Spectra of Algebraic Fields

Andrey Frolov
Kazan State University

Iskander Kalimullin
Kazan State University

Russell Miller,
Queens College & Graduate Center
CUNY

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Spectrum of a Structure

Defns: For a countable structure $\mathcal{S}$ with domain ω, the *Turing degree of* $\mathcal{S}$ is the Turing degree of the atomic diagram of $\mathcal{S}$. The *spectrum of* $\mathcal{S}$ is the set

$$\{\deg(\mathcal{A}) : \mathcal{A} \cong \mathcal{S}\}$$

of all Turing degrees of copies of $\mathcal{S}$.

Many general results are known about spectra.
Thm. (Knight): For all nontrivial structures, the spectrum is closed upwards under $\leq_T$.

Algebraic Fields

Defn: A field F is *algebraic* if it is an algebraic (but possibly infinite) extension of its prime subfield. Equivalently, F is a subfield of either $\overline{\mathbb{Q}}$ or $\overline{\mathbb{Z}/(p)}$, the algebraic closures of the prime fields.

Thm. (FKM): The spectra of algebraic fields of characteristic 0 are precisely the sets of the form

$$\{\boldsymbol{d} : T \text{ is c.e. in } \boldsymbol{d}\}$$

where T ranges over all subsets of ω.

The same holds for infinite algebraic fields of characteristic > 0.

Normal Extensions of $\mathbb{Q}$

A simple case: let $F \supseteq \mathbb{Q}$ be a normal algebraic extension. Enumerate the irreducible polynomials $p_0(X), p_1(X), \ldots$ in $\mathbb{Q}[X]$. (So for each i, F contains either all roots of p_i, or no roots of p_i.) Define

$$T_F^* = \{i : (\exists a \in F) p_i(a) = 0\}.$$

Claim: $\text{Spec}(F) = \{\boldsymbol{d} : T_F^* \text{ is c.e. in } \boldsymbol{d}\}$.

$\subseteq$ is clear: any presentation of F allows us to enumerate T_F^*.
$\supseteq$: Given a $\boldsymbol{d}$-oracle, start with $E_0 = \mathbb{Q}$. Whenever an i enters T_F^*, check whether E_s yet contains any root of $p_i(X)$. If so, do nothing; if not, enumerate all roots of p into E_{s+1}. (Use a computable presentation of $\overline{\mathbb{Q}}$ as a guide.) This builds $E \cong F$ with $E \leq_T \boldsymbol{d}$.

Converse

Problem: Not all $T \subseteq \omega$ can be T_F^*. If $(X^2 - 2)$ and $(X^2 - 3)$ both have roots in F, then so does $(X^2 - 6)$.

Solution: Consider only polynomials $(X^2 - p)$ with p prime. Given T, let F be generated over $\mathbb{Q}$ by $\{\sqrt{p_n} : n \in T\}$. Then

$$\mathrm{Spec}(F) = \{\boldsymbol{d} : T \text{ is c.e. in } \boldsymbol{d}\}.$$

So, for every $T \subseteq \omega$, this spectrum can be realized.

All Algebraic Fields

Defn: Given F, define T_F similarly to T_F^*, but reflecting non-normality:

$$\begin{array}{ll} T_F : & \underbrace{1 \quad 0 \quad 0} \quad \underbrace{1 \quad 1 \quad 0 \quad 0} \quad \underbrace{0 \quad 0 \quad 0} \cdots \\ p_i : & \quad X^3 - 7 \qquad X^4 - X^2 + 1 \quad \cdots \end{array}$$

Problem: Suppose that first $(X^2 - 3)$ requires a root $\sqrt{3}$ in F, and later $(X^4 - X^2 + 1)$ requires a root x in F. But

$$X^4 - X^2 + 1 = (X^2 + X\sqrt{3} + 1)(X^2 - X\sqrt{3} + 1),$$

and T_F does not say which factor should have x as a root.

Solution

Let $\langle q_{j0}(X), q_{j1}(X,Y)\rangle_{j\in\omega}$ list all pairs in $(\mathbb{Q}[X] \times \mathbb{Q}[X,Y])$ s.t.:

- $\mathbb{Q}[X]/(q_{j0})$ is a field, and
- q_{j1}, viewed as a polynomial in Y, is irreducible in $(\mathbb{Q}[X]/(q_{j0}))[Y]$.

In the example above, q_{j0} would be $(X^2 - 3)$ and q_{j1} could be either factor of $(X^4 - X^2 + 1)$.

Defn: Given F, let U_F be the set:

$$\{j : (\exists x, y \in F)[q_{j0}(x) = 0 = q_{j1}(x,y)]\}$$

and let $V_F = T_F \oplus U_F$. So every presentation of F can enumerate V_F.

Construction of $E \cong F$

Fix F, and suppose $\boldsymbol{d}$ enumerates V_F. When T_F demands that k roots of some $p_i(X)$ enter E, we find $j \in U_F$ such that q_{j0} is the minimal polynomial of a primitive generator x of E_s over $\mathbb{Q}$ (so that $E_s \cong \mathbb{Q}[X]/(q_{j0})$), and $q_{j1}(Y)$ divides $p_i(Y)$ in $(\mathbb{Q}[X]/(q_{j0}))[Y]$. Extend our E_s to E_{s+1} by adjoining a root of $q_{j1}(Y)$. Since $j \in U_F$, E_{s+1} embeds into F via some f_{s+1}.

Now all f_s agree on $\mathbb{Q}$ ($\subseteq E_s$). The least element $x_0 \in E = \cup_s E_s$ has only finitely many possible images in F, so some infinite subsequence of $\langle f_s \rangle_{s \in \omega}$ agrees on $\mathbb{Q}[x_0]$. Likewise, some infinite subsequence of this subsequence agrees on $\mathbb{Q}[x_0, x_1]$, etc. This embeds E into F. But T_F ensures that E has as many roots of each $p_i(X)$ as F does, so the embedding is an isomorphism.

Corollaries

Thm. (Richter): There exists $A \subseteq \omega$ such that there is no least degree $\boldsymbol{d}$ which enumerates A.
Cor. (Calvert-Harizanov-Shlapentokh): There exists an algebraic field whose spectrum has no least degree.

Thm. (Coles-Downey-Slaman): For every $T \subseteq \omega$ there is a degree $\boldsymbol{b}$ which enumerates T, such that all $\boldsymbol{d}$ enumerating T satisfy $\boldsymbol{b}' \leq \boldsymbol{d}'$.
Cor.: Every algebraic field F has a jump degree, i.e. a degree $\boldsymbol{c}$ such that all $\boldsymbol{d} \in \mathrm{Spec}(F)$ have $\boldsymbol{d}' \leq \boldsymbol{c}$ and some $\boldsymbol{d} \in \mathrm{Spec}(F)$ has $\boldsymbol{d}' = \boldsymbol{c}$. In particular, $\boldsymbol{c}$ is the degree of the enumeration jump of V_F.

Cor.: No algebraic field has spectrum $\{\boldsymbol{d} : \boldsymbol{0} < \boldsymbol{d}\}$. Indeed, $(\forall \boldsymbol{d}_0)(\exists \boldsymbol{d}_1 \not\leq \boldsymbol{d}_0)$ s.t. every algebraic field F with $\{\boldsymbol{d}_0, \boldsymbol{d}_1\} \subseteq \mathrm{Spec}(F)$ is computably presentable.