G}^*$  and  $\mathcal{H}^*$  are disjoint and form the cardinal sum  $\mathcal{B} \cong \mathcal{A}$ . By our assumption, the image of  $R$ , the unique isomorphism from  $\mathcal{G}$  onto  $\mathcal{H}$ , is  $\Sigma_\alpha^0(\mathcal{H}^*)$ . Now, we extend  $R$  to an isomorphism  $f$  from  $\mathcal{G}^*$  onto  $\mathcal{H}^*$ , still  $\Delta_\alpha^0(\mathcal{H}^*)$ .

For each pair  $a, b$  in  $\mathcal{G}$ , we can find the  $R$ -images  $a', b'$  in  $\mathcal{H}$ . Let  $\mathcal{U}_{(a,b)}$  be the structure  $(U_{(a,b)}, \dots)$ , and let  $\mathcal{V}_{(a',b')}$  be the corresponding part of  $\mathcal{H}^*$ . Using  $\Delta_\alpha^0$ , we can determine whether  $\mathcal{G} \models a \rightarrow b$ , so we can determine which  $\mathcal{B}_i$  is isomorphic to  $\mathcal{U}_{(a,b)}$ . We can find an  $\mathcal{H}^*$ -computable index for the corresponding structure  $\mathcal{V}_{(a',b')}$ . Since  $\mathcal{B}_i$  is uniformly relatively  $\Delta_\alpha^0$  categorical, we can find a  $\Delta_\alpha^0(\mathcal{H}^*)$  index for an isomorphism  $f_{(a,b)}$  from  $\mathcal{U}_{(a,b)}$  onto  $\mathcal{V}_{(a',b')}$ . The union of  $R$  with these  $f_{(a,b)}$  is a  $\Delta_\alpha^0(\mathcal{H}^*)$  isomorphism from  $\mathcal{G}^*$  onto  $\mathcal{H}^*$ . Therefore,  $\mathcal{G}^*$  is relatively  $\Delta_\alpha^0$  categorical, a contradiction.  $\square$

Here is our lifting of the result of Goncharov on structures with finite computable dimension.

**Theorem 6.3.** *For each computable successor ordinal  $\alpha$  and each finite  $n$ , there is a computable structure with  $\Delta_\alpha^0$  dimension  $n$ .*

*Proof.* First, we relativize Theorems 2.2 and 4.3 to  $\Delta_\alpha^0$ , getting a rigid graph structure  $\mathcal{G}$  with just  $n$   $\Delta_\alpha^0$  copies, up to  $\Delta_\alpha^0$  isomorphism. Then we apply Lemma 5.5 to pass from  $\mathcal{G}$  to a computable structure  $\mathcal{G}^*$  with  $\Delta_\alpha^0$  dimension  $n$ .  $\square$

Here is our lifting of the result of Slaman and Wehner.

**Theorem 6.4.** *For each computable successor ordinal  $\alpha$ , there is a structure with copies in just the Turing degrees of sets  $X$  such that  $\Delta_\alpha^0(X)$  is not  $\Delta_\alpha^0$ . In particular, for each finite  $n$ , there is a structure with copies in just the non- $\text{low}_n$  degrees.*

*Proof.* We can relativize Theorems 2.3 and 4.4 to  $\Delta_\alpha^0$ , getting a graph structure  $\mathcal{G}$  (not rigid) such that the degrees of copies of  $\mathcal{G}$  are just the degrees of sets that are not  $\Delta_\alpha^0$ . Next, we apply Lemma 5.5 to pass from  $\mathcal{G}$  to a structure  $\mathcal{G}^*$ , where the degrees of copies of  $\mathcal{G}^*$  are just the degrees of sets  $X$  such that  $\Delta_\alpha^0(X)$  is not  $\Delta_\alpha^0$ . If  $\alpha = n + 1$ , where  $n$  is finite, then the degrees of copies of  $\mathcal{G}^*$  are the degrees of sets  $X$  such that  $\Delta_{n+1}^0(X)$  is not  $\Delta_{n+1}^0$ ; i.e.,  $X^{(n)} \not\leq_T \emptyset^{(n)}$ .  $\square$

## 7 Pairs of structures

In this section, we prove Proposition 5.2. Recall that we broke the proof into three parts. We need to verify that the orderings  $\omega$  and  $\omega^*$  work for the case  $\alpha = 2$ , and we need to

prove Lemmas 5.3 and 5.4 (which are restated below as Lemmas 7.1 and 7.2). The analysis of various order types draws heavily on the work of Ash [2].

First, we consider the orderings  $\omega$  and  $\omega^*$ . The orderings can be distinguished by finitary  $\Pi_2$  sentences saying that there is no first, or last, element. Since both orderings are infinite, we have  $\omega \leq_1 \omega^*$  and  $\omega^* \leq_1 \omega$ . Each ordering is rigid, with a c.e. defining family consisting of finitary  $\Sigma_2$  formulas  $\varphi_n(x)$  saying that there are exactly  $n$  elements to the left, or right, of  $x$ . Similarly, any tuple of elements  $\bar{x}$  in  $\omega$  or  $\omega^*$  can be defined by a conjunction of such formulas. These properties imply that  $\omega$  and  $\omega^*$  each have a formally  $\Sigma_1^0$  Scott family without parameters.

To see that  $\{\omega, \omega^*\}$  is 2-friendly, fix computable copies of  $\omega$  and  $\omega^*$  in which we can determine the size of the interval  $(x, y)$  for any  $x < y$ . Recall the following two facts concerning the  $\leq_\gamma$  relations on linear orders. First,  $L_0 \leq_1 L_1$  if and only if either both orders are infinite or  $L_0$  is at least as big as  $L_1$ . Second,  $(L_0, \bar{a}) \leq_\gamma (L_1, \bar{b})$  iff, writing  $L_0 = A_0 + a_1 + A_1 + \dots + a_n + A_n$  and  $L_1 = B_0 + b_1 + B_1 + \dots + b_n + B_n$ , we have  $A_i \leq_\gamma B_i$ , for all  $i = 0, \dots, n$ . From these two facts and the existence of our nice copies of  $\omega$  and  $\omega^*$ , it is clear that we can enumerate the  $\leq_1$  relation between tuples in these models.

