Classes of structures with universe a subset of ω_1

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Abstract

We continue recent work on computable structure theory in the setting of ω_1. We prove the analog of a result from [5] saying that isomorphism of computable structures lies "on top" among Σ^1_1 equivalence relations relations on ω. Our equivalence relations are on ω_1. In the standard setting, Σ^1_1 sets are characterized in terms of paths through trees. In the setting of ω_1, we use a new characterization of Σ^1_1 sets that involves clubs in ω_1.

1 Introduction

There is some recent work on computable structure theory in the setting of ω_1 [9], [3], [10]. We assume at least that all subsets of ω are constructible, and in some places, we assume that all subsets of ω_1 are constructible. The basic definitions come from "α-recursion" theory, where $\alpha = \omega_1$ (see [16]).

Definition 1.1.

- *A set or relation on ω_1 is* computably enumerable*, or* c.e.*, if it is defined in $(L_{\omega_1}, \in)$ by a Σ_1-formula $\varphi(\bar{c}, x)$, with finitely many parameters—a Σ_1 formula is finitary, with only existential and bounded quantifiers.*
- *A set or relation is* computable *if it and its complement are both computably enumerable.*
- *A (partial) function is* computable *if its graph is c.e.*

Results of Gödel give us a $1-1$ function g from ω_1 onto L_{ω_1} such that the relation $g(\alpha) \in g(\beta)$ is computable. The function g gives us ordinal codes for

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sets, so that computing on ω_1 is really the same as computing on L_{ω_1}. There is also a computable function ℓ taking α to the code for L_α. Using the fact that L_{ω_1} is closed under α-sequences for any countable ordinal α, we may allow relations and functions of arity α, where α is any countable ordinal.

As in the standard setting, we have indices for c.e. sets. We have a c.e. set C of codes for pairs $(\varphi, \overline{c})$, representing Σ_1 definitions—$\varphi(\overline{u}, x)$ is a Σ_1-formula and $\overline{c}$ is a tuple of parameters appropriate for $\overline{u}$. We have a computable function h mapping ω_1 onto C. The ordinal α is a *c.e. index* for the set X if $h(\alpha)$ is the code for a pair $(\varphi, \overline{c})$, where $\varphi(\overline{c}, x)$ is a Σ_1 definition of X in $(L_{\omega_1}, \in)$. We write W_α for the c.e. set with index α. Suppose W_α is determined by the pair $(\varphi, \overline{c})$; i.e., $\varphi(\overline{c}, x)$ is a Σ_1 definition. We say that *x is in W_α at stage β*, and we write $x \in W_{\alpha,\beta}$, if L_β contains x, the parameters $\overline{c}$, and witnesses making the formula $\varphi(\overline{c}, x)$ true. The relation $x \in W_{\alpha,\beta}$ is computable. Let $U \subseteq (\omega_1)^2$ consist of the pairs (α, β) s.t. $\beta \in W_\alpha$. Then U is m-complete c.e. It is not computable, since the "halting set" $K = \{\alpha : \alpha \in W_\alpha\}$ is c.e. and not computable.

In the setting of ω_1, we have a good notion of relative computability.

Definition 1.2.

- *A relation is* c.e. relative to X *if it is Σ_1-definable in $(L_{\omega_1}, \in, X)$.*
- *A relation is* computable relative to X *if it and its complement are both c.e. relative to X.*
- *A (partial or total) function is* computable relative to X *if the graph is c.e. relative to X.*

A *c.e. index* for R relative to X is an ordinal α s.t. $g(h(\alpha)) = (\varphi, \overline{c})$, where φ is a Σ_1 formula (in the language with $\in$ and a predicate symbol for X), and $\varphi(\overline{c}, x)$ defines R in $(L_{\omega_1}, \in, X)$. We write W_α^X for the c.e.set with index α relative to X. As in the standard setting, we have a universal c.e. set of partial computations using oracle information. Let U consist of the codes for triples (σ, α, β) s.t. $\sigma \in 2^\rho$ (for some countable ordinal ρ), and for X with characteristic function extending σ, $\beta \in W_\alpha^X$. Then U is c.e.

Definition 1.3. *The* jump *of X is $X' = \{(\alpha, x) : x \in W_\alpha^X\}$.*

We can iterate the jump function through countable levels. We let $X^{(0)} = X$, $X^{(\alpha+1)} = (X^{(\alpha)})'$, and for limit α, $X^{(\alpha)}$ is the set of codes for pairs (β, x) s.t. $\beta < \alpha$ and $x \in X^{(\beta)}$. As L_{ω_1} is closed under countable sequences, it follows that for countable limit λ, $X^{(\lambda)}$ is the least upper bound of the $X^{(\alpha)}$ for $\alpha < \lambda$, in the ordering of relative computability.

