Hilbert's Tenth Problem for Subrings of the Rationals

Russell Miller

Queens College & CUNY Graduate Center

Special Session on
Computability Theory & Applications

AMS Sectional Meeting
Loyola University, Chicago, IL
4 October 2015

HTP: Hilbert's Tenth Problem

Definition

For a ring R, *Hilbert's Tenth Problem for R* is the set

$$\mathrm{HTP}(R) = \{p \in R[X_0, X_1, \ldots] : (\exists \vec{a} \in R^{<\omega})\ p(a_0, \ldots, a_n) = 0\}$$

of all polynomials (in several variables) with solutions in R.

So $\mathrm{HTP}(R)$ is c.e. relative to (the atomic diagram of) R.

Hilbert's formulation demanded a decision procedure for $\mathrm{HTP}(\mathbb{Z})$.

Theorem (RPDM, 1970)

$\mathrm{HTP}(\mathbb{Z})$ is undecidable: indeed, $\mathrm{HTP}(\mathbb{Z}) \equiv_1 \emptyset'$, and there is a polynomial $f \in \mathbb{Z}[Y, X_1, \ldots, X_k]$ such that

$$(\forall n)\ [n \in \emptyset' \iff f(n, X_1, \ldots, X_k) \in \mathrm{HTP}(\mathbb{Z})].$$

The Turing degree of $\mathrm{HTP}(\mathbb{Q})$ remains an open question.

Subrings R_W of $\mathbb{Q}$

A subring R of $\mathbb{Q}$ is characterized by the set of primes p such that $\frac{1}{p} \in R$. For each $W \subseteq P = \{$ all primes $\}$, set

$$R_W = \left\{ \frac{m}{n} \in \mathbb{Q} \ : \ (\forall p)\ [p \text{ divides } n \implies p \in W] \right\}$$

to be the subring $\mathbb{Z}[W^{-1}]$ generated by inverting all $p \in W$.

We often move effectively between $W \subseteq P$ and $\{n : p_n \in W\} \subseteq \omega$.

Notice that R_W is computably presentable precisely when W is c.e., while R_W is a computable subring of $\mathbb{Q}$ iff W is computable.

HTP(R_W) vs. W'

It is immediate that HTP(R_W) $\leq_1$ W'. The PMDR result shows that 1-equivalence can hold: when $W = \emptyset$, we have HTP($R_\emptyset$) $\equiv_1$ $\emptyset'$.

It is possible to have $W' \not\equiv_T$ HTP(R_W): let W be c.e. and nonlow. Then R_W is computably presentable, so HTP(R_W) is c.e. Hence HTP(R_W) $\leq_1$ $\emptyset' <_T W'$ for such sets W.

In fact, HTP(R_W) $\equiv_T$ W is also possible, e.g. when $W = \emptyset'$.

Generalizing This Idea

Definition

A set W is *relatively c.e.* if there exists $V <_T W$ such that W is c.e. relative to V.

Theorem (Jockusch 1977; Kurtz 1981)

The relatively c.e. sets form a comeager class (Jockusch) of measure 1 (Kurtz).

Recall the topology: the basic open subsets of 2^ω are the intervals

$$\mathcal{U}_\sigma := \{X \subseteq \omega : \sigma \subset X\},$$

for all $\sigma \in 2^{<\omega}$. The *measure* of $\mathcal{U}_\sigma$ is $2^{-|\sigma|}$.

A class $\mathcal{M} \subseteq 2^\omega$ is *nowhere dense* if the closure $\text{cl}(\mathcal{M})$ contains no interval $\mathcal{U}_\sigma$. The *meager sets* are the elements of the Σ-ideal generated by these: all countable unions of nowhere dense sets.

Corollaries of the Jockusch & Kurtz Results

Corollary (M.)

The class $\{W \subseteq P : W' \leq_1 \text{HTP}(R_W)\}$ is meager, of measure 0. Therefore, so is the class of those W such that W' is *polynomially definable* in R_W, by an $f \in \mathbb{Z}[Y, X_1, \ldots, X_k]$ with

$$(\forall n)\ [n \in W' \iff f(n, X_1, \ldots, X_k) \in \text{HTP}(R_W).$$

Proof: If W is relatively c.e., then W is c.e. in some V with $W \not\leq_T V$. But then $W' \not\leq_1 V'$. However, HTP(R_W) is also c.e. in V, so HTP$(R_W) \leq_1 V'$. Thus $W' \not\leq_1$ HTP(R_W).

So the MRDP proof for the case $W = \emptyset$ is anomalous.

Can We Do Better?

W is *relatively c.e. and non-low* if there exists some $V <_T W$ in which W is c.e., but with $W' \not\leq_T V'$. If this holds, then $\text{HTP}(R_W) \leq_1 V'$, and so $W' \not\leq_T \text{HTP}(R_W)$.

Can We Do Better?

W is *relatively c.e. and non-low* if there exists some $V <_T W$ in which W is c.e., but with $W' \not\leq_T V'$. If this holds, then $\text{HTP}(R_W) \leq_1 V'$, and so $W' \not\leq_T \text{HTP}(R_W)$.

However, this is far more rare. The class

$$\mathbf{GL}_1 := \{W : W \oplus \emptyset' \equiv_T W'\}$$

of *generalized-low*$_1$ sets is comeager of measure 1.
If $W \in \mathbf{GL}_1$ and W is c.e. in V, then

$$W' \equiv_T W \oplus \emptyset' \leq_T V'.$$

So almost all W fail to be relatively c.e. and non-low.

Enumeration Reducibility

Fact

W and HTP(R_W) are always e-equivalent, via uniform reductions.

$W \leq_1$ HTP(R_W) via $p \mapsto (pX - 1)$.

To see that HTP(R_W) $\leq_e W$, given a polynomial f, start enumerating solutions $\vec{x}$ of f in $\mathbb{Q}$. Each time we find one, we add an axiom: if W contains the primes necessary for the denominators in $\vec{x}$, then $f \in$ HTP(R_W).

$W' \nleq_e \mathrm{HTP}(R_W)$ for most W

Theorem

The class of all $W \subseteq P$ with $W' \leq_e \mathrm{HTP}(R_W)$ is meager, of measure 0.

Proof: Suppose $W' \leq_e \mathrm{HTP}(R_W)$. Now $P - W = \overline{W} \leq_e W'$, so $\overline{W} \leq_e \mathrm{HTP}(R_W)$.

But if V can enumerate W, then it can enumerate $\mathrm{HTP}(R_W)$, since $\mathrm{HTP}(R_W) \leq_e W$. Hence V can enumerate $\overline{W}$, and thus $W \leq_T V$. It follows that W cannot be relatively c.e. The Jockusch-Kurtz results complete the proof.