In the rest of this section, we will prove Lemmas 5.3 and 5.4. Recall that for any ordinal  $\delta > 0$ ,

$$\xi_\delta = \sum_{\gamma < \delta} \mathbb{Z}^\gamma \cdot \omega = \omega + \mathbb{Z} \cdot \omega + \mathbb{Z}^2 \cdot \omega + \dots + \mathbb{Z}^\gamma \cdot \omega + \dots .$$

We will also use the ordering  $\eta_\delta = \xi_\delta + \xi_\delta^*$ . Whenever we mention  $\xi_\delta$  or  $\eta_\delta$ , we assume that  $\delta > 0$ . We repeat the statements of the lemmas so the reader can avoid flipping back and forth between this section and Section 5.

**Lemma 7.1.** *For all  $\beta \geq 1$ ,*

1.  $\xi_\beta | (\xi_\beta + \mathbb{Z}^\beta) \equiv_{2\beta} (\xi_\beta + \mathbb{Z}^\beta) | \xi_\beta$ ,
2. each of  $\xi_\beta | (\xi_\beta + \mathbb{Z}^\beta)$  and  $(\xi_\beta + \mathbb{Z}^\beta) | \xi_\beta$  satisfies a computable  $\Pi_{2\beta+1}$  sentence not true in the other,
3. the pair  $\{\xi_\beta | (\xi_\beta + \mathbb{Z}^\beta), (\xi_\beta + \mathbb{Z}^\beta) | \xi_\beta\}$  is  $(2\beta + 1)$ -friendly,
4.  $\xi_\beta | (\xi_\beta + \mathbb{Z}^\beta), (\xi_\beta + \mathbb{Z}^\beta) | \xi_\beta$  are uniformly relatively  $\Delta_{2\beta+1}^0$  categorical.

**Lemma 7.2.** *For all  $\beta \geq 1$ ,*

1.  $\mathbb{Z}^\beta \cdot \omega \equiv_{2\beta+1} \mathbb{Z}^\beta \cdot \omega^*$ ,
2. each of  $\mathbb{Z}^\beta \cdot \omega$  and  $\mathbb{Z}^\beta \cdot \omega^*$  satisfies a computable  $\Pi_{2\beta+2}$  sentence not true in the other,
3.  $\{\mathbb{Z}^\beta \cdot \omega, \mathbb{Z}^\beta \cdot \omega^*\}$  is  $(2\beta + 2)$ -friendly,
4.  $\mathbb{Z}^\beta \cdot \omega, \mathbb{Z}^\beta \cdot \omega^*$  are uniformly relatively  $\Delta_{2\beta+2}^0$  categorical.

For the rest of this section, we use  $\Pi_\gamma^c$  to abbreviate the expression “computable  $\Pi_\gamma$ ”. We begin by working toward part 2 of Lemmas 7.1 and 7.2. We need to see how complicated certain statements are in the various structures we have introduced. We assume that all points  $x, y$  we discuss come from a structure of the form  $\mathbb{Z}^\delta, \mathbb{Z}^\delta \cdot \omega, \mathbb{Z}^\delta \cdot \omega^*, \xi_\delta, \xi_\delta + \mathbb{Z}^\delta, \eta_\delta$ , or the reverse ordering of one of these structures.

**Lemma 7.3.** *For points  $x \leq y$ , the statement “ $x, y$  are in the same copy of  $\mathbb{Z}^\alpha$ ” is  $\Sigma_{2\alpha}^c$  and the statement “there are exactly  $n$  copies of  $\mathbb{Z}^\alpha$  between  $x$  and  $y$ ” is  $\Sigma_{2\alpha+2}^c$ .*

*Proof.* We proceed by induction on  $\alpha$ . For  $\alpha = 1$ , “ $x, y$  are in the same  $\mathbb{Z}$ ” is given by

$$\bigvee_{n>0} (|[x, y]| = n)$$

where  $|[x, y]| = n$  is the standard finitary  $\Sigma_2$  statement saying that the closed interval between  $x$  and  $y$  has size  $n$ . The disjunction is clearly  $\Sigma_2^c$ , as required. We abbreviate this formula by  $x \sim_{\mathbb{Z}} y$ .

To say “there are exactly  $n$  copies of  $\mathbb{Z}$  between  $x$  and  $y$ ”, we say that there exist  $x_0 < x_1 < \dots < x_{n+1}$  such that  $x = x_0, y = x_{n+1}$ , and

$$\bigwedge_{i<n} (x_i \not\sim_{\mathbb{Z}} x_{i+1}) \& \bigwedge_{i<n} (\forall z) (x_i \leq z \leq x_{i+1} \rightarrow (x_i \sim_{\mathbb{Z}} z \vee z \sim_{\mathbb{Z}} x_{i+1})).$$

This formula has the form  $\exists(\Sigma_0 \wedge \Pi_2^c \wedge \Pi_3^c)$  and hence is  $\Sigma_4^c$ , as required. We abbreviate this formula by  $x \sim_{\mathbb{Z}, n} y$ . Notice that when we say “there are exactly  $n$  copies of  $\mathbb{Z}$  between  $x$  and  $y$ ,” we mean that there are  $n$  complete copies. That is, for  $n = 0$ , this sentence says that  $x$  and  $y$  are in the same or adjacent copies of  $\mathbb{Z}$ .

For  $\alpha = \delta + 1$ , we say “ $x, y$  are in same  $\mathbb{Z}^\alpha$ ” by

$$x \sim_{\mathbb{Z}^\delta} y \vee \bigvee_{n>0} (x \sim_{\mathbb{Z}^\delta, n} y).$$

Here, we are using the obvious generalizations of the abbreviations given above in the  $\alpha = 1$  case. This formula has the form  $\Sigma_{2\delta}^c \vee \Sigma_{2\delta+2}^c$  and hence is  $\Sigma_{2\alpha}^c$ , as required. We abbreviate this formula by  $x \sim_{\mathbb{Z}^\alpha} y$ .

To say that “there are exactly  $n$  copies of  $\mathbb{Z}^\alpha$  between  $x$  and  $y$ ” we say there exist  $x_0 < x_1 < \dots < x_{n+1}$  such that  $x = x_0$  and  $y = x_{n+1}$  and

$$\bigwedge_{i<n} (x_i \not\sim_{\mathbb{Z}^\alpha} x_{i+1}) \& \bigwedge_{i<n} (\forall z) (x_i \leq z \leq x_{i+1} \rightarrow (x_i \sim_{\mathbb{Z}^\alpha} z \vee z \sim_{\mathbb{Z}^\alpha} x_{i+1})).$$

This formula has the form  $\exists(\Sigma_0 \wedge \Pi_{2\alpha}^c \wedge \Pi_{2\alpha+1}^c)$  and hence is  $\Sigma_{2\alpha+2}^c$ , as required. We abbreviate this formula by  $x \sim_{\mathbb{Z}^\alpha, n} y$ .

For a limit ordinal  $\alpha$ , we say “ $x, y$  are in same  $\mathbb{Z}^\alpha$ ” by

$$\bigvee_{\delta<\alpha} (x \sim_{\mathbb{Z}^\delta} y).$$

Since  $x \sim_{\mathbb{Z}^\delta} y$  is  $\Sigma_{2\delta}^c$  and  $2\delta < \alpha$ , this formula is  $\Sigma_\alpha^c = \Sigma_{2\alpha}^c$ , as required. Saying “ $x, y$  have exactly  $n$  copies of  $\mathbb{Z}^\alpha$  between them” is exactly as in the successor ordinal case.  $\square$

**Lemma 7.4.** For  $\mathbb{Z}^\beta \cdot \omega$  and  $\mathbb{Z}^\beta \cdot \omega^*$ , the statements “there is a least copy of  $\mathbb{Z}^\beta$ ” and “there is a greatest copy of  $\mathbb{Z}^\beta$ ” are both  $\Sigma_{2\beta+2}^c$ .

*Proof.* “There is a least copy of  $\mathbb{Z}^\beta$ ” is

$$(\exists y) (\forall x < y) (x \sim_{\mathbb{Z}^\beta} y)$$

which is  $\Sigma_{2\beta+2}^c$ , as required. The formula for a greatest copy is similar.  $\square$

Lemma 7.4 proves 2 in Lemma 7.2. We delay verifying 2 for Lemma 7.1 until we have analyzed the complexity of some more statements. To check 1 from Lemmas 7.1 and 7.2, we need to understand the back-and-forth relations on our structures, which means that we need to understand how the structures can be partitioned by a finite number of points. We mention two useful facts, both of which were pointed out by Ash [2].

**Facts:** For any  $\gamma > 0$  and  $n \in \omega$ ,

1.  $\mathbb{Z}^\gamma = \xi_\gamma^* + 1 + \xi_\gamma$ ,
2.  $\xi_\gamma + \mathbb{Z}^\gamma \cdot n + \xi_\gamma^* = \eta_\gamma \cdot (n + 1)$ .

The first equality follows by a simple examination of the linear orders involved. The second equality is obtained by applying the first equality to rewrite  $\mathbb{Z}^\gamma \cdot n$  as  $(\xi_\gamma^* + 1 + \xi_\gamma) \cdot n$  and then rearranging the parentheses. Notice that before rearranging the parentheses, we can absorb the 1’s into either  $\xi_\gamma$  or  $\xi_\gamma^*$ .

We next examine the proper initial and final segments of the orders in which we are interested.

- For  $\mathbb{Z}^\beta$ , the only proper initial segment is  $\xi_\beta^*$ , and the only proper end segment is  $\xi_\beta$ .
- For  $\xi_\beta$ , the proper initial segments are either finite or  $\eta_\gamma \cdot (n + 1)$  for some  $\gamma < \beta$  and  $n \in \omega$ . The only proper final segment is  $\xi_\beta$ .
- For  $\mathbb{Z}^\beta \cdot \omega$ , the proper initial segments have the form  $\mathbb{Z}^\beta \cdot n + \xi_\beta^*$ . We can have  $n = 0$ , in which case the initial segment is just  $\xi_\beta^*$ . The only proper final segment is  $\xi_\beta + \mathbb{Z}^\beta \cdot \omega$ , which is, of course,  $\xi_{\beta+1}$ .
- Since  $\mathbb{Z}^\beta \cdot \omega^*$  is  $(\mathbb{Z}^\beta \cdot \omega)^*$ , the only proper initial segment is  $\xi_{\beta+1}^*$ , and the proper final segments have the form  $\xi_\beta + \mathbb{Z}^\beta \cdot n$ . Again, we can have  $n = 0$ , in which case the end segment is  $\xi_\beta$ .

We use this information to tell us how various orders can be partitioned by a finite number of points. For the first example, we provide a detailed explanation, and we leave the similar explanations for the rest to the reader. If  $\mathbb{Z}^\beta$  is partitioned by a finite number of points, then we have

$$\mathbb{Z}^\beta = \xi_\beta^* + 1 + \sigma_1 + 1 + \sigma_2 + 1 + \cdots + 1 + \sigma_{k-1} + 1 + \xi_\beta$$

if and only if each  $\sigma_i$  is either finite or  $\eta_\gamma \cdot (n + 1)$  for some  $\gamma < \beta$  and  $n \in \omega$ . To obtain this equivalence, notice that  $\xi_\beta^*$  is the only proper initial segment of  $\mathbb{Z}^\beta$ . The remaining final segment is  $\xi_\beta$ , so  $\sigma_1$  must be an initial segment of  $\xi_\beta$ . Hence, we give the required form for such an initial segment. After partitioning off  $\sigma_1$ , the remaining end segment is again  $\xi_\beta$ . Therefore,  $\sigma_2$  must have the prescribed form. Continuing this process, we see that each  $\sigma_i$  must be an initial segment of  $\xi_\beta$ , and hence we get the equivalence.

For  $\xi_\beta$ ,

$$\xi_\beta = \sigma_0 + 1 + \sigma_1 + 1 + \cdots + 1 + \sigma_{k-1} + 1 + \xi_\beta$$

if and only if each  $\sigma_i$  is either finite or  $\eta_\gamma \cdot (n + 1)$  for some  $\gamma < \beta$  and  $n \in \omega$ . For  $\mathbb{Z}^\beta \cdot \omega$ ,

$$\mathbb{Z}^\beta \cdot \omega = \mathbb{Z}^\beta \cdot m + \xi_\beta^* + 1 + \sigma_1 + 1 + \cdots + 1 + \sigma_{k-1} + 1 + \xi_{\beta+1}$$

if and only if each  $\sigma_i$  is either finite or  $\eta_\gamma \cdot (n + 1)$  for some  $\gamma \leq \beta$  and  $n \in \omega$ . Notice that in this case, in contrast to the earlier cases, we can have  $\gamma = \beta$ . Also, as explained above, we can have  $m = 0$  in this case. For  $\mathbb{Z}^\beta \cdot \omega^*$ ,

$$\mathbb{Z}^\beta \cdot \omega^* = \xi_{\beta+1}^* + 1 + \sigma_1 + 1 + \cdots + 1 + \sigma_{k-1} + 1 + \xi_\beta + \mathbb{Z}^\beta \cdot m$$

if and only if each  $\sigma_i$  is either finite or  $\eta_\gamma \cdot (n + 1)$  for some  $\gamma \leq \beta$  and  $n \in \omega$ . Again, we can have  $\gamma = \beta$  in this case, and we can also have  $m = 0$ , as explained above. For  $\eta_\beta \cdot n$ ,

$$\eta_\beta \cdot n = \sigma_0 + 1 + \sigma_1 + 1 + \cdots + 1 + \sigma_k$$

if and only if each  $\sigma_i$  is either finite, or  $\eta_\gamma \cdot (m + 1)$  for some  $\gamma < \beta$  and  $m \in \omega$ , or  $\eta_\beta \cdot (m + 1)$  for some  $m < n$ . Furthermore, the coefficients of the  $\eta_\beta$  terms must sum to  $n$ . We use this information to prove part 2 of Lemma 7.1.

**Lemma 7.5.** *Let  $x$  be an element of  $\xi_\beta + \mathbb{Z}^\beta$ . The properties “ $x$  is in the  $\mathbb{Z}^\beta$  summand” and “ $x$  is in the  $\xi_\beta$  summand” are both  $\Sigma_{2\beta+1}^c$ .*

*Proof.* To say that  $x$  is the  $\mathbb{Z}^\beta$  summand, we say

$$(\exists y < x)(y \not\sim_{\mathbb{Z}^\beta} x).$$

To see that this formula is correct, assume that  $x$  is in  $\mathbb{Z}^\beta$ . Let  $y$  be an element in the  $\xi_\beta$  summand and notice that the interval  $(y, x)$  is  $\eta_\beta$ . However,  $\eta_\beta$  is not a bounded interval in  $\mathbb{Z}^\beta$  and therefore  $y \not\sim_{\mathbb{Z}^\beta} x$ . Next, assume that  $x$  is not in the  $\mathbb{Z}^\beta$  summand. Then for any  $y < x$ , we have that  $y$  and  $x$  are in  $\xi_\beta$ . Since  $\xi_\beta$  occurs as a final segment in  $\mathbb{Z}^\beta$ , we have that  $y \sim_{\mathbb{Z}^\beta} x$ , as required.

To say that  $x$  is in the  $\xi_\beta$  summand, we say

$$(\exists y > x)(x \not\sim_{\mathbb{Z}^\beta} y).$$

The analysis that this is the correct statement is similar to the argument for being in the  $\mathbb{Z}^\beta$  summand. (Although it will not be important for our discussion, these two formulas actually show that the properties in the lemma are  $\Delta_{2\beta+1}^c$ .)  $\square$

Lemma 7.5 shows that (2) from Lemma 7.1 holds for  $\xi_\beta | (\xi_\beta + \mathbb{Z}^\beta)$  and  $(\xi_\beta + \mathbb{Z}^\beta) | \xi_\beta$ . Consider the  $\Sigma_{2\beta+1}^c$  sentence

$$(\exists x, y) (x < y \wedge x \not\sim_{\mathbb{Z}^\beta} y).$$

This sentence is satisfied in  $\xi_\beta + \mathbb{Z}^\beta$ , but it is false in  $\xi_\beta$ . Since it is false in the first cardinal sum and true in the second cardinal sum of  $\xi_\beta | (\xi_\beta + \mathbb{Z}^\beta)$ , and it is true in the first cardinal sum and false in the second cardinal sum of  $(\xi_\beta + \mathbb{Z}^\beta) | \xi_\beta$ , we have separated these structures, as required.

For  $x$  in  $\xi_\beta$  and  $\gamma < \beta$ , we say “ $x$  is in a copy of  $\mathbb{Z}^\gamma$ ” if  $x$  is an element of the  $\mathbb{Z}^\gamma \cdot \omega$  summand in  $\xi_\beta$ . We say “ $x$  is in the least copy of  $\mathbb{Z}^\gamma$ ” if  $x$  is in the first of these  $\omega$  many copies of  $\mathbb{Z}^\gamma$ . We define the phrase “ $x$  is in the  $n$ -th copy of  $\mathbb{Z}^\gamma$ ” to be the obvious extension of this concept. We regard being in the least copy as being in the first copy rather than the 0-th copy. When we speak of an element in the  $n$ -th copy, we will always assume that  $n \geq 1$ .

**Lemma 7.6.** *For  $x$  in  $\xi_\beta$ ,  $\gamma < \beta$  and  $n \geq 1$ , there are  $\Sigma_{2\beta+1}^c$  formulas which say “ $x$  is in a copy of  $\mathbb{Z}^\gamma$ ,” “ $x$  is in the least copy of  $\mathbb{Z}^\gamma$ ,” and “ $x$  is in the  $n$ -th copy of  $\mathbb{Z}^\gamma$ ”.*

*Proof.* To say “ $x$  is in a copy of  $\mathbb{Z}^\gamma$ ”, we consider two cases. If  $\gamma + 1 < \beta$ , then there are ordinals  $\delta$  such that  $\gamma < \delta < \beta$ , which contribute summands of the form  $\mathbb{Z}^\delta \cdot \omega$  to  $\xi_\beta$ . Therefore, we have to say

$$(\exists y) (y < x \wedge y \not\sim_{\mathbb{Z}^\gamma} x) \wedge (\forall y) (y < x \rightarrow y \sim_{\mathbb{Z}^{\gamma+1}} x).$$

This formula is  $\exists(\Pi_{2\gamma}^c) \wedge \forall(\Sigma_{2\gamma+2}^c)$ , which turns out to be  $\Pi_{2\gamma+3}^c$ . As long as  $\gamma + 1 < \beta$ , this is still  $\Sigma_{2\beta+1}^c$ . However, if  $\gamma + 1 = \beta$ , then  $\mathbb{Z}^\gamma \cdot \omega$  is the last summand in  $\xi_\beta$ , so we only have to say

$$(\exists y < x) (y \not\sim_{\mathbb{Z}^\gamma} x).$$

This formula is  $\Sigma_{2\gamma+1}^c$ , and hence is  $\Sigma_{2\beta+1}^c$ . It is clear, by arguments similar to those already given, that these formulas have the correct meaning.

To say “ $x$  is in the least copy of  $\mathbb{Z}^\gamma$ ,” we say  $x$  is in a copy of  $\mathbb{Z}^\gamma$  and

$$(\forall y_0 < y_1 < x) ((y_1 \not\sim_{\mathbb{Z}^\gamma} x) \rightarrow y_0 \sim_{\mathbb{Z}^\gamma} y_1).$$

The displayed part of this formula is  $\forall(\Pi_{2\gamma}^c \rightarrow \Sigma_{2\gamma}^c)$ , which is  $\Pi_{2\gamma+1}^c$ . So, we are still within  $\Sigma_{2\beta+1}^c$ . For  $n > 1$ , to say “ $x$  is in the  $n$ -th copy of  $\mathbb{Z}^\gamma$ ,” we say there is  $y < x$  such that  $y$  is in the least copy of  $\mathbb{Z}^\gamma$  and  $y \sim_{\mathbb{Z}^\gamma, n-2} x$ . That is, to say that  $x$  is in the second copy of  $\mathbb{Z}^\gamma$ , we need to say that there is an element  $y < x$  such that  $y$  is in the least copy of  $\mathbb{Z}^\gamma$  and there are zero copies of  $\mathbb{Z}^\gamma$  between  $y$  and  $x$ . The second conjunct in this formula is  $\Sigma_{2\gamma+2}^c$ , and hence the whole formula is  $\Sigma_{2\beta+1}^c$ .  $\square$

We will use the following three facts from Ash [2].

**Lemma 7.7.**

1. If  $\beta > \gamma > 0$  and  $n, m \geq 1$ , then  $\eta_\beta \cdot m <_{2\gamma+1} \eta_\gamma \cdot n$ .
2. If  $\beta > 0$ , then  $\xi_\beta + \mathbb{Z}^\beta <_{2\beta+1} \xi_\beta$ .

3. If  $m > n \geq 1$  and  $\gamma > 0$ , then  $\eta_\gamma \cdot m <_{2\gamma+1} \eta_\gamma \cdot n$ .

We have already verified the fact that there is a strict inequality in 2 of this lemma. Lemma 7.7 follows by an analysis similar to the ones given below for other order types. Notice that 2 in Lemma 7.7 tells us that  $\xi_\beta | (\xi_\beta + \mathbb{Z}^\beta) \equiv_{2\beta} (\xi_\beta + \mathbb{Z}^\beta) | \xi_\beta$ . Therefore, 1 from Lemma 7.1 holds.

**Lemma 7.8.** For all  $\beta > 0$ ,  $\xi_{\beta+1} <_{2\beta+1} \xi_\beta$ .

*Proof.* First, notice that the statement

$$(\exists x, y) (x < y \wedge x \not\prec_{\mathbb{Z}^\beta} y)$$

shows that this inequality must be strict. To check  $\xi_{\beta+1} \leq_{2\beta+1} \xi_\beta$ , suppose that we are given a partition of  $\xi_\beta$  by finitely many points. We have already seen what such a partition looks like, and so we know that we can partition  $\xi_{\beta+1}$  to match these intervals exactly, except for the final interval. The final interval in  $\xi_\beta$  is  $\xi_\beta$ , while in  $\xi_{\beta+1}$  it is  $\xi_{\beta+1}$ . Therefore, we have to check that  $\xi_\beta \leq_{2\beta} \xi_{\beta+1}$ .

To show  $\xi_\beta \leq_{2\beta} \xi_{\beta+1}$ , suppose we are given a partition of  $\xi_{\beta+1}$  by finitely many points. We know from our analysis above that we can pick points in  $\xi_\beta$  which exactly match the intervals given from  $\xi_{\beta+1}$ , except for two cases. First, the final intervals still have the form  $\xi_\beta$  and  $\xi_{\beta+1}$ . Second, we could have intervals in  $\xi_{\beta+1}$  of the form  $\eta_\beta \cdot (n+1)$ , and we must be able to match those by intervals of the form  $\eta_\gamma \cdot (n+1)$ , with  $\gamma < \beta$ , from  $\xi_\beta$ .

To see that we can choose appropriate intervals of the form  $\eta_\gamma \cdot (n+1)$ , we consider two cases. If  $\beta$  is a successor ordinal, let  $\gamma$  be such that  $\gamma+1 = \beta$ . By 1 from Lemma 7.7, we have that  $\eta_\beta \cdot (n+1) \leq_{2\gamma+1} \eta_\gamma \cdot (n+1)$ . Since  $2\gamma+1 = 2\beta-1$ , we have the exact relation required for the back-and-forth relation. If  $\beta$  is a limit ordinal, then  $2\beta = \beta$  and so we are required to verify that  $\eta_\beta \cdot (n+1) \leq_\delta \eta_\gamma \cdot (n+1)$  for some given  $\delta < \beta$ . However, we can pick  $\gamma < \beta$  such that  $\gamma > \delta$ . Again, applying 1 from Lemma 7.7, we have  $\eta_\beta \cdot (n+1) \leq_{2\gamma+1} \eta_\gamma \cdot (n+1)$ . Since  $\delta < 2\gamma+1$ , we have met the requirement for the back-and-forth relation.

We are still left with having to verify that the final intervals  $\xi_{\beta+1}$  and  $\xi_\beta$  match up correctly. However, notice that we have reduced the level of the back-and-forth relation required between them. Continuing this process of matching intervals other than the final interval, we eventually reach the  $\leq_1$  relation, which is satisfied between  $\xi_\beta$  and  $\xi_{\beta+1}$  since both are infinite.  $\square$

A similar argument gives the following slightly stronger result.

**Lemma 7.9.** For all  $\beta \geq 1$  and all  $n \in \omega$ ,  $\xi_{\beta+1} <_{2\beta+1} \xi_\beta + \mathbb{Z}^\beta \cdot n$ .

We are now ready to verify 1 from Lemma 7.2 for the structures  $\mathbb{Z}^\beta \cdot \omega$  and  $\mathbb{Z}^\beta \cdot \omega^*$ .

**Lemma 7.10.**  $\mathbb{Z}^\beta \cdot \omega \leq_{2\beta+1} \mathbb{Z}^\beta \cdot \omega^*$ .

*Proof.* Suppose we are given a partition of  $\mathbb{Z}^\beta \cdot \omega^*$  by finitely many points. Since we know what this partition looks like, we know that we can partition  $\mathbb{Z}^\beta \cdot \omega$  to match the intervals exactly, except for the initial and end intervals. To see that the end intervals match up, we need to see

that  $\xi_\beta + \mathbb{Z}^\beta \cdot n \leq_{2\beta} \xi_{\beta+1}$ . This inequality follows from Lemma 7.9. To verify that the initial intervals match up, we need to see that  $\xi_{\beta+1}^* \leq_{2\beta} \mathbb{Z}^\beta \cdot n + \xi_\beta^*$ . By Lemma 7.9 we know that  $\xi_{\beta+1} \leq_{2\beta} \xi_\beta + \mathbb{Z}^\beta \cdot n$ . However, it is also the case that  $(\xi_\beta + \mathbb{Z}^\beta \cdot n)^* = \mathbb{Z}^\beta \cdot n + \xi_\beta^*$ . Therefore, the initial intervals match up correctly since for any  $\delta$ , if  $L_0 \leq_\delta L_1$  then  $L_0^* \leq_\delta L_1^*$ .  $\square$

**Lemma 7.11.**  $\mathbb{Z}^\beta \cdot \omega^* \leq_{2\beta+1} \mathbb{Z}^\beta \cdot \omega$ .

*Proof.* This follows from Lemma 7.10, together with the fact that for all  $\delta$ , if  $L_0 \leq_\delta L_1$ , then  $L_0^* \leq_\delta L_1^*$ .  $\square$

We next turn our attention to 4 from Lemmas 7.1 and 7.2 and verify the appropriate categoricity results. Lemma 7.12 shows that 4 holds for  $\mathbb{Z}^\beta \cdot \omega$ , and an almost identical argument shows that it holds for  $\mathbb{Z}^\beta \cdot \omega^*$ .

**Lemma 7.12.**  $\mathbb{Z}^\beta \cdot \omega$  has a formally  $\Sigma_{2\beta+2}^0$  Scott family with no parameters.

*Proof.* Let  $\bar{x} = (x_0, \dots, x_n)$  be a tuple such that  $x_0 < \dots < x_n$ . (We blur the distinction between variables and elements from  $\mathbb{Z}^\beta \cdot \omega$ . That is, when giving the formula below we think of  $\bar{x}$  as a tuple of elements, but technically we are defining a formula with free variables that determines the orbit of this tuple.) To determine the orbit of  $\bar{x}$ , we take a conjunction of the following statements.

1. We say  $x_0$  is in the  $l$ -th copy of  $\mathbb{Z}^\beta$  for the appropriate  $l$ . By our work above, this statement is  $\Sigma_{2\beta+2}^c$ .
2. If  $x_i$  and  $x_{i+1}$  are not in the same copy of  $\mathbb{Z}^\beta$  and there are exactly  $n$  copies of  $\mathbb{Z}^\beta$  between  $x_i$  and  $x_{i+1}$ , then we say  $x_i \sim_{\mathbb{Z}^\beta, n} x_{i+1}$ . This statement is  $\Sigma_{2\beta+2}^c$ .
3. If  $x_i$  and  $x_{i+1}$  are in the same  $\mathbb{Z}^\beta$ , then fix the minimum  $\alpha_i \leq \beta$  such that  $x_i$  and  $x_{i+1}$  are in the same  $\mathbb{Z}^{\alpha_i}$ . Notice that  $\alpha_i = \delta_i + 1$  for some  $\delta_i$ . Add a clause saying  $x_i \sim_{\mathbb{Z}^{\alpha_i}} x_{i+1} \wedge x_i \sim_{\mathbb{Z}^{\delta_i, n}} x_{i+1}$  for the appropriate number of copies of  $\mathbb{Z}^{\delta_i}$  between  $x_i$  and  $x_{i+1}$ . This statement is  $\Sigma_{2\alpha_i}^c$ , and hence is  $\Sigma_{2\beta+2}^c$ .

We have explained why this formula is  $\Sigma_{2\beta+2}^c$  and it is clear that any two tuples satisfying this formula are automorphic. Furthermore, every tuple must satisfy a formula of this form. Therefore, the Scott family consists of all formulas obtained in this way, corresponding to different tuples.  $\square$

**Lemma 7.13.** Both  $\xi_\beta$  and  $\xi_\beta + \mathbb{Z}^\beta$  have formally  $\Sigma_{2\beta+1}^0$  Scott families with no parameters.

*Proof.* We prove that  $\xi_\beta + \mathbb{Z}^\beta$  has an appropriate Scott family. (The proof for  $\xi_\beta$  is almost identical, except that we do not need to say whether points lie in the  $\xi_\beta$  summand or the  $\mathbb{Z}^\beta$  summand, and we do not need to include the clause below, which concerns pairs of points in the  $\mathbb{Z}^\beta$  summand.)

Let  $\bar{x}$  be a tuple of elements in  $\xi_\beta + \mathbb{Z}^\beta$  such that  $x_0 < x_1 < \dots < x_n$ . We describe a formula which determines the orbit of  $\bar{x}$ .

1. For each  $x_i$ , we say whether it is in the  $\xi_\beta$  summand or the  $\mathbb{Z}^\beta$  summand.



2. For each  $x_i$  in the  $\xi_\beta$  summand, we say it is in the  $n$ -th copy of  $\mathbb{Z}^\gamma$  for the appropriate  $\gamma < \beta$  and  $n \geq 1$ .
3. For each pair of points  $x_i, x_{i+1}$  that are both in  $\mathbb{Z}^\beta$ , we determine the interval between  $x_i$  and  $x_{i+1}$  exactly as in step 3 in the proof of Lemma 7.12.
4. For each pair of points  $x_i, x_{i+1}$  that are both in  $\xi_\beta$  and are both in the  $n$ -th copy of  $\mathbb{Z}^\gamma$ , we determine the interval between  $x_i$  and  $x_{i+1}$  as in step 3 in the proof of Lemma 7.12, except that we work in  $\mathbb{Z}^\gamma$  instead of  $\mathbb{Z}^\beta$ .

Notice that if  $x_i, x_{i+1}$  are either not both in  $\xi_\beta$ , or not both in  $\mathbb{Z}^\beta$ , or are both in  $\xi_\beta$  but not in the same copy of  $\mathbb{Z}^\gamma$ , then we do not need to add any more information about the interval between them.

Finally, we check that this formula is  $\Sigma_{2\beta+1}^c$ . We have already verified this property for steps 1 and 2. For steps 3 and 4, the formula is  $\Sigma_{2\alpha_i}^c$  for some  $\alpha_i \leq \beta$ . Therefore, the entire formula is  $\Sigma_{2\beta+1}^c$ .  $\square$

It remains to verify 3 from Lemmas 7.1 and 7.2. For 3 in Lemma 7.2, we need to show that there are computable copies of  $\mathbb{Z}^\beta \cdot \omega$  and  $\mathbb{Z}^\beta \cdot \omega^*$ , which have the property that for all  $\gamma < 2\beta+2$ , the set of pairs  $(\bar{a}, \bar{b})$  with  $\bar{a} \in \mathbb{Z}^\beta \cdot \omega$ ,  $\bar{b} \in \mathbb{Z}^\beta \cdot \omega^*$ ,  $|\bar{a}| = |\bar{b}|$  and  $(\mathbb{Z}^\beta \cdot \omega, \bar{a}) \leq_\gamma (\mathbb{Z}^\beta \cdot \omega^*, \bar{b})$  is uniformly c.e. in  $\gamma$ . Recall that  $(\mathbb{Z}^\beta \cdot \omega, \bar{a}) \leq_\gamma (\mathbb{Z}^\beta \cdot \omega^*, \bar{b})$  if and only if each partitioned interval in  $\mathbb{Z}^\beta \cdot \omega$  is  $\leq_\gamma$  the corresponding interval in  $\mathbb{Z}^\beta \cdot \omega^*$ . This reduces our work to describing the  $\leq_\gamma$  relations between all possible subintervals of  $\mathbb{Z}^\beta \cdot \omega$  and  $\mathbb{Z}^\beta \cdot \omega^*$ , and to constructing nice computable copies of these structures in which we can compute the initial and final segments determined by each point, and we can compute the bounded interval determined by any pair of points.

We have already seen the relations  $\leq_\gamma$  on all the subintervals of  $\mathbb{Z}^\beta \cdot \omega$  and  $\mathbb{Z}^\beta \cdot \omega^*$ . These are

- (i)  $\xi_{\beta+1} <_{2\beta+1} \xi_\beta + \mathbb{Z}^\beta \cdot m$  for all  $m$ ,
- (ii)  $\xi_{\beta+1}^* <_{2\beta+1} \mathbb{Z}^\beta \cdot m + \xi_\beta^*$  for all  $m$ ,
- (iii)  $\eta_\gamma \cdot (n+1) <_1 m$  for any  $m$  and  $n$ ,
- (iv)  $n <_1 m$  if and only if  $n > m$ ,
- (v)  $\eta_\gamma \cdot m <_{2\gamma+1} \eta_\gamma \cdot n$  for all  $m > n \geq 1$ , and
- (vi)  $\eta_\alpha \cdot m <_{2\gamma+1} \eta_\gamma \cdot n$  for all  $m, n \geq 1$  and  $\alpha > \gamma$ .

The first two statements follow from Lemma 7.9, the third and fourth statements follow from considering the size of the orders, and the last two statements were proved by Ash and stated in Lemma 7.7.

It remains to show that there are nice computable copies of  $\mathbb{Z}^\beta \cdot \omega$  and  $\mathbb{Z}^\beta \cdot \omega^*$ . Suppose we could construct a nice copy of  $\mathbb{Z}^\beta$  in which we understood the relationship between any pair of points. We could use this copy of  $\mathbb{Z}^\beta$  to build uniform copies of  $\mathbb{Z}^\beta \cdot \omega$  and  $\mathbb{Z}^\beta \cdot \omega^*$  in the sense that for each point we would know which copy of  $\mathbb{Z}^\beta$  that point sits in. In particular, we would know the initial and final segments determined by each point and the intervals between any pair of points. These copies of  $\mathbb{Z}^\beta \cdot \omega$  and  $\mathbb{Z}^\beta \cdot \omega^*$  would establish 3 in Lemma 7.2. Therefore, it suffices to show there is a computable copy of  $\mathbb{Z}^\beta$  in which we can determine the interval between any pair of points, in terms of the orderings given above. That is, for any two points

$x < y$ , we need to be able to say whether the interval they determine is finite or infinite. If it is finite, we need to give the size. If it is infinite, we need to give the minimum  $\alpha \leq \beta$  for which  $x \sim_\alpha y$  and the exact number of copies of  $\mathbb{Z}^\delta$  between  $x$  and  $y$ , where  $\delta$  is such that  $\delta + 1 = \alpha$ .

To build a nice copy of  $\mathbb{Z}^\beta$ , we need (temporarily) to be careful about the distinction between a linear order with the same order type as  $\mathbb{Z}$  (which we denote here by  $\zeta$ ) and the integers with their usual arithmetic structure and with a distinguished element for 0 (which we denote by  $\mathbb{Z}$  to make the difference clear). The domain of our nice copy of  $\zeta^\beta$  is the set of functions  $f : \beta \rightarrow \mathbb{Z}$  with finite support. That is, we consider functions which take value 0 on all but finitely many arguments  $\gamma < \beta$ . Notice that, in addition to using the distinguished element 0 in an important way, we can talk about subtraction in expressions such as  $f(\gamma) - g(\gamma)$ , involving these functions. We immediately revert back to using  $\mathbb{Z}^\beta$  to denote the linear order, but warn the reader that we will make use of the arithmetical relations that hold among elements of this special copy of  $\mathbb{Z}^\beta$ .

The ordering relation is defined by  $f < g$  if and only if  $f(\gamma) < g(\gamma)$ , where  $\gamma < \beta$  is the largest value on which  $f$  and  $g$  disagree. From a computable copy of  $\beta$ , it is easy to use finite sequences to build a copy of  $\mathbb{Z}^\beta$  in terms of these functions with finite support. Since  $\beta$  is a computable ordinal, we can assume that we have a computable ordering of type  $\beta$  that is well connected to the ordinal notations in the sense that there is an ordinal notation  $b$  for  $\beta$ , and from the element of our ordering corresponding to  $\alpha < \beta$ , we can effectively determine the ordinal notation  $a <_{\mathcal{O}} b$  for the ordinal  $\alpha$ . That is, we will work with ordinals in the usual way and assume that they are translated into ordinal notations.

Given such a coding of  $\mathbb{Z}^\beta$ , fix  $f, g \in \mathbb{Z}^\beta$  with  $f < g$ . We need to show how to determine the appropriate information about the interval between  $f$  and  $g$ . Let  $\delta$  be the largest ordinal on which  $f$  and  $g$  disagree and let  $\alpha = \delta + 1$ . We know that  $f(\delta) < g(\delta)$  and that  $f(\gamma) = g(\gamma)$  for all  $\gamma \geq \alpha$ . We claim that  $f \sim_{\mathbb{Z}^\alpha} g$  and  $f \sim_{\mathbb{Z}^{\delta, n}} g$ , where  $n = g(\delta) - f(\delta) - 1$ .

To see that  $f \sim_{\mathbb{Z}^\alpha} g$ , consider the set of all functions in  $\mathbb{Z}^\beta$  which agree with  $f$  and  $g$  on all  $\gamma \geq \alpha$ . These functions form a convex interval of type  $\mathbb{Z}^\alpha$ , which contains both  $f$  and  $g$ . Therefore,  $f \sim_{\mathbb{Z}^\alpha} g$ . To see that  $f \sim_{\mathbb{Z}^{\delta, n}} g$ , notice that the copy of  $\mathbb{Z}^\delta$  in which  $f$  ( $g$ , respectively) sits is given by the set of  $h$  which agree with  $f$  ( $g$ , respectively) for all  $\gamma \geq \delta$ . The copies of  $\mathbb{Z}^\delta$  between  $f$  and  $g$  correspond exactly to the numbers  $m$  such that  $f(\delta) < m < g(\delta)$ . That is, for any such  $m$ , the set of functions  $h$  for which  $h(\delta) = m$  and  $h$  agrees with  $f$  and  $g$  for all  $\gamma \geq \alpha$  forms a convex copy of  $\mathbb{Z}^\delta$  which sits completely between  $f$  and  $g$ . Furthermore, any function  $h$  with  $h(\delta) \leq f(\delta)$  or  $h(\delta) \geq g(\delta)$  does not sit in a copy of  $\mathbb{Z}^\delta$  contained completely between  $f$  and  $g$ . There are  $g(\delta) - f(\delta) - 1$  many such  $m$ , and hence  $f \sim_{\mathbb{Z}^{\delta, n}} g$ . Therefore, we can determine the status of all intervals in our nice copy of  $\mathbb{Z}^\beta$ , which finishes the proof of 3 from Lemma 7.2.

To perform a similar analysis for 3 in Lemma 7.1, we need to see how  $\xi_\beta + \mathbb{Z}^\beta$  can be partitioned by a finite number of points. There are two possible partitions. If the first  $k$  partition points are in  $\xi_\beta$ , then

$$\xi_\beta + \mathbb{Z}^\beta = \sigma_0 + 1 + \cdots + 1 + \sigma_{k-1} + 1 + \xi_\beta + \xi_\beta^* + 1 + \sigma_{k+1} + 1 + \cdots + 1 + \sigma_l + 1 + \xi_\beta$$

if and only if each  $\sigma_i$  is either finite or  $\eta_\gamma \cdot (n + 1)$  for some  $\gamma < \beta$  and  $n \in \omega$ . The  $\xi_\beta + \xi_\beta^*$  interval (which is equal to  $\eta_\beta$ ) consists of the final segment of the  $\xi_\beta$  summand and the initial

segment of the  $\mathbb{Z}^\beta$  summand. If the first partition point is in  $\mathbb{Z}^\beta$ , then

$$\xi_\beta + \mathbb{Z}^\beta = \xi_\beta + \xi_\beta^* + 1 + \sigma_1 + 1 + \cdots + 1 + \sigma_k + 1 + \xi_\beta$$

if and only if each  $\sigma_i$  is either finite or  $\eta_\gamma \cdot (n+1)$  for some  $\gamma < \beta$  and  $n \in \omega$ . The  $\xi_\beta + \xi_\beta^*$  term consists of all of the  $\xi_\beta$  summand and the initial segment of the  $\mathbb{Z}^\beta$  summand. As above, it can be rewritten as  $\eta_\beta$ .

We have already specified all the back-and-forth relations  $\leq_\gamma$  between the subintervals occurring in  $\xi_\beta$  and  $\xi_\beta + \mathbb{Z}^\beta$ , for all  $\gamma < 2\beta + 1$ . As above, it suffices to show that we can construct computable copies of  $\xi_\beta$  and  $\xi_\beta + \mathbb{Z}^\beta$ , in which we know for each point whether it is in  $\xi_\beta$  or  $\mathbb{Z}^\beta$ , and we know the interval determined by any pair of points. However, given a nice copy of  $\mathbb{Z}^\beta$  as described above, it is straightforward to put these copies together in a uniform way to form nice copies of  $\xi_\beta$  and  $\xi_\beta + \mathbb{Z}^\beta$ .

## 8 Problems

**Problem 1.** *For a computable limit ordinal  $\alpha$ , is there a computable structure that is  $\Delta_\alpha^0$  categorical but not relatively  $\Delta_\alpha^0$  categorical?*

**Problem 2.** *For a computable limit ordinal  $\alpha$ , is there a computable structure  $\mathcal{A}$  with a relation  $R$  that is intrinsically  $\Sigma_\alpha^0$  but not relatively intrinsically  $\Sigma_\alpha^0$ .*

**Problem 3.** *If  $\mathcal{A}$  is  $\Delta_1^1$  categorical, must it be relatively  $\Delta_1^1$  categorical?*

Soskov [19] showed that for a computable (or hyperarithmetical) structure  $\mathcal{A}$  and a relation  $R$  on  $\mathcal{A}$ , if  $R$  is invariant under automorphisms of  $\mathcal{A}$ , and hyperarithmetical, then it is definable by a computable infinitary formula. Hence, intrinsically  $\Delta_1^1$  and relatively intrinsically  $\Delta_1^1$  relations are the same.

**Problem 4.** *For a computable limit ordinal  $\alpha$  and finite  $n$ , is there a structure with  $\Delta_\alpha^0$  dimension  $n$ ?*

**Problem 5.** *Is it true that for any computable successor ordinal  $\alpha$ , there is a rigid computable structure which is  $\Delta_\alpha^0$  categorical but not relatively  $\Delta_\alpha^0$  categorical?*

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