Functors and Effective Interpretations in Model Theory

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(Joint work with many researchers.)

A First Example

Background: a structure A with domain ω is *computable* if all of its functions and relations are computable. Such an A is *computably categorical* if, for every computable structure B which is classically isomorphic to A, there is a computable isomorphism from A onto B.

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Theorem (M-Park-Poonen-Schoutens-Shlapentokh)

For every countable graph G, there exists a countable field $\mathscr{F}(G)$ with the same computable-model-theoretic properties as G.

Construction of $\mathscr{F}(G)$

We use two curves X and Y, defined by integer polynomials:

$$X : p(u,v) = u^4 + 16uv^3 + 10v^4 + 16v - 4 = 0$$
$$Y : q(T,x,y) = x^4 + y^4 + 1 + T(x^4 + xy^3 + y + 1) = 0$$

Let $G = (\omega, E)$ be a graph. Set $K = \mathbb{Q}(\Pi_{i\in\omega} X)$ to be the field generated by elements $u_0 < v_0 < u_1 < v_1, \ldots$, with $\{u_i : i \in \omega\}$ algebraically independent over $\mathbb{Q}$, and with $p(u_i, v_i) = 0$ for every i. The element u_i in $K \subseteq \mathscr{F}(G)$ will represent the node i in G.

Next, adjoin to K elements x_{ij} and y_{ij} for all $i > j$, with $\{x_{ij} : i > j\}$ algebraically independent over K, and with

$$q(u_i u_j, x_{ij}, y_{ij}) = 0 \text{ if } (i,j) \in E$$
$$q(u_i + u_j, x_{ij}, y_{ij}) = 0 \text{ if } (i,j) \notin E.$$

We write Y_t for the curve defined by $q(t,x,y) = 0$ over $\mathbb{Q}(t)$. So the process above adjoins the function field of either $Y_{u_i u_j}$ or $Y_{u_i + u_j}$, for each $i > j$. $\mathscr{F}(G)$ is the extension of K generated by all x_{ij} and y_{ij}.

Reconstructing G From $\mathscr{F}(G)$

Lemma

Let $G = (\omega, E)$ be a graph, and build $\mathscr{F}(G)$ as above. Then:

(i) $X(\mathscr{F}(G)) = \{(u_i, v_i) : i \in \omega\}$.

(ii) If $(i,j) \in E$, then $Y_{u_i u_j}(\mathscr{F}(G)) = \{(x_{ij}, y_{ij})\}$ and $Y_{u_i + u_j}(\mathscr{F}(G)) = \emptyset$.

(iii) If $(i,j) \notin E$, then $Y_{u_i u_j}(\mathscr{F}(G)) = \emptyset$ and $Y_{u_i + u_j}(\mathscr{F}(G)) = \{(x_{ij}, y_{ij})\}$.

This is the heart of the proof. (i) says that $p(u,v) = 0$ has no solutions in $\mathscr{F}(G)$ except the ones we put there, so we can enumerate

$$\{u_i : i \in \omega\} = \{u \in \mathscr{F}(G) : (\exists v \in \mathscr{F}(G)) p(u,v) = 0\}.$$

Similarly, (ii) and (iii) say that the equations $q(u_i u_j, x, y) = 0$ and $q(u_i + u_j, x, y) = 0$ have no unintended solutions in $\mathscr{F}(G)$. So, given i and j, we can determine from $\mathscr{F}(G)$ whether $(i,j) \in E$: search for a solution to either $q(u_i u_j, x, y) = 0$ or $q(u_i + u_j, x, y) = 0$.

Interpretations

One can readily view this construction as a way of *interpreting* the graph G in the field $\mathscr{F}(G)$. The domain of G (within $\mathscr{F}(G)$) is defined by the formula

$$(\exists v)\; p(u, v) = 0,$$

under the relation of equality, and the edge relation on such u_0, u_1 is defined by

$$E(u_0, u_1) \iff (\exists x \exists y)\; q(u_0 u_1, x, y) = 0;$$

$$\neg E(u_0, u_1) \iff (\exists x \exists y)\; q(u_0 + u_1, x, y) = 0.$$

Since the domain, E, and $\neg E$ are all defined by Σ_1 formulas, the interpretation may be considered *effective*.

Consequences in Computable Model Theory

Definition

The *isomorphism problem* for a class $\mathfrak{S}$ of computable structures (e.g. $\mathfrak{S} = \{$ all computable graphs $\}$) is the set of all pairs of isomorphic members of $\mathfrak{S}$:

$$\{(i,j) \in \omega^2 : \varphi_i \text{ and } \varphi_j \text{ are the characteristic functions of the atomic diagrams of isomorphic members of } \mathfrak{S}\}.$$

Since the isomorphism problem for computable graphs is known to be Σ^1_1-complete, this re-proves the known result that the isomorphism problem for computable fields is also Σ^1_1-complete.

Here we only needed that $\mathscr{F}$ respects isomorphism. The Friedman-Stanley embedding did the same.

Consequences: Spectra of Structures

Definition

The *spectrum* of S is the set of all Turing degrees of copies of S:

$$\text{Spec}(S) = \{\deg(M) : M \cong S \ \& \ \text{dom}(M) = \omega\}.$$

Corollary

For every countable structure A, there exists a field F with the same Turing degree spectrum as A:

$$\begin{aligned}\text{Spec}(A) &= \{\deg(B) : B \cong A \ \& \ \text{dom}(B) = \omega\} \\ &= \{\deg(E) : E \cong F \ \& \ \text{dom}(E) = \omega\} \\ &= \text{Spec}(F).\end{aligned}$$

This follows because $\mathscr{F}$ respects isomorphism, with $\mathscr{F}(G) \equiv_T G$, and $\mathscr{F}$ has a computable left inverse taking copies of $\mathscr{F}(G)$ to copies of F.

Categoricity Spectra & Computable Dimension

Definition

If S is computable, the *computable dimension* of S is the number of computable isomorphism classes of computable structures isomorphic to S. If this equals 1, then S is *computably categorical*.

$\boldsymbol{d}$-computable dimension is similar, still for a computable structure S but with $\boldsymbol{d}$-computable isomorphisms.

Definition

The *categoricity spectrum* of a computable structure S is the set of all Turing degrees $\boldsymbol{d}$ such that S is $\boldsymbol{d}$-computably categorical.

Consequences: Categoricity Spectra & Dimension

Corollary

For every computable structure A, there exists a computable field F with the same categoricity spectrum as A and (for each Turing degree $\boldsymbol{d}$) the same $\boldsymbol{d}$-computable dimension as A.

That is, for every Turing degree $\boldsymbol{d}$, A is $\boldsymbol{d}$-computably categorical if and only if F is $\boldsymbol{d}$-computably categorical.

This requires the functoriality of the map $\mathscr{F}$: we use the fact that a $\boldsymbol{d}$-computable isomorphism $g : G \to \widehat{G}$ gives rise to a $\boldsymbol{d}$-computable $\mathscr{F}(g) : \mathscr{F}(G) \to \mathscr{F}(\widehat{G})$. So it is important that $\mathscr{F}$ is a **functor**, not just a map on structures.
Moreover, if F is computable and $F \cong \mathscr{F}(G)$, then F is computably isomorphic to $\mathscr{F}(\widehat{G})$ for some computable $\widehat{G} \cong G$. This yields the required reverse implication.

Functoriality

Our procedure $\mathscr{F}$ can also be viewed as a **functor.** Not only does it build a field $\mathscr{F}(G)$ from a graph G, but also, given an isomorphism $g : G_0 \to G_1$, it builds an isomorphism $\mathscr{F}(g) : \mathscr{F}(G_0) \to \mathscr{F}(G_1)$, respecting composition and preserving the identity map. g tells us where each pair (u_i, v_i) from $\mathscr{F}(G_0)$ should be mapped in $\mathscr{F}(G_1)$, and this in turn determines the map on all x_{ij} and y_{ij}, effectively. So

$$\mathscr{F}(g) = \Phi_*^{G_0 \oplus g \oplus G_1}.$$

Now we are thinking of our collection of all countable graphs as a category, under isomorphisms, and the same for fields. ($\mathscr{F}$ would be a functor even with monomorphisms, not just isomorphisms.)

Consequences: Computable Categoricity

Downey, Kach, Lempp, Lewis, Montalbán, and Turetsky have recently proven that computable categoricity for trees is Π^1_1-complete.

Corollary

The property of computable categoricity for computable fields is Π^1_1-complete. That is, the set

$$\{e \in \omega : \varphi_e \text{ computes a computably categorical field}\}$$

is a Π^1_1 set, and every Π^1_1 set is 1-reducible to this set.

Again, functoriality of $\mathscr{F}$ is essential to this result.

The Friedman-Stanley Embedding

Given a graph G with domain ω, H. Friedman and Stanley defined the field $\mathscr{FS}(G)$. Let $X_0, X_1, \ldots$ be algebraically independent over $\mathbb{Q}$. Let F_0 be the field generated by $\cup_n \overline{\mathbb{Q}(X_n)}$. Then set

$$\mathscr{FS}(G) = F_0[\sqrt{X_m + X_n} \ : \ (m, n) \in G].$$

Thus $\mathscr{FS}(G)$ is computable in G, uniformly, and an isomorphism $g : G \to H$ gives an isomorphism $\mathscr{FS}(g) : \mathscr{FS}(G) \to \mathscr{FS}(H)$. Indeed $G \cong H \iff \mathscr{FS}(G) \cong \mathscr{FS}(H)$.

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However, $\mathscr{FS}(G)$ may be computably presentable, even when G is not. And $\mathscr{FS}(G)$ is never computably categorical, even when G is. So this $\mathscr{FS}$ does not preserve the properties we want. The functor $\mathscr{FS}$ is neither computable, nor *full*: not all isomorphisms $\mathscr{FS}(G) \to \mathscr{FS}(H)$ are of the form $\mathscr{FS}(g)$.

Other Possible Functors

Another example is given by Victor Ocasio Gonzalez (recent PhD student of Knight), using ideas of Dave Marker and others.

Theorem (Ocasio)

There is a computable functor (Φ, Φ_*) from the category of countable linear orders L into that of countable real closed fields F. Moreover, there is a computable functor (Ψ, Ψ_*) which is a left inverse of (Φ, Φ_*).

Given L, Φ builds the real closure F of the ordered field $\mathbb{Q}(a_0, a_1, \ldots)$, where $(\forall i)(\forall n)\ n < a_i$ in F and

$$i < j \text{ in } L \iff a_i < a_j \text{ in } F \iff (\forall m) a_i^m < a_j \text{ in } L.$$

So L is the linear order of the positive nonstandard elements of F, modulo the equivalence $a \sim b \iff (\exists m \in \omega)[a < b^m \ \&\ b < a^m]$.

$$\cdots \quad RC(\mathbb{Q}) \quad [a_0]_\sim \quad [a_1]_\sim \quad [a_2]_\sim \quad \cdots$$

Inverse of Ocasio's Functor?

For each L, the field $F = \Phi^L$ is built in a straightforward way, with the odd numbers in $\omega = \text{dom}(F)$ serving as the elements a_i in F. Therefore, there is a computable functor (Ψ, Ψ_*) which is a left inverse of (Φ, Φ_*).

However, this Ψ does *not* extend to all other F isomorphic to fields of the form Φ^L. The interpretation of L in F uses Σ_2^c formulas: computable infinitary Σ_2^0 formulas. Therefore, picking out representatives $a_0, a_1, \ldots$ in a copy of F requires the jump of the atomic diagram of F.

Ocasio uses this to show that, for every (infinite) L, there is a RCF F such that

$$\text{Spec}(F) = \{\boldsymbol{d} : \boldsymbol{d}' \in \text{Spec}(L)\}.$$

For $\supseteq$, he takes an arbitrary $\boldsymbol{d}$-computable approximation to L, and builds a $\boldsymbol{d}$-computable copy of F from the approximation.

Computable Infinitary Formulas

Recall the **computable infinitary** formulas in $L_{\omega_1\omega}$:

- All finite quantifier-free formulas (with constants from the domain ω) are Σ_0^c, and also Π_0^c.
- If $\alpha_0, \alpha_1, \ldots$ is a computable list of Π_n^c formulas, then

$$\exists n\, (\alpha_n)$$

  is Σ_{n+1}^c, and its negation is Π_{n+1}^c. (Since we allow constants from ω, this allows quantification $\exists x$ over the structure's domain.)
- Taking unions at limit ordinals defines Σ_θ^c iteratively for all $\theta < \omega_1^{CK}$.

These arise very naturally in computable model theory. For instance, the following Σ_2^c formula defines the standard part of a nonstandard model of $\mathrm{Th}(\omega, <)$:

$$\exists \langle y_1, \ldots, y_m \rangle\ \forall z (z < x \implies (z = y_1 \text{ or } \cdots \text{ or } z = y_m)).$$

More Marker Ideas

A similar process uses the ENI-DOP for the theory $\mathbf{DCF}_0$ to show that, for every countable, automorphically nontrivial graph G, there is a countable differentially closed field K such that

$$\mathrm{Spec}(K) = \{\boldsymbol{d} : \boldsymbol{d}' \in \mathrm{Spec}(G)\}.$$

Indeed, we have a converse, established by a priority construction:

Theorem (Marker-M.)

Every model of $\mathbf{DCF}_0$ of low Turing degree is isomorphic to a computable DCF.

Corollary (Marker-M.)

The spectra of differentially closed fields of characteristic 0 are exactly the preimages, under the jump operation, of the spectra of graphs.

From Graphs to Differentially Closed Fields

Once again, this can be seen as a construction of a computable functor from graphs to models of $\mathbf{DCF}_0$. It has a computable inverse functor, but this inverse is only defined on the image, not on a class closed under isomorphism.

As with the Ocasio functor, this one is best described as building a DCF K such that the given graph G has an interpretation in K by Σ_2^c formulas. Nodes $n \in G$ are represented by elements of a decidable infinite set of indiscernibles a_n in $\hat{\mathbb{Q}}$. The existence of an edge between m and n is coded by:

$$(\exists (x, y) \in E^{\#}_{a_m a_n})[x, y \text{ transcendental over } \mathbb{Q}\langle a_m + a_n \rangle]$$

where $E^{\#}_{a_m a_n}$ is the Manin kernel for an elliptic curve involving a_m and a_n. Thus this is a Σ_2^c formula, though not a finitary formula.

Effective Interpretation

Definition (Montalbán)

Let A be an L-structure, and B be any structure. Let us assume that L is a relational language $L = \{P_0, P_1, P_2, ...\}$ where P_i has arity $a(i)$; so $A = (A; P_0^A, P_1^A, ...)$ and $P_i^A \subseteq A^{a(i)}$.

We say that A is *effectively interpretable* in B if, in B, there is

- a uniformly r.i.c.e. set $D_A^B \subseteq B^{<\omega}$ (the domain of the interpretation),
- a uniformly r.i. computable relation $\eta \subseteq B^{<\omega} \times B^{<\omega}$ which is an equivalence relation on D_A^B (interpreting equality),
- a uniformly r.i. computable sequence of relations $R_i \subseteq (B^{<\omega})^{a(i)}$, closed under the equivalence η within D_A^B (interpreting P_i),
- and a function $f_A^B \colon D_A^B \to A$ which induces an isomorphism:

$$(D_A^B/\eta; R_0, R_1, ...) \cong (A; P_0^A, P_1^A, ...).$$

With parameters, Montalbán notes, this is equivalent to Σ-definability.

Functors

Definition

Let $\mathfrak{C}$ be a category in which the objects are countable structures with domain ω (in a single computable language) and the morphisms are maps; and let $\mathfrak{D}$ be another such category (possibly with a different language). A (type-2) *computable functor* from $\mathfrak{C}$ into $\mathfrak{D}$ consists of two Turing functionals Φ and Φ_* such that:

- for all $A \in \mathfrak{C}$, $\Phi^A \in \mathfrak{D}$; and
- for all morphisms $f : A \to B$ in $\mathfrak{C}$, $\Phi_*^{A \oplus f \oplus B}$ is a morphism from Φ^A to Φ^B in $\mathfrak{D}$; and
- these define a functor from $\mathfrak{C}$ into $\mathfrak{D}$.

For instance, any time we have an interpretation of B in A by Σ_1^c-formulas, we automatically get a functor

$$\text{Iso}(A) := \{\text{isomorphic copies of } A \text{ with domain } \omega\} \quad \longrightarrow \quad \text{Iso}(B).$$

Connecting Semantics with Syntax

Let $\text{Iso}(A)$ be the category of all structures (with domain ω) isomorphic to A, with isomorphisms as the morphisms.

Theorem (Harrison-Trainor, Melnikov, M., Montalbán)

B is effectively interpretable in A if and only if there is a computable functor (Φ, Φ_*) from $\text{Iso}(A)$ into $\text{Iso}(B)$.

$\Longleftarrow$: First code $A^{<\omega} \times \omega$ into $A^{<\omega}$: represent $(a_0, \ldots, a_j, n)$ by all tuples $(a_0, \ldots, a_j)\hat{\ } a^{n+1}$ with $a_j \neq a$.

A pair $(\vec{a}, n)$ enters the domain D_B^A if $\Phi_*^{\Delta(\vec{a}) \oplus \text{id} \upharpoonright |\vec{a}| \oplus \Delta(\vec{a})}(n)\downarrow = n$.

Since $\Phi_*^{\Delta(A) \oplus \text{id} \oplus \Delta(A)}$ is the identity on Φ^A, every n has an $\vec{a}$ with $(\vec{a}, n) \in D_B^A$. Intuitively, $\Delta(\vec{a})$ was enough information for Φ_* to recognize the element n in $\widehat{B} = \Phi^{\widehat{A}}$ whenever $\Delta(\widehat{A})$ extends $\Delta(\vec{a})$.

Notice that this is a *computable infinitary* Σ_1 relation on tuples.

Equivalence on tuples

Roughly: for tuples $\vec{a}$, $\vec{a}'$, we define $(\vec{a}, n) \sim (\vec{a}', n')$ if, for some $m > \max(\vec{a}, \vec{a}')$, some permutation $\sigma \in \Sigma_m$ has $\sigma(\vec{a}) = \vec{a}'$ and

$$\Phi_*^{\Delta(\mathrm{dom}(\sigma))\oplus\sigma\oplus\Delta(\mathrm{rg}(\sigma))}(n) = n' \quad \& \quad \Phi_*^{\Delta(\mathrm{rg}(\sigma))\oplus\sigma^{-1}\oplus\Delta(\mathrm{dom}(\sigma))}(n') = n.$$

Again, this is a *computable infinitary* Σ_1 relation on tuples.

Of course, to be an effective interpretation, this process should avoid using $\Delta(A)$. In the above, choosing a tuple $\vec{a}$ really means choosing a finite atomic diagram for that many elements. The Σ_1^c formula says that, if you find that finite atomic diagram within an oracle $\Delta(\widehat{A})$, then you should consider these two tuples from $D_B^{\widehat{A}}$ to represent the same element in the interpretation.

Bi-Interpretability

In the MPPSS construction, B was an arbitrary graph G, and A was the field $\mathscr{F}(G)$ which we built from G. In this construction, there were *two* computable functors: $\mathscr{F}$ uses the graph G to build the field $\mathscr{F}(G)$, and then we saw that $\mathscr{F}$ has a computable left-inverse functor $\mathscr{G}$ which, given any copy of $\mathscr{F}(G)$, produces a copy of G. The graph G and the field $\mathscr{F}(G)$ always satisfy:

Definition (Montalbán)

Structures A and B effectively interpretable in each other are *effectively bi-interpretable* if the compositions

$$f_B^A \circ \tilde{f}_A^B : D_B^{D_A^B} \to B \text{ and } f_A^B \circ \tilde{f}_B^A : D_A^{D_B^A} \to A$$

are uniformly relatively intrinsically computable in B and A.

(Recall: f_B^A is an isomorphism onto B from the interpretation D_B^A of B within A.)

Bi-Interpretability and Functors

Theorem (HTM[3])

Structures A and B are effectively bi-interpretable if and only if there exist computable functors $\mathscr{F} : \mathrm{Iso}(A) \to \mathrm{Iso}(B)$ and $\mathscr{G} : \mathrm{Iso}(B) \to \mathrm{Iso}(A)$ such that $\mathscr{F} \circ \mathscr{G}$ and $\mathscr{G} \circ \mathscr{F}$ are effectively isomorphic to the identity functors in their categories.

The technical term *"effectively isomorphic"* means that there is a computable natural transformation from $\mathscr{G} \circ \mathscr{F}$ to the identity functor on $\mathrm{Iso}(A)$, and likewise in $\mathrm{Iso}(B)$.

Ultimately the MPPSS theorem shows that, for every graph G, there is a field $\mathscr{F}(G)$ which is effectively bi-interpretable with G, and that the formulas used in the interpretations (equivalently, the algorithms for the computable functors) are uniform for all graphs G. Moreover, the relation $\sim$ is just equality. This is sufficient to transfer from G to $\mathscr{F}(G)$ all the computable model theoretic properties seen earlier.

Current Work

Question: what about those more complicated interpretations?

Intepretations using Σ_2^c formulas (e.g. Ocasio's interpretation of a LO L in a RCF F_L) can readily be viewed as functors into the *jump*.

Defn. (various researchers), roughly stated

The *jump* A' of a countable structure A has the same domain as A and includes the same predicates, but also has a predicate for every Σ_1^c formula (with free variables $v_1, \ldots, v_n$) in the language of A. That predicate holds of $\vec{a}$ in A' iff the formula holds of $\vec{a}$ in A.

This includes predicates such as "the length of the tuple $\vec{a}$ lies in $\emptyset'$," which are not really structural. We get $\text{Spec}(A') = \{\boldsymbol{d}' : \boldsymbol{d} \in \text{Spec}(A)\}$.

The Σ_2^c interpretation of L in F_L is naturally an effective interpretation of L in the jump F_L', and thus corresponds to a computable functor.

Current Work: Graphs vs. Linear Orders

The Σ_2^c interpretations given here suggest that DCF's are connected to graphs, and RCF's to linear orders. These are the opposite sides of a basic divide in computable model theory: linear orders and related classes (e.g. Boolean algebras) are the main classes known not to be complete for many of the properties we have discussed: spectra, computable categoricity, etc.

The Marker-Miller theorem shows that $\mathbf{DCF}_0$ models are not complete, but still ties them closely to graphs. However:

Conjecture (M-Ocasio)

Every graph G has a Σ_2^c-interpretation in some RCF F_G.

Specific multiplicative classes $[x]$ in F_G are identified by Σ_2^c-formulas:

$$(\exists y, z \in [x])(\forall q \in \mathbb{Q}^+)[y < z^q \iff \sqrt{2} < q].$$

Current Work: Noncomputable Infinