Real Computable Manifolds and Homotopy Groups

Wesley Calvert[1]* and Russell Miller[2]**

[1] Department of Mathematics and Statistics
Murray State University
Murray, KY 42071 U.S.A.
Wesley.Calvert@murraystate.edu
http://campus.murraystate.edu/academic/faculty/wesley.calvert
[2] Queens College of CUNY
65-30 Kissena Blvd., Flushing, NY 11367 USA
and
The CUNY Graduate Center
365 Fifth Avenue, New York, NY 10016 USA
Russell.Miller@qc.cuny.edu
http://qcpages.qc.cuny.edu/~rmiller

Abstract. Using the model of real computability developed by Blum, Cucker, Shub, and Smale, we investigate the difficulty of determining the answers to several basic topological questions about manifolds. We state definitions of real-computable manifold and of real-computable paths in such manifolds, and show that, while BSS machines cannot in general decide such questions as nullhomotopy and simple connectedness for such structures, there are nevertheless real-computable presentations of paths and homotopy equivalence classes under which such computations are possible.

Key words: Computability, Blum-Shub-Smale computability, homotopy, manifold.

1 Introduction

A notable shortcoming of the standard (Turing) model of computation is that it does not produce a theory of effectiveness relevant to uncountable structures. Since these structures are a routine part of the practice of pure and applied mathematics, a growing body of literature has addressed effective mathematics on uncountable structures (see, for instance [1, 7–9, 12, 15, 17, 18]). In addition

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to all of these, there is a large literature which we shall not review here on computable analysis, which has its own large arsenal of approaches to effectiveness on geometric structures. The present paper will begin to describe the use of one of the proposed models for uncountable structures to explore the effective homotopy theory of manifolds.

In [2], Blum, Shub, and Smale introduced a notion of computation based on full-precision real arithmetic, which received its canonical form in [1]. Let $\mathbb{R}^\infty$ be the set of finite sequences of elements from $\mathbb{R}$, and $\mathbb{R}_\infty$ the bi-infinite direct sum

$$\bigoplus_{i\in\mathbb{Z}} \mathbb{R}.$$

Definition 1 A machine M over $\mathbb{R}$ is a finite connected directed graph, containing five types of nodes: input, computation, branch, shift, and output, with the following properties:

1. The unique input node has no incoming edges and only one outgoing edge.
2. Each computation and shift node has exactly one output edge and possibly several input branches.
3. Each output node has no output edges and possibly several input edges.
4. Each branch node η has exactly two output edges (labeled 0_η and 1_η) and possibly several input edges.
5. Associated with the input node is a linear map $g_I : \mathbb{R}^\infty \to \mathbb{R}_\infty$.
6. Associated with each computation node η is a rational function $g_\eta : \mathbb{R}_\infty \to \mathbb{R}_\infty$.
7. Associated with each branch node η is a polynomial function $h_\eta : \mathbb{R}_\infty \to \mathbb{R}$.
8. Associated with each shift node is a map $\sigma_\eta \in \{\sigma_l, \sigma_r\}$, where $\sigma_l(x)_i = x_{i+1}$ and $\sigma_r(x)_i = x_{i-1}$.
9. Associated with each output node η is a linear map $O_\eta : \mathbb{R}_\infty \to \mathbb{R}^\infty$.

Each machine computes a partial function from $\mathbb{R}^\infty$ to $\mathbb{R}^\infty$ in the natural way. Such a function is said to be $\mathbb{R}$-computable. There is a list $\langle \varphi_{\boldsymbol{e}} \rangle$ of all $\mathbb{R}$-computable functions, indexed by finite tuples $\boldsymbol{e}$ from $\mathbb{R}^\infty$, such that each $\varphi_{\boldsymbol{e}}$ may be computed uniformly (on any input) from its index $\boldsymbol{e}$.

In the present paper, we will explore manifolds which are given effectively (in the sense of $\mathbb{R}$-computation). In the remainder of the present section, we will describe this sense of effectiveness exactly, and will recall some relevant definitions from topology. In section 2, we will describe some ineffectiveness results in homotopy theory: that nullhomotopy and simple-connectedness are both undecidable. In section 3, we will show that certain important standard computations — notably the computation of the fundamental group — are still, modulo the difficulties in section 2, computable.

Definition 2 A *real-computable d-manifold M* consists of real-computable i, j, j', k, the *inclusion functions*, satisfying the following conditions for all $m, n \in \omega$.

- If $i(m,n)\downarrow = 1$, then $\varphi_{j(m,n)}$ is a total real-computable homeomorphism from $\mathbb{R}^d$ into $\mathbb{R}^d$, and $\varphi_{j'(m,n)} = \varphi^{-1}_{j(m,n)}$, and $k(m,n)\downarrow = k(n,m)\downarrow = m$.

– If $i(m,n)\downarrow= 0$, then $k(m,n)\downarrow= k(n,m)\downarrow\in\omega$ with $i(k(m,n),m) = 1 = i(k(m,n),n)$ and for all $p\in\omega$, if $i(p,m) = i(p,n) = 1$, then $i(p,k(m,n)) = 1$, and for all $q\in\omega$, if $i(m,q) = i(n,q) = 1$, then $i(k(m,n),q) = 1$ with

$$\mathrm{range}(\varphi_{j(m,q)})\cap\mathrm{range}(\varphi_{j(n,q)}) = \mathrm{range}(\varphi_{j(k(m,n),q)}).$$

– If $i(m,n)\notin\{0,1\}$, then $i(m,n)\downarrow= i(n,m)\downarrow= -1$, and

$$(\forall p\in\omega)[i(p,m)\neq 1 \text{ or } i(p,n)\neq 1],$$

and for all $q\in\omega$, if $i(m,q)$ and $i(n,q)$ both lie in $\{0,1\}$, then

$$\mathrm{range}(\varphi_{j(k(m,q),q)})\cap\mathrm{range}(\varphi_{j(k(n,q),q)}) = \emptyset.$$

– For all $q\in\omega$, if $i(m,n) = i(n,q) = 1$, then $i(m,q) = 1$ and

$$\varphi_{j(n,q)}\circ\varphi_{j(m,n)} = \varphi_{j(m,q)}.$$

Here we explain how these abstract conditions are to be understood. First, each natural number m represents a chart U_m for the manifold M, which is to say, a nonempty open subset of M homeomorphic to $\mathbb{R}^d$ via some map

$$\alpha_m : U_m\to\mathbb{R}^d.$$

In fact, though, we do not give this homeomorphism, since we do not wish to attempt to present the points in the manifold globally in an effective way. Instead we simply understand $\mathbb{R}^d$ to represent the chart U_m, and understand the manifold M to be the union of this countable collection of charts. The meat of the definition lies in the inclusion functions, which describe the inclusion relations among the charts. For each m and n, $i(m,n)$ equals 1, 0, or -1, according as either $U_m\subseteq U_n$, or $\emptyset\subsetneq U_m\cap U_n\subsetneq U_m$, or $U_m\cap U_n = \emptyset$, and the conditions translate as follows.

First, if $i(m,n) = 1$, then $j(m,n)$ is the index for a real-computable map from $\mathbb{R}^d$ into $\mathbb{R}^d$ which describes the inclusion map $U_m\hookrightarrow U_n$, in the sense that $\alpha_n^{-1}\circ\varphi_{j(m,n)}\circ\alpha_m$ is the inclusion map. Moreover, we also have an index $j'(m,n)$ for its inverse.

If $i(m,n) = 0$, then $U_m\not\subseteq U_n$, but the two charts have nonempty intersection $U_{k(m,n)}$. (Notice that this implies that our collection of charts must be closed under finite intersection, and that the intersection of two charts must be connected.) The conditions ensure that $U_{k(m,n)}$ contains every other chart U_p which is a subset of both U_m and U_n, and also that if a chart U_q contains both U_m and U_n, then it must contain $U_{k(m,n)}$, which must be the intersection of U_m with U_n within U_q.

Finally, if $i(m,n) = -1$, then the conditions ensure that $U_m\cap U_n$ does not contain any other U_p, and that for any U_q intersecting both U_m and U_n, we have $U_m\cap U_n = \emptyset$ within U_q.

The last condition is the obvious rule for composition of inclusion maps: if $U_m\subseteq U_n\subseteq U_q$, then

$$\alpha_q^{-1}\circ\varphi_{j(m,q)}\circ\alpha_m = (\alpha_q^{-1}\circ\varphi_{j(n,q)}\circ\alpha_n)\circ(\alpha_n^{-1}\circ\varphi_{j(m,n)}\circ\alpha_m).$$

An earlier version of this definition indexed the charts by finite tuples $\boldsymbol{x} \in \mathbb{R}^\infty$ of real numbers, with the condition that the indices must be precisely the tuples from some fixed real-computable subset of $\mathbb{R}^\infty$. This allowed the possibility of an uncountable collection of charts, and also allowed those charts to be enumerated in a far less effective way. Since most definitions of manifold require the manifold to be covered by a countable collection of charts, we feel that Definition 2 reflects the general topological situation more accurately. (Since the property of being covered by countably many charts is equivalent to that of having a countable basis, manifolds having these properties are often called *second-countable* manifolds in the literature.) It also gives a more precise enumeration of the charts, allowing us greater ability to search through the charts to find the one we need, and this permits us to prove stronger theorems about our real-computable manifolds, at the cost of excluding those which are not second-countable or whose presentations are insufficiently effective to fit our definition. We regard the strength of our results as a solid justification for our choice of the stronger definition.

Our definition of manifold is also strict in requiring that the intersection of any two charts must be another chart (or else empty), and hence must be connected. For manifolds in general this is not usually required, but on the other hand, it is possible to take a manifold in which the intersection of two charts need not be connected and to produce a new, finer cover of it by charts whose pairwise intersections are always connected. We leave open the question of how effectively such a transformation of the cover can be accomplished. Our definition does facilitate the decidability of homotopy for certain specific classes of computable paths, a question which we conjecture is not decidable in the more general context. This conjecture would imply that a general cover cannot always be effectively converted into a cover with pairwise intersections all connected.

Our definition of manifold is quite abstract, in the sense that it essentially ignores the intended underlying topological space M entirely, giving a set of conditions on the charts (each of which is a copy of $\mathbb{R}^d$), rather than the corresponding conditions on the space M itself. We refer the reader to [24, p. 30, Example 8] for a noneffective precursor to this definition. That example also provides a condition on the charts which is equivalent to connectedness of M:

Definition 3 The real-computable manifold M is *connected* if there is no proper nonempty subset S of ω such that

- whenever $i(m,n) = 0$ with $m, n \in S$, we also have $k(m,n) \in S$; and
- whenever $i(m,n) = 0$ with $m, n \notin S$, we also have $k(m,n) \notin S$; and
- whenever $i(m,n) = 1$ with $m \in S$, we also have $n \in S$; and
- whenever $i(m,n) = 1$ with $m \notin S$, we also have $n \notin S$.

If M is not connected, then we define its connected components to be the maximal subsets of ω satisfying all four of the above properties. (Intuitively, the connected components of M are the unions of the charts corresponding to these maximal subsets.)

Another reasonable variation on our definition would require the enumeration of the charts to terminate; that is, that the sequence of U_m be only finitely long.

This would, no doubt, bring the study closer to the intuition of the classical problems, in which one wants the manifold not only to be given, but to *have been given*, in a perfective aspect. Even here there are still undecidable problems to be considered: our Theorem 7 suggests this, as did earlier and well-known work of Markov [16]. However, much of this study is likely to be not of computability, but of complexity. This is, no doubt, an interesting study (it has been pursued in the more limited context of affine and projective varieties; see [20, 6, 5, 4]), but not the one we choose to pursue here.

Classically, a *path* in a manifold M is a continuous map of the closed unit interval into M. A path is said to be a *loop* if and only if its endpoints coincide. Two paths p_1 and p_2 are said to be *homotopic* if there is a continuous map $h : I \times I \to M$ such that $h(t, 0) = p_1(t)$ and $h(t, 1) = p_2(t)$. The class of loops in M up to homotopy equivalence forms a group under concatenation, called the *fundamental group*, or $\pi_1(M)$. A loop in the identity class is said to be *nullhomotopic*. A manifold M such that $\pi_1(M)$ is trivial is said to be simply connected. Since we hope for a broad audience for the present paper, we offer the torus $T^2 = S^1 \times S^1$ as an example. There are many loops which are nullhomotopic on T^2, for instance the very small ones which do not wrap around either the hole in the middle or the tube running through T^2. The fundamental group is Abelian, and is freely generated by a loop that runs once around the hole and a loop that runs once around the tube. More detailed background on these issues can be found in [13].

With our abstract definition, we are forced into a more involved definition of path.

Definition 4 A *path* in a real-computable manifold M, with inclusion functions i, j, and k, consists of a pair of functions $g : [0, 1] \to \omega$ and $h : [0, 1] \to \mathbb{R}^d$ such that there exists a finite sequence $0 = t_0 < t_1 < \cdots < t_n = 1$ of real numbers such that, for all $m < n$:

- $g{\upharpoonright}[t_m, t_{m+1})$ is constant, with $g(1) = g(t_{n-1})$; and
- $h{\upharpoonright}[t_m, t_{m+1})$ is continuous (and right-continuous at t_m), and h is also left-continuous at 1; and
- $i(g(t_m), g(t_{m+1})) \in \{0, 1\}$; and

$$\lim_{x \to t_{m+1}^-} \varphi_{j(k(g(t_m),g(t_{m+1})),g(t_{m+1}))}(\varphi^{-1}_{j(k(g(t_m),g(t_{m+1})),g(t_m))}(h(x))) = h(t_{m+1}).$$

If g and h are both real-computable functions, then we call f a *computable path.* If $g(0) = g(1)$ and $h(0) = h(1)$, then the path is a *loop.*

The intuition here is that the t_m are values at which the path switches from one chart to another. Of course, they are not uniquely defined, but by compactness of $[0, 1]$, we need only finitely many such points to express the entire path. We do *not* require any such sequence of points to be computable. Of course, any finite sequence of real numbers is immediately real-computable, but for an infinite collection of paths $p_{\boldsymbol{c}}$, indexed by $\boldsymbol{c}$ in some real-computable set C, we will call the collection *computable* if there is a single real-computable p satisfying

$p(\boldsymbol{c}, x) = p_{\boldsymbol{c}}(x)$, and *strongly computable* if an appropriate n and $t_1 < \cdots < t_{n-1}$ for each $p_{\boldsymbol{c}}$ can be computed uniformly in $\boldsymbol{c}$.

Weyl himself accepted the difficulty of carrying out homotopy theory on manifolds defined in this way as a drawback of this definition. He noted that Weil's criticism [23] of the introduction of homology via the Eilenberg-Steenrod axioms applied also to this definition of a manifold. In essence, Sections 2 and 3 of the present paper argue the two sides of this criticism. Weil claims that "triangulation is a quite trivial matter," and gives the structure of the fundamental group. This may be contrasted with a more abstract presentation, like the definition above, in which the computation of the fundamental group — in the views of both Weil and Weyl — is presumably more difficult.

A much more typical approach, especially in algebraic topology, is to replace a manifold with some combinatorial structure — a triangulation, a simplicial complex, a CW complex, or something similar. This is certainly the approach taken in much of the literature on applying classical computation to decision problems in topology [3, 16, 19, 10, 21, 11, 14]. Indeed, this approach was necessary there: while the combinatorial structures could be countable (and even discrete), at least for a reasonably well-behaved manifold, the manifolds themselves were uncountable, and thus inaccessible to direct computation by Turing machines.

Thus, the classical results on computation of properties of manifolds somehow implicitly assume Weil's claim that "triangulation is a quite trivial matter." The problem of how one gets the combinatorial representation of the space is, at least, put outside the question of computing invariants. One understanding of Section 2 is that triangulation — at least, when starting with a manifold in the way we have defined it — is quite difficult. On the other hand, one understanding of Section 3 is that once one can do that, the computation of the fundamental group is indeed possible.

Our notation mirrors that of Turing computability, in that we index real-computable functions (effectively) as $\varphi_{\boldsymbol{e}}$, where $\boldsymbol{e}$ is allowed to range over $\mathbb{R}^\infty$. Of course, the programs of such functions are finite, but they are allowed to use finitely many real parameters within those programs, yielding 2^ω-many such programs. This precludes procedures which simultaneously run all real-computable functions on an input (let alone on all possible inputs), but diagonalization arguments are often still possible. We have a real-computability version of Kleene's Recursion Theorem.

Theorem 5 (Recursion Theorem) *For each real-computable function f with domain $\mathbb{R}^\infty$, there exists an $\boldsymbol{x} \in \mathbb{R}^\infty$ such that $\varphi_{\boldsymbol{x}} = \varphi_{f(\boldsymbol{x})}$.*

Both this theorem and its proof are very similar to Kleene's original theorem for Turing computability; see for example [22, II.3.1]. Indeed, the proof there also adapts to show, in our context, that the tuple $\boldsymbol{x}$ can be real-computed uniformly from an index $\boldsymbol{e}$ such that $f = \varphi_{\boldsymbol{e}}$.

2 The Undecidability of Nullhomotopy and Simple Connectedness

Although this section is dedicated to proofs of undecidability, it will be useful for us to begin with a positive result about paths through computable manifolds.

Lemma 1. *Every loop f in a computable manifold M is homotopic to a computable loop $\overline{f}$ there. Indeed, such an $\overline{f}$ can be computed by a real-computable function whose only real parameters are the base point of f and the parameters of the inclusion functions defining M.*

Proof. Let $f = \langle g, h \rangle$ be the original loop, with $0 = t_0 < t_1 < \cdots < t_n = 1$ as described in Definition 4. We first give the intuition for the proof, without which the details would be baffling. Each segment $h\restriction[t_m, t_{m+1}]$ of the loop f lies within a single chart $U_{g(t_m)}$, which is simply connected, being homeomorphic to $\mathbb{R}^d$. Within $U_{g(0)} \cap U_{g(t_1)}$ we pick an arbitrary point whose coordinates in $U_{g(0)}$ are all rational, and we define our computable loop $\overline{f}$ to begin with a line segment (in the $U_{g(0)}$ coordinates) from $h(0)$ to this point, given linearly in the variable $t \in [0, t_1]$. Continuing from this point, we add a new line segment (in the $U_{g(t_1)}$ coordinates) from there to a rational point in $U_{g(t_1)} \cap U_{g(t_2)}$, and so on, with the last line segment going from a point in $U_{g(t_{n-2})} \cap U_{g(t_{n-1})}$ back to the base point in $U_{g(1)} = U_{g(0)}$. By simple connectedness of each $U_{g(t_m)}$ and each $U_{k(g(t_m),g(t_{m+1}))} = U_{g(t_m)} \cap U_{g(t_{m+1})}$, this path $\overline{f}$ is homotopic to the original f.

Notice that n and the points $t_1, \ldots, t_{n-1}$ are not given to us in any uniform way. However, since each $U_{g(t_m)} \cap U_{g(t_{m+1})}$ is open, its preimage under f is also open, and so we may assume that the points t_m have all been chosen (nonuniformly) to be rational. Now define $\overline{g} = g$, which is clearly real-computable since g is piecewise constant. The key to the proof is our program for computing $\overline{h}$, the second component of the computable loop $\overline{f}$.

Using the density of $\mathbb{Q}^d$ in $\mathbb{R}^d$, we start by selecting a point $\boldsymbol{x}_1$ in $\mathbb{Q}^d \cap \mathrm{range}(\varphi_{j(k(g(t_0),g(t_1)),g(t_0))})$, and defining $\overline{h}\restriction[t_0, t_1)$ to be the linear function from $\boldsymbol{x}_0 = h(0)$ to $\boldsymbol{x}_1$. Next, define $\overline{h}(t_1)$ to equal

$$\varphi_{j(k(g(t_0),g(t_1)),g(t_1))}(\varphi^{-1}_{j(k(g(t_0),g(t_1)),g(t_0))}(\boldsymbol{x}_1)).$$

Since $\boldsymbol{x}_1 = \lim_{t \to t_1^-} \overline{h}(t)$, this allows $\overline{f}$ to satisfy the continuity requirement in the definition of path. Moreover, $\overline{h}(t_1)$ can be computed with no new real parameters: we use i, j, k, the base point $\boldsymbol{x}_0$, and the rational point $\boldsymbol{x}_1$.

We then continue inductively, picking an arbitrary point $\boldsymbol{x}_{m+1} \in \mathbb{Q}^d \cap \mathrm{range}(\varphi_{j(k(g(t_m),g(t_{m+1})),g(t_m))})$, and defining $\overline{h}\restriction[t_m, t_{m+1})$ to be the linear function from $\overline{h}(t_m)$ to this point, then defining $\overline{h}(t_{m+1})$ to equal

$$\varphi_{j(k(g(t_m),g(t_{m+1})),g(t_{m+1}))}(\varphi^{-1}_{j(k(g(t_m),g(t_{m+1})),g(t_m))}(\boldsymbol{x}_m)).$$

When we reach the case $m + 1 = n$, of course, $t_{m+1} = 1$, so we no longer pick a rational point, but simply define $\boldsymbol{x}_n = \overline{h}(t_n) = \overline{h}(0)$, and let $\overline{h}\restriction[t_{n-1}, t_n]$ be the linear function from $\overline{h}(t_{n-1})$ to $\overline{h}(t_n)$. Thus we really do define a loop in M. Moreover, each line segment in the computable path $\overline{f}$ is homotopic to the corresponding segment $f\upharpoonright[t_m, t_{m+1}]$, since each $U_{g(t_m)}$ is simply connected. This proves Lemma 1.

Theorem 6 *Let M be any real-computable manifold which is connected but not simply connected. Then nullhomotopy of real-computable loops in M is not real-decidable: there is no real-computable function ψ such that for every index $\boldsymbol{c}$ of a computable loop $f = \varphi_{\boldsymbol{c}}$ in M, $\psi(\boldsymbol{c})$ halts with output 1 if f is nullhomotopic in M, but halts with output 0 if f is not nullhomotopic in M.*

Of course, the converse is trivial: if M is simply connected, then nullhomotopy is decidable. Also, the hypothesis of connectedness of M is only to simplify the proof; for arbitrary M, any connected component M'can be presented as a real-computable manifold, and if M' is not simply connected, then the theorem applies to M', hence gives the same result for M.

Proof. By assumption there is a loop in M, say with base point $\langle n, \boldsymbol{p}\rangle$, which is not nullhomotopic. By Lemma 1, it is homotopic to a computable loop $f = \langle g, h\rangle$ with the same base point. Using Theorem 5, the Recursion Theorem for real computability, for any real-computable ψ, we define a computable function $\varphi_{\boldsymbol{e}}$ which "knows its own index $\boldsymbol{e}$" and can feed that index to ψ:

$$\varphi_{\boldsymbol{e}}(t) = \begin{cases} f((t-1)2^{s+1}+2), & \text{if } t \in [\frac{2^s-1}{2^s}, \frac{2^{s+1}-1}{2^{s+1}}) \\ & \& \ \psi(\boldsymbol{e})\downarrow = 1 \text{ in exactly } s \text{ steps} \\ \langle n, \boldsymbol{p}\rangle, & \text{if not.} \end{cases}$$

The "if not" case automatically includes $t = 1$, so this $\varphi_{\boldsymbol{e}}$ is a computable loop, and is the constant loop $\langle n, \boldsymbol{p}\rangle$ (hence nullhomotopic) unless $\psi(\boldsymbol{e})\downarrow = 1$. If convergence to 1 happens in exactly s steps, then $\varphi_{\boldsymbol{e}}$ is homotopic to f, since on the interval $[\frac{2^s-1}{2^s}, \frac{2^{s+1}-1}{2^{s+1}}]$ it copies the entire loop f, while staying constant everywhere else. Because f is not nullhomotopic, neither is this $\varphi_{\boldsymbol{e}}$. Thus ψ does not correctly decide nullhomotopy of $\varphi_{\boldsymbol{e}}$. Indeed, the use of the Recursion Theorem is effective, so we may compute, uniformly in an index for ψ, the index $\boldsymbol{e}$ of the counterexample $\varphi_{\boldsymbol{e}}$ produced above.

It was important, in the foregoing proof, that the index $\boldsymbol{e}$ for the path $\varphi_{\boldsymbol{e}}$ did *not* need to include the finite sequence $0 = t_0 < t_1 < \cdots < t_n = 1$ whose existence is required by Definition 4. To give such a sequence, we would have been forced to choose n at some finite stage, thereby closing off the path and giving up our strategy of waiting out ψ until it made up its mind about the value of $\psi(\boldsymbol{e})$. In the proof, this sequence still exists, but it is not computable from $\boldsymbol{e}$.

Theorem 7 *Simple connectedness of real-computable manifolds is not decidable by any real-computable function. Specifically, there is no real-computable ψ such that for every real-computable M given by inclusion functions $i = \varphi_{\boldsymbol{c}}$, $j = \varphi_{\boldsymbol{d}}$, $j' = \varphi_{\boldsymbol{d}'}$, and $k = \varphi_{\boldsymbol{e}}$, $\psi(\boldsymbol{c}, \boldsymbol{d}, \boldsymbol{d}', \boldsymbol{e})$ converges to 1 if M is simply connected, and converges to 0 if M is not simply connected.*

Proof. Again the Recursion Theorem is key. Fixing any real-computable ψ, we define our parameters $\boldsymbol{c}$, etc., so that they start out by giving a basic non-simply-connected manifold, with $U_0 \cap U_1 = U_3$, $U_0 \cap U_2 = U_4$, $U_1 \cap U_2 = U_5$, and all other intersections empty. In particular, $U_0 \cap U_1 \cap U_2 = U_3 \cap U_2 = \emptyset$, so $\cup_{m \leq 5} U_m$ is not simply connected. Then the program given by our parameters bides its time while running ψ on input $\langle \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{d}', \boldsymbol{e} \rangle$. If there is a stage s at which this computation halts and outputs 0, then our program adds a new chart U_s containing all of $\cup_{m \leq 5} U_m$, thus making M simply connected. If there is no such s, then we never add any more new charts. So ψ does not decide simple connectedness of this M.

We remark that a stronger statement is possible: simple connectedness is not decidable by any function of the form $\lim_s \theta(\boldsymbol{c}, \boldsymbol{d}, \boldsymbol{d}', \boldsymbol{e}, s)$ with θ real-computable. (By analogy to Turing computability, we might say that no real-Δ^0_2 function decides simple connectedness.) One sees this by noting that in addition to the diagonalization strategy described above, we can extend a simply connected finite union of charts to a larger union which is not simply connected. So it is possible for M to flip back and forth every time θ changes its guess about the simple connectedness of M, forcing $\lim_s \theta$ to diverge.

3 Determining the Fundamental Group

A reasonable objection to the results of the previous section is that they somehow involve either a path no one would choose in computing a fundamental group or a manifold whose fundamental group no one would care to know. The present section will show that we can (non-uniformly) identify a system of loops, complete up to homotopy, which are free of the pathologies identified in the previous section, and that from such a system we can compute a presentation of the fundamental group.

Theorem 8 *Let M be an $\mathbb{R}$-computable manifold. Then there exists an $\mathbb{R}$-computable function S_M, defined on the naturals, such that the set $\{S_M(n) : n \in \omega\}$ consists of a set of indices for loops, and contains exactly one representative from each homotopy equivalence type.*

Proof. Fix a nice base point in M, such as the origin $\mathbf{0}$ in the chart U_0. The fundamental group $\pi_1(M)$ is countable, and using Lemma 1, we may (nonuniformly) choose a set of representatives for the homotopy-equivalence classes, all with this base point $\langle 0, \mathbf{0} \rangle$, such that each can be real-computed uniformly from a single code number in ω (coding two finite sequences of rational numbers, namely $\langle t_m \rangle_{m \leq n}$ and $\langle \boldsymbol{x}_m \rangle_{m \leq n}$ in the proof of Lemma 1, along with the n itself) using only the parameters of the inclusion functions for M. (The base point is now all rational.) Consequently, there exists a single additional real parameter, encoding all these natural numbers, using which a Blum-Shub-Smale machine can accept any input $n \in \omega$ and output the n-th of these representatives of the homotopy classes. That is, this machine computes precisely a function S_M as described in Theorem 8.

In [3], Brown proved that it was possible, from a presentation of a manifold as a finite simplicial complex, to compute a presentation for each of the homotopy groups of the manifold, including π_1. The following is a similar result (although the proof is quite different), showing that from the data S_M — analogous to a triangulation — we can pass to a presentation of π_1. As was observed in the earlier discussion from Weil and Weyl, here we have (at least superficially) a good deal less information than is given by presenting the manifold as a simplicial complex.

Proposition 1 *For every real-computable manifold M, with S_M as in Theorem 8, there exists a real-computable function $c_M : \omega \times \omega \to \omega$ which, on any input $\langle u, v\rangle$, outputs the unique w such that $S_M(u) * S_M(v) \simeq S_M(w)$.*

Proof. Since we can search through all $w \in \omega$ by dovetailing, it suffices to give a BSS machine which halts on input $\langle u, v, w\rangle$ iff $S_M(u) * S_M(v) \simeq S_M(w)$. Here we present such a machine. Notice that if F is a homotopy between two loops, then F itself has compact image in a manifold M, and therefore is contained within the union of finitely many charts in M. Due to space limitations, we will content ourselves here with pointing out the key to the inductive argument, which is as follows.

Let α and β be paths in M from point a to point b. Assume that $a \in U_m \cap U_p$ and $b \in U_n \cap U_p$, and that β lies entirely within U_p, while α has an initial segment contained in U_m and the remainder contained in U_n. We claim that $\alpha \simeq \beta$ within the submanifold $(U_m \cup U_n \cup U_p)$ iff $U_m \cap U_n \cap U_p \neq \emptyset$. The forward direction is essentially topology, of course, and we omit the details here. For the converse, suppose there exists a point $c \in U_m \cap U_n \cap U_p$, and let γ be a path from a to x lying within $U_m \cap U_p$, and δ a path from x to b within $U_n \cap U_p$. (This is possible because $U_m \cap U_p = U_{k(m,p)}$ is homeomorphic to $\mathbb{R}^d$, hence path-connected. Our presentation of M with all pairwise intersections of charts simply connected constitutes a very strong presentation!) Also, fix t such that $\alpha(t) \in U_m \cap U_n$, and let θ be a path from x to $\alpha(t)$ lying within $U_m \cap U_n$. Then

$$\alpha \simeq (\alpha\restriction[0,t]) * \theta^{-1} * \theta * (\alpha\restriction[t,1]) \simeq \gamma * \delta$$

since γ and $(\alpha\restriction[0,t] * \theta^{-1})$ both lie within U_m, hence are homotopic, and likewise δ and $(\theta * \alpha\restriction[t,1])$. On the other hand, $(\gamma * \delta)$ is contained within U_p, hence is homotopic to β. This proves $\alpha \simeq \beta$.

The subsequent inductive argument essentially involves proving that every homotopy can be viewed as a finite string of such operations, deforming the original path α into an α' which goes through the same sequence of charts, up to a change by one (either one old chart replaced by two new ones, or two old ones replaced by one new one, in the sequence of charts intersected by α'). The crucial point about our presentations of loops from Lemma 1, therefore, as used in Theorem 8, is that for each loop, we know a sequence of charts containing that loop (including the order in which the loop intersects those charts). Of course, this sequence is not at all unique, but we need only know one such sequence for each of our representatives $S_M(u)$, and all that information was encoded into a single real parameter used in computing S_M. Moreover, that parameter also provides the same information about the concatenation $(S_M(u) * S_M(v))$ of any two of those representatives.

Of course, for arbitrary m and n, the inclusion function i for M tells us whether $U_m \cap U_n = \emptyset$: just check whether $i(m,n) = -1$. Therefore, we may search through increasingly large finite sets of charts, and increasingly long finite perturbations of a given concatenation $(S_M(u) * S_M(v)$ through those charts, and if indeed $(S_M(u) * S_M(v)) \simeq S_M(w)$, then eventually we will find a sequence of charts and paths through those charts to prove it. On the other hand, since S_M maps ω (or an initial segment, when $\pi_1(M)$ is finite) bijectively onto $\pi_1(M)$, there is exactly one $S_M(w)$ for which we will ever find such a proof, and when we have found it, we know that $(S_M(u) * S_M(v)) \simeq S_M(w)$.

Corollary 1 *Let M be an $\mathbb{R}$-computable manifold. There is a uniform procedure to pass from an index for S_M to indices for a real-computable presentation of the group $\pi_1(M)$ and for computing its word problem.*

References

1. L. Blum, F. Cucker, M. Shub & S. Smale, *Complexity and Real Computation* (Springer, 1997).
2. L. Blum, M. Shub, & S. Smale, On a theory of computation and complexity over the real numbers, *Bulletin of the American Mathematical Society (New Series)* **21** (1989), 1-46.
3. E.H. Brown, Computability of Postnikov Complexes, *Annals of Mathematics* **65** (1957), 1-20.
4. P. Bürgisser, F. Cucker & P. Jacobé de Naurois; The complexity of semilinear problems in succinct representation, *Computational Complexity* **15** 2006, 197–235.
5. P. Bürgisser, F. Cucker & M. Lotz; The complexity to compute the Euler characteristic of complex Varieties, *Comptes Rendus Mathématique, Académie des Sciences, Paris* **339** (2004), 371-376.
6. P. Bürgisser & M. Lotz; The complexity of computing the Hilbert polynomial of smooth equidimensional complex projective varieties, *Foundations of Computational Mathematics* **7** (2007) 51-86.
7. W. Calvert, On three notions of effective computation over $\mathbb{R}$, to appear in *Logic Journal of the IGPL*.
8. W. Calvert & J.E. Porter, Some results on $\mathbb{R}$-computable structures, preprint, 2009.
9. Ю. Л. Ершов, Определимость и Вичислимость, Сибирская Школа Алгебры и Логики (Научная Книга, 1996).
10. W. Haken; Theorie der Normalflächen, *Acta Mathematica* **105** (1961) 245-375.
11. W. Haken; Ein Verfahren zur Aufspaltung einer 3-Mannigfaltigkeit in irruduzible 3-Mannigfaltigkeiten *Matematische Zeitschrift* **76** (1961) 427-467.
12. J.D. Hamkins, R. Miller, D. Seabold, & S. Warner, An introduction to infinite time computable model theory, in *New Computational Paradigms*, eds. S.B. Cooper, B. Löwe, & A. Sorbi (Springer, 2008) 521–557.
13. A. Hatcher, *Algebraic Topology* (Cambridge University Press: Cambridge, 2001).
14. G. Hemion; On the classification of homeomorphisms of 2-manifolds and the classification of 3-manifolds, *Acta Mathematica* **142** (1979) 123-155.

15. G. Hjorth, B. Khoussainov, A. Montalban and A. Nies, ¿From automatic structures to Borel structures, preprint (2008).
16. A. Markov; Unsolvability of certain problems in topology, *Doklady Akademii Nauk SSSR* **123** (1958) 978-980.
17. R. Miller, Locally computable structures, in *Computation and Logic in the Real World - Third Conference on Computability in Europe, CiE 2007*, eds. B. Cooper, B. Löwe, & A. Sorbi, *Lecture Notes in Computer Science* **4497** (Springer-Verlag: Berlin, 2007), 575-584.
18. А.С. Морозов, Элементарные подмодели параметризуемых моделей, Сибирский Математический Журнал **47** (2006), 595-612.
19. A. Nabutovsky & S. Weinberger; Algorithmic aspects of homeomorphism problems, *Tel Aviv Topology Conference (1998)*, Contemporary Mathematics 231, American Mathematical Society, 1999, 245-250.
20. P. Scheiblechner; On the complexity of deciding connectedness and computing Betti numbers of a complex algebraic variety, *Journal of Complexity*, **23** (2007), 359-397.
21. H. Schubert; Bestimmung ber Primfaktorzerlegung von Verkettungen, *Mathematische Zeitschrift* **76** (1961) 116-148.
22. R.I. Soare; *Recursively Enumerable Sets and Degrees* (Springer-Verlag: New York, 1987).
23. A. Weil, Review of *Introduction to the theory of algebraic functions of one variable*, by C. Chevalley, *Bulletin of the American Mathematical Society* **57** (1951), 384-398.
24. H. Weyl, *The concept of a Riemann surface*, 3rd ed. (Addison-Wesley, 1955)