1.1 Computable structures

We consider structures with universe a subset of ω_1. As in the standard setting, we usually identify a structure with its atomic diagram. A structure is *computable* if the atomic diagram is computable. We see that the ordered field of reals has a computable copy with universe ω_1. If we think of the reals as a subset of L_{ω_1}, where each number is represented by a rational cut, this is a computable structure. The field of complex numbers has a computable copy. We may even add exponential functions such as exp, noting that any analytic function is determined by the countable sequence of coefficients of a power series.

In the standard setting, Morley [14] and Millar [13] showed that for any countable complete decidable elementary first order theory T, there is a decidable saturated model iff there is a computable enumeration of the complete types consistent with T. In the setting of ω_1, we have the following.

Proposition 1.4. *For any countable complete elementary first order theory T (with infinite models), T has a decidable saturated model with universe ω_1.*

In the standard setting, the first non-computable ordinal, ω_1^{CK}, is the next admissible ordinal after ω. In the setting of ω_1, the first non-computable ordinal comes much before the next admissible after ω_1. Shore gave a proof of this, which is included in [9][1]. In the standard setting, the *Harrison ordering* is a computable ordering of type $\omega_1^{CK}(1+\eta)$. This ordering has initial segments isomorphic to all computable well orderings. In the setting of ω_1, we have the following.

Theorem 1.5 (Greenberg-Knight-Shore)**.** *There is a computable ordering $\mathcal{H}$ with initial segments isomorphic to all computable ordinals.*

Sketch of proof. We take a uniformly computable list of linear orderings, representing all computable isomorphism types, and carry out a finite-injury priority construction to produce $\mathcal{H}$ with an initial segment that is a sum of intervals representing the well ordered $\mathcal{A}_\alpha$, in order, followed by various other intervals that are not well ordered.

□

The following result holds in the standard setting [5].

Theorem 1.6 (Fokina-Friedman-Harizanov-Knight-McCoy-Montalbán)**.** *For any Σ^1_1 equivalence relation E on ω, there is a uniformly computable sequence of trees $(T_n)_{n\in\omega}$ (subtrees of $\omega^{<\omega}$) such that*

$$mEn \Leftrightarrow T_m \cong T_n \ .$$

In [5], the result for trees is used to show that isomorphism on computable members of certain other classes lies on top in the same way: notably torsion-free Abelian groups and Abelian p-groups.

We shall lift Theorem 1.6 to the setting of ω_1.

[1] Here is the argument: Let α be the least admissible after ω_1. Then the set of computable wellorderings of ω_1 is an element of L_α and the function f that takes such a wellordering to its length is Σ_1 definable over L_α; it follows that the range of f is bounded in α.

Theorem 1.7. *Assume $V = L$. For any Σ^1_1 equivalence relation E on ω_1, there is a uniformly computable sequence of structures $M^*(\alpha)_{\alpha<\omega_1}$ (with universe ω_1) such that $\alpha E \beta$ iff $M^*(\alpha) \cong M^*(\beta)$.*

2 Σ^1_1 sets

Recall that in the standard setting, a set $S \subseteq \omega$ is Σ^1_1 if there is a computable relation $R(x, u)$ such that

$$n \in S \Leftrightarrow (\exists f \in \omega^\omega)\,(\forall s \in \omega)\, R(n, f \upharpoonright s)\ .$$

Kleene showed the following.

Theorem 2.1 (Kleene)**.** *If S is Σ^1_1, then there is a uniformly computable sequence of trees $(T_x)_{x\in\omega}$ such that $x \in S$ iff T_x has a path.*

In the standard setting, a computable tree with no path has a tree rank that is a computable ordinal. The ordinal tree ranks were crucial to the proof of Theorem 1.6. In our setting, we do not have enough computable ordinals, so we will need a new idea. We take the following as our definition of Σ^1_1 subset of ω_1.

Definition 2.2. *A set $S \subseteq \omega_1$ is Σ^1_1 if there is a computable relation R, on ordinals and functions $f \in \omega_1^{\omega_1}$, such that $x \in S$ iff $(\exists f \in \omega_1^{\omega_1})\,(\forall \beta \in \omega_1)\, R(x, f \upharpoonright \beta)$.*

Lemma 2.3. *For any Σ^1_1 set $S \subseteq \omega_1$, there is a uniformly computable sequence $(T_x)_{x<\omega_1}$ of subtrees of $\omega_1^{<\omega_1}$ such that $x \in S$ iff T_x has an ω_1-branch.*

Proof. We do just what Kleene did. Let T_x consist of those $\sigma \in \omega_1^{<\omega_1}$ such that $\forall \beta < length(\sigma)\, R(x, \sigma \upharpoonright \beta)$.

□

We show that all Σ^1_1 sets $S \subseteq \omega_1$ are m-reducible to the isomorphism relation on computable subtrees of $\omega_1^{<\omega_1}$. In fact, there is a special tree T such that for any Σ^1_1 set S, there is a uniformly computable sequence of trees $(T_x)_{x<\omega_1}$ such that $x \in S$ iff $T_x \cong T$.

Description of the special tree T

The tree T has just one node $\emptyset$ at level 0. This node has $\aleph_1$ successors. For each node above level 0, there are $\aleph_1$ copies. Half of the copies are terminal, while the other half have $\aleph_1$ successors. Let T be the set of functions σ from countable ordinals to $\omega_1 \times \{0, 1, 2\}$ such that if σ has last term $(\beta, 0)$, then σ is terminal, and if σ has limit length α, with terms $(\beta, 1)$ for arbitrarily large $\beta < \alpha$, then σ is also terminal. The elements of T are the sequences σ mapping countable ordinals α to $\omega_1 \times \{0, 1, 2\}$ such that if there is a term $(\beta, 0)$, then σ has length $\beta + 1$, and if there are infinitely many terms $(\beta_i, 1)$ and $\beta = \sup\{\beta_i\}$, then σ has length β.

In [5], we combined subtrees of $\omega^{<\omega}$, using a kind of product. We define the analogous product for subtrees of $\omega_1^{<\omega_1}$.

Definition 2.4. *Suppose T_1, T_2 are subtrees of $\omega_1^{<\omega_1}$. Then $T_1^* T_2$ is the subtree of $(\omega_1 \times \omega_1)^{<\omega_1}$ consisting of the functions τ such that for some $\sigma_1 \in T_1$ and $\sigma_2 \in T_2$, both of length α, τ has length α and for all $\beta < \alpha$, $\tau(\beta) = (\sigma_1(\beta), \sigma_2(\beta))$.*

It is easy to see that the tree $T_1^* T_2$ has an ω_1-branch iff T_1 and T_2 each have an ω_1-branch.

Lemma 2.5. *Let T be the special tree defined above. For any tree $P \subseteq \omega_1^{<\omega_1}$, if P has an ω_1-branch, then $P^* T \cong T$, and if P has no ω_1-branch, then $P^* T$ also has no ω_1-branch.*

Combining the two lemmas, we get the following.

Proposition 2.6. *For any Σ_1^1 set $S \subseteq \omega_1$, there is a uniformly computable sequence of trees $(T_\alpha)_{\alpha < \omega_1}$ such that $\alpha \in S$ iff $T_\alpha \cong T$.*

The structures that we produce for our main result (Theorem 1.7) are not members of any familiar class. The structures in the range of our embedding will each code a sequence of sets $(X_\beta)_{\beta < \omega_1}$, up to an equivalence relation $\sim$, which is defined as follows.

Definition 2.7. *For $X, Y \subseteq \omega_1$, $X \sim Y$ iff $X \Delta Y$ is not stationary.*

Lemma 2.8. *For any Σ_1^1 set $X \subseteq \omega_1$, there is a uniformly computable sequence $(S_\alpha)_{\alpha < \omega_1}$ of subsets of ω_1 such that $\alpha \in X$ iff S_α contains a club.*

Proof. Choose a uniformly computable sequence of trees $(T_\alpha)_{\alpha < \omega_1}$ as in Lemma 2.3. Thus $\alpha \in S$ iff T_α has an ω_1-branch. For ordinals $\alpha < \beta \leq \omega_1$ we let T_α^β be the interpretation of the tree T_α in L_β, using its Δ_1 definition. In particular, $T_\alpha^{\omega_1} = T_\alpha$.

Now let S_α be the set of countable ordinals $\beta > \alpha$ such that for some countable $\gamma > \beta$,

1. $L_\gamma \models ZF^-$ (ZF minus Power Set),
2. $\omega_1^{L_\gamma} = \beta$,
3. T_α^β is a tree which has a branch of length β in L_γ.

First, suppose that T_α has an ω_1-branch b. We must show that S_α contains a club.

Suppose that M is a countable elementary substructure of L_{ω_2} such that $b \in M$. Then the transitive collapse, denoted by $\overline{M}$, has the form L_γ. Let $\beta = \omega_1^{\overline{M}}$. Since b is an ω_1-branch through the tree $T_\alpha = T_\alpha^{\omega_1}$, $b \restriction \beta$ is a β-branch through the tree T_α^β that belongs to L_γ and therefore γ witnesses that β belongs to S_α.

Now form a continuous chain $(M_i)_{i < \omega_1}$ of countable elementary substructures of L_{ω_2}. Let $\overline{M}_i$ be the transitive collapse of M_i. Then $\overline{M}_i = L_{\gamma_i}$, for some countable ordinal γ_i. Let $\beta_i = \omega_1^{L_{\gamma_i}}$. Then the sequence $(\beta_i)_{i < \omega_1}$ enumerates a club c in ω_1. For each i, the image of b under the transitive collapse of M_i, $\pi_i(b)$, is a β_i-branch through $T_\alpha^{\beta_i}$ belonging to L_{γ_i}, witnessing that β_i belongs to S_α. Thus c is the required club.

Conversely, we must show that if T_α has no ω_1-branch, then S_α does not contain a club. Suppose that c is a club and we will show that some element of c does not belong to S_α. Let M be the least elementary substructure of L_{ω_2} such that $c, \alpha, \omega_1 \in M$. In L_{ω_2}, T_α has no ω_1-branch, so the same holds in M. Again, we take the transitive collapse $\pi(M) = \overline{M} = L_{\overline{\gamma}}$. We have $\beta = \omega_1^{\overline{M}} \in c$ and T_α^β has no β-branch in $L_{\overline{\gamma}}$. We claim that β does not belong to S_α. Indeed, suppose otherwise and that the ordinal γ witnesses this. Then γ must be greater than $\overline{\gamma}$, as T_α^β has no β-branch in $L_{\overline{\gamma}}$. But if γ is greater than $\overline{\gamma}$, then β is countable in L_γ, as M was chosen to be the least elementary substructure of L_{ω_2} containing the parameters c, α, ω_1. We have reached the desired contradiction.

□

Let E be a Σ_1^1 equivalence relation on ω_1. We identify pairs of ordinals with single ordinals and let S be as above, so that $\alpha E \beta$ iff $S_{\alpha,\beta}$ contains a club. For any $X \subseteq \omega_1$, let $L(X)$ be the $\aleph_1$-like linear order formed by stacking ω_1 many copies of the rational order, and at limit stage α putting in a supremum iff $\alpha \in X$.

Lemma 2.9. *For $X, Y \subseteq \omega_1$, $L(X) \cong L(Y)$ iff $X \sim Y$.*

Proof. Suppose $L(X) \cong L(Y)$. Then for a club C of α's, $\alpha \in X$ iff $\alpha \in Y$. Therefore, $X \sim Y$. Suppose $L(X) \not\cong L(Y)$. Then there is no club C of α's such that $\alpha \in X$ iff $\alpha \in Y$. If there were, then we could build an isomorphism between $L(X)$ an $L(Y)$. This means that $X \Delta Y$ is stationary, so $X \not\sim Y$.

□

We use ideas from [5]. For any finite chain $c = (\alpha, \gamma_1, \gamma_2, \ldots, \gamma_n, \beta)$, let

$$S(c) = S_{\alpha,\gamma_1} \cap S_{\gamma_1,\gamma_2} \cap \ldots \cap S_{\gamma_n,\beta}$$

If $\alpha' E \alpha$, then $S_{\alpha',\alpha}$ contains a club. Therefore, for each finite chain c from α to β, $S_{\alpha',\alpha} \cap S(c) \sim S(c)$. It follows that if we define $S^*(\alpha, \beta)$ to be the set of the $S(c)$, where c is a chain starting with α and ending with β, and $\alpha E \alpha'$, then $S^*(\alpha, \beta)$ agrees with $S^*(\alpha', \beta)$, in the sense that they have the same elements modulo the ideal of nonstationary sets. Let $M(\alpha, \beta)$ be the structure that is the "free union" of ω_1 copies of the linear orders $L(X)$ for $X \in S^*(\alpha, \beta)$. One way to make this precise is to let $M(\alpha, \beta)$ consist of two disjoint sets A, B of size ω_1, with a relation $R(a, b_0, b_1)$ for a in A and b_0, b_1 in B so that for each fixed a, $R(a, -, -)$ defines a linear order of B isomorphic to one of the $L(X)$, for $X \in S^*(\alpha, \beta)$, and each such order occurs for exactly ω_1-many such a in A.

Alternatively, we may let $M(\alpha, \beta)$ have an equivalence relation with an ordering on each equivalence class, so that for each set $X \in S^*(\alpha, \beta)$, the ordering $L(X)$ is copied in uncountably many equivalence classes, and for each equivalence class, the ordering on the equivalence class is isomorphic to $L(X)$ for some $X \in S^*(\alpha, \beta)$. We note that in either case, the language of the structures $M(\alpha, \beta)$ is finite.

Lemma 2.10.

1. *If $\alpha E \alpha'$, then for all β, $M(\alpha, \beta) \cong M(\alpha', \beta)$.*

2. *If it is not the case that $\alpha E \alpha'$, then $M(\alpha, \alpha) \ncong M(\alpha', \alpha)$.*

Proof. (1) is clear. For (2), we note that if it is not the case that $\alpha E \alpha'$, then there is no set $X \in S^*(\alpha, \alpha')$ that contains a club, but there is such a set in $S^*(\alpha, \alpha)$. From this it follows that $M(\alpha, \alpha)$ is not isomorphic to $M(\alpha, \alpha')$.

□

The structures $M(\alpha, \beta)$ have a finite language. Finally, let $M^*(\alpha)$ be the *sequence* (not the free union) of structures $M(\alpha, \beta)$, for $\beta < \omega_1$. We could add to the language a disjoint family of unary predicates $(U_\beta)_{\beta < \omega_1}$, where U_β is the universe of a copy of $M(\alpha, \beta)$. We would like to keep the language finite. So, instead of this, we let $M^*(\alpha)$ be a structure that includes a copy of ω_1 with the usual ordering, and has a predicate associating to each $\beta < \omega_1$ one of a family of sets, disjoint from ω_1 and disjoint from each other. We put a copy of $M(\alpha, \beta)$ on the set associated with β.

Lemma 2.11. *For all α, α', $\alpha E \alpha'$ iff $M^*(\alpha) \cong M^*(\alpha')$.*

Proof. If $\alpha E \alpha'$, then for all β, $M(\alpha, \beta) \cong M(\alpha', \beta)$. Then $M^*(\alpha) \cong M^*(\alpha')$. If it is not the case that $\alpha E \alpha'$, then $M(\alpha, \alpha) \ncong M(\alpha', \alpha)$. Then we have $M^*(\alpha) \ncong M^*(\alpha')$.

□

Our structures $M^*(\alpha)$ are in a finite relational language, and we may use standard coding tricks to transform them into undirected graphs. We represent each element by a point attached to a triangle. For an n-place relation symbol R, we represent each n-tuple of elements by a special point, attached by chains of length $1, 2, \ldots, n$ to the points representing the elements.

3 Turing computable embeddings

H. Friedman and Stanley [6] introduced the notion of Borel embedding for comparing classification problems for classes of countable structures. The notion of Turing computable embedding [2] allows some finer distinctions. Here we define the analogue of Turing computable embedding for structures with universe a subset of ω_1. We write $K \leq_{tc} K'$ if there is a computable operator Φ taking structures in K to structures in K' such that for $\mathcal{A}, \mathcal{B} \in K$, $\mathcal{A} \cong \mathcal{B}$ iff $\Phi(\mathcal{A}) \cong \Phi(\mathcal{B})$.

In the standard setting, the class of undirected graphs lies on top among classes of countable structures under $\leq_{tc}$. The same is true in our setting. Let L be a computable relational language—L may be uncountable, and it may include symbols of arity α for computable ordinals α. When we say that the language is computable, we mean that the set of relation symbols is computable, and we have a computable function assigning a countable ordinal arity to each symbol. Let $Mod(L)$ be the class of L structures with universe a subset of ω_1. Let UG be the class of undirected graphs.

Proposition 3.1. $Mod(L) \leq_{tc} UG$

Proof. We first give a transformation that replaces the language L by one with just finitely many relations of finite arity. We have a predicate U, for elements of the structure M. We have a predicate O with an ordering of type ω_1. We have another predicate S for special points. For each predicate symbol R, say of arity α, and each α-tuple in U, we have a special point in S. There is a relation Q that holds of $x \in O$, $p \in S$ and $a \in U$ if p is the special point corresponding to some R of arity α and some $\sigma \in M^\alpha$ just in case $a = \sigma(\beta)$ and x is the β^{th} element of O. We let T be the set of special points in S such that the relation R in M of the tuple. The unary predicates U, O, and S are disjoint, and the universe of our structure M^* is the union. Beyond these, we have a binary relation—the ordering on O, a ternary relation Q, and the set $T \subseteq S$. So, the language is finite. It is not difficult to see that $M_1 \cong M_2$ iff $M_1^* \cong M_2^*$. We can apply Marker's transformation to pass from the structures M^* to undirected graphs, still preserving isomorphism.

□

In [6], H. Friedman and Stanley give a Borel embedding of undirected graphs into fields (of arbitrary characteristic). The embedding is effective. Moreover, we can use the same idea to give an embedding in our uncountable setting.

Proposition 3.2. *If K is the class of undirected graphs, and K' is the class of fields of characteristic* 0 *(or any other desired characteristic), then $K \leq_{tc} K'$.*

Proof. Let F^* be a large algebraically closed field of the desired characteristic, with independent transcendentals b_α, for $\alpha < \omega_1$. For a graph G, with universe a subset of ω_1, we let $\Phi(G)$ be the subfield of F^* generated by the elements b_α, for $\alpha \in G$, the elements algebraic over a single one of these b_α, and elements $\sqrt{d + d'}$, where for some b_α and b_β that we have included, there is an edge between the corresponding graph elements, and d is inter-algebraic with b_α while d' is inter-algebraic with b_β.

□

In [6], H. Friedman and Stanley give a Borel embedding of undirected graphs into linear orderings. This embedding is effective. We can use the same idea, with modifications, to give an embedding in our uncountable setting.

Proposition 3.3. *If K is the class of undirected graphs and K' is the class of linear orderings, then $K \leq_{tc} K'$.*

Proof. By Proposition 1.4, we have a saturated model Q of the theory of dense linear orderings without endpoints. (We are assuming $V = L$, so we have CH.) We want more than this. Consider the theory of a structure whose universe is the union of disjoint predicates U and V, where on V, there is a dense linear ordering without endpoints, and there is a function mapping f from V onto U such that for each $u \in U$, $f^{-1}(u)$ is dense in V. We consider a computable saturated model Q^* of this theory, with universe ω_1. We have a type ω_1 ordering of U, inherited from the ordering on ω_1, and we identify the elements of U with the countable ordinals. Let Q_α be $f^{-1}(u)$, where u is the α^{th} element of U. The sets Q_0 and Q_1 play a special role. We make a list of the atomic types of countable graphs t_α, $\alpha < \omega_1$. Consider the lexicographic ordering on $(Q^*)^{<\omega_1}$. The ordering corresponding to a given graph G will be a sub-ordering $\Phi(G)$ of this. Consider the sequences σ of length $2\beta + 2$ such that for some β-tuple $\overline{a}$ from G, satisfying the atomic type t_α, we have

1. for $\gamma < \beta$, $\sigma(2\gamma) \in Q_0$, and $\sigma(2\gamma + 1) \in Q_{2+a_\gamma}$,
2. $\sigma(2\beta) \in Q_1$,
3. $\sigma(2\beta + 1)$ is an element of U identified with an ordinal less than α.

The elements of $\Phi(G)$ are the sequences σ of the form above. We say that σ *represents* $\overline{a}$ if σ is related to $\overline{a}$ in the way described. The ordering $L(G)$ is made up of intervals having the order type α, for the various atomic types t_α realized in G. It takes some effort to show that $G_1 \cong G_2$ iff $\Phi(G_1) \cong \Phi(G_2)$.

First, suppose $G_1 \cong G_2$ via f. To show that $\Phi(G_1) \cong \Phi(G_2)$, we consider the set of countable partial isomorphisms p, where for some countable family of $\sigma_i \in L(G_1)$, where σ_i represents $\overline{a}_i \in G_1$,

1. if $p(\sigma_i) = \tau_i$, where σ_i represents $\overline{a}$, then τ_i represents the corresponding sequence $f(\overline{a}_i)$,
2. if σ_i represents $\overline{a}_i$, realizing type t_α, then p maps the full interval of type α containing σ_i to the corresponding interval of type α containing τ_i.

The fact that Q^* is saturated allows us to show that the family of countable partial isomorphisms has the back-and-forth property.

Next, suppose $L(G_1) \cong L(G_2)$ via f. We must define an isomorphism g from G_1 onto G_2. The universes of G_1 and G_2 are subsets of ω_1, so we have lists of elements $(a_\alpha)_{\alpha<\omega_1}$ and $(b_\alpha)_{\alpha<\omega_1}$. At step α, we have a countable partial isomorphism with a_α is in the domain and b_α in the range. Take $\sigma \in L(G_1)$ representing the sequence (a_0), of atomic type t_δ. Then $\sigma = r_0 q_0 r_1 \beta$, where $r_0 \in Q_0$, $q_0 \in Q_{2+a_0}$, $r_1 \in Q_1$, and $\beta < \delta$. Then $f(\sigma)$ is a sequence of length 4, of the form r_0', q_0', r_1', β, where $r_0' \in Q_0$, $q_0' \in Q_d$, $r_1' \in Q_{2+d}$, for some $d \in G_2$. We let $g(a_0) = d$. If $b_0 = d$, then we are done. Otherwise, we consider a sequence $\tau \in L(G_2)$ of length 6 such that $\tau \supseteq r_0' q_0'$ and τ represents (d, b_0). The pre-image of this τ under f has the form $(r_0 q_0 r_1'' q_1, r_2, \delta'')$, where $q_1 \in Q_{2+c}$, for some c. We let $g(c) = b_0$. We continue in this way, letting f guide us in finding the appropriate image or pre-image for the next element.

$\square$

4 Results on fields

Here we consider arbitrary ω_1-computable fields of characteristic 0. The domain of each field is either ω_1 or possibly just ω, and the field operations are all ω_1-computable. We believe that our results carry over equally well to fields of positive characteristic.

Lemma 4.1. *Every ω_1-computable field has a computable transcendence basis over its prime subfield Q (which is ω_1-computable, being countable).*

Proof. For each $\alpha \in F$ we define $\alpha \in B$ iff

$$(\forall \langle \beta_1, \ldots, \beta_n \rangle \in \alpha^{<\omega})(\forall p \in Q[X_1, \ldots, X_n, Y])$$
$$[p(\beta_1, \ldots, \beta_n, \alpha) = 0 \implies p(\beta_1, \ldots, \beta_n, Y) = 0].$$

This statement quantifies only over countable sets which we can enumerate uniformly and know when we have finished enumerating each one. It says that α lies in B iff α satisfies no nonzero polynomial over the subfield $Q(\beta : \beta < \alpha)$ generated by all elements $< \alpha$. Clearly this B is a transcendence basis for F.

$\square$

Corollary 4.2. *The field C is relatively ω_1-computably categorical.*

Proof. Given any two ω_1-computable fields $E \cong F \cong C$, use the lemma to find computable transcendence bases B for E and C for F. Let f be any computable bijection from B onto C (for instance, let $f(\alpha)$ be the least element of C which is $> f(\beta)$ for every $\beta < \alpha$). This f extends effectively to an isomorphism from E onto F: for each element $x \in E - B$ in order, find the minimal polynomial $p(X)$ of x over $Q(B \cap x)$ (using the splitting algorithm provided by Kronecker for purely transcendental field extensions of Q), and map x to the least root in F of the image of $p(X)$ in $F[X]$ under the map f on the coefficients of $p(X)$. By normality of F over $Q(C)$, at every step this map still extends to an isomorphism from E into F, so we always find such a root in F. Moreover, since f maps B onto the transcendence basis C for F, f must map E onto all of F: every $y \in F$ has a minimal polynomial $p(X) \in Q(C)[X]$ of some degree d, and the roots $x_1, \ldots, x_d$ of its preimage in $E[X]$ must map one-to-one to the d-many roots of $p(X)$ in F, forcing $y \in rg(f)$.

The foregoing proof relativizes to the degree of any field $E \cong C$, yielding relative ω_1-computable categoricity. $\square$

Theorem 4.3. *Let F be any ω_1-computable field with a subfield K isomorphic to C, and assume that F is countably generated over K. Then F is relatively ω_1-computably categorical.*

Proof. Say $F = K(C)$, where C is countable and $C \cap K = \emptyset$. In general K will not be computable. Notice that for each $c \in C$, F cannot contain the algebraic closure of $K(c)$, because this field is not countably generated over K. So we may fix a countable set $S \subseteq K$ with the property that, for every $c \in C$, S contains some tuple $x_0, \ldots, x_n$ such that F does not contain the algebraic closure of the set $\{x_0, \ldots, x_n, c\}$. Therefore, an arbitrary element $x \in F$ lies in K iff K contains the algebraic closure of $S \cup \{x\}$. Since $S \cup \{x\}$ is countable, we will recognize at some countable stage that F contains this algebraic closure (if indeed $x \in F$). Therefore, K is computably enumerable within F.

We next define a subfield F_0 of K as follows. Write $C = \{c_1, c_2, \ldots\}$. For each $i > 0$, if c_i is transcendental over the field $K(c_1, \ldots, c_{i-1})$, then add nothing to F_0; otherwise, add to F_0 a finite set of elements $y_1, \ldots, y_n$ from K such that the minimal polynomial of c_i over $K(c_1, \ldots, c_{i-1})$ has coefficients in $Q(y_1, \ldots, y_n, c_1, \ldots, c_{i-1})$. Since C is countable, this only adds countably many elements in all, and we let $F_0 \subseteq K$ be the algebraic closure of the subfield of K generated by these elements along with the elements of S. Thus F_0 is also countable.

We claim that every automorphism of K which fixes F_0 pointwise extends to an automorphism of F which is the identity on C. To see this, let h_0 be such an automorphism of K. Define h to extend h_0 by setting $h(c_i) = c_i$ for all i. We claim that this h extends to an automorphism of all of F. For each $s > 0$, if c_s is transcendental over $K(c_1, \ldots, c_{s-1})$, then it is clear that setting $h_s(c_s) = c_s$ extends to an automorphism h_s of $K(c_1, \ldots, c_s)$. If c_s is algebraic over $K(c_1, \ldots, c_{s-1})$, then by our choice of F_0, the minimal polynomial $p(X)$ of c_s over all of $K(c_1, \ldots, c_{s-1})$ lies in $F_0(c_1, \ldots, c_{s-1})[X]$. So

$$\begin{aligned} K(c_1, \ldots, c_s) &\cong K(c_1, \ldots, c_{s-1})[X]/(p(X)) \\ &\cong (F_0(c_1, \ldots, c_{s-1})[X]/(p(X)))(K - K_0). \end{aligned}$$

Since h_s is the identity on $F_0(c_1, \ldots, c_{s-1})$, it is clearly an automorphism of the last of these fields, and so it is also an automorphism of the first field, as desired. Thus, the union h of all these h_s is an automorphism of $K(C)$, which is to say, of F.

Now let E be any field isomorphic to F, with domain ω_1, and suppose ρ is a noncomputable isomorphism from F onto E. We give the details for the case where E is computable; they relativize directly to an arbitrary E. Let E_0 be the countable image $\rho(F_0)$, and let $T = \rho(S)$. We start be defining $f_0 = \rho \restriction (F_0(C))$, which is computable because F_0 and C are countable. Next, we enumerate K as defined above, and similarly enumerate its image $\rho(K)$ within E, using the set T. At stage $\sigma + 1$, we wait for a new element x to appear in K (using our enumeration) on which f_σ is not defined. When this happens, we find the first element y to appear in our enumeration of $\rho(K)$ which is not already in $rg(f_\sigma)$, and define $f_{\sigma+1}(x) = y$. At this stage we also find all elements of F which are algebraic over the portion of K which has appeared so far (including x) but not in the domain of f_σ, and define $f_{\sigma+1}$ of each of these to be a root of the corresponding polynomial in E. (We can do this effectively, simply enumerating

F until all of the countably many polynomials over this portion of K have their full complement of roots.) Thus we extend the domain of $f_{\sigma+1}$ to include a larger algebraically closed subfield $K_{\sigma+1}$ of K than previously. In addition, we extend $f_{\sigma+1}$ to have the appropriate values on all elements generated by C over $K_{\sigma+1}$; again, it is not difficult to find all of these elements in countably many steps. This completes stage $\sigma + 1$.

It is clear that this defines a map $f = \cup f_\sigma$ on all of F, whose restriction to K is an isomorphism from K onto $\rho(K)$ (because we always chose the next new element of $\rho(K)$ in our enumeration to be the image of the next new element of K). The map f is also defined and equal to ρ on C (as well as on F_0), and is defined on all elements generated by C over K as well. That is, f is defined on all of F. Since ρ and f are equal on F_0, our argument above shows that the automorphism $(\rho^{-1} \circ f) \restriction K$ of K extends to an automorphism τ of all of F, which is the identity on C. But then $f = \rho \circ \tau$, so f is an isomorphism from F onto E as desired. □

At the other extreme from algebraically closed fields, namely fields purely transcendental over Q, the opposite result holds.

Proposition 4.4. *The purely transcendental field extension* $F = Q(X_\alpha \ : \ \alpha \in \omega_1)$ *is not* ω_1*-computably categorical.*

Proof. We may assume that F is a presentation with the transcendence basis $\{X_\alpha \ : \ \alpha < \omega_1\}$ computable. (Lemma 4.1 only guarantees the existence of some computable transcendence basis, not necessarily of one generating the entire field.) We buld a computable field $E \cong F$ with no computable isomorphism from E onto F. X_α will be our witness that the computable function ϕ_α is not such an isomorphism.

At the start, we build E_0 to be F itself, although we only use half the elements of ω_1 to do so. (Let E_0 be the isomorphic image of F under the map $\lambda + n \mapsto \lambda + 2n$ for all limit ordinals λ.) We write $y_\alpha \in E_0$ for the image of x_α under this map. Then, for each α, we wait for $\phi_\alpha(y_\alpha)$ to converge, say to some $z_\alpha \in F$. When this happens, we find $\beta_1, \ldots, \beta_n$ such that $z_\alpha \in Q(x_{\beta_1}, \ldots, x_{\beta_n})$, and ask whether the polynomial $p(X) = X^2 - z_\alpha$ factors over the subfield $Q(x_{\beta_1}, \ldots, x_{\beta_n})$. (Kronecker gives a splitting algorithm for this field, since we know that x_{β_i} to be algebraically independent over Q.) If so, then z_α has a square root in F, and so we do not change anything in E, but define $y'_\alpha = y_\alpha$. If not, then we adjoin to E a new element y'_α whose square in E is y_α, and use half of the currently unused elements to close E under the field operations. This completes the construction.

Now $E = Q(y'_\alpha : \alpha < \omega_1)$ is isomorphic to F via the map $y'_\alpha \mapsto x_\alpha$. However, if $\phi_\alpha(y_\alpha)\downarrow$, then y_α has a square root in E iff $\phi_\alpha(y_\alpha)$ has no square root in F. Thus no ϕ_α can be an isomorphism from E onto F.

□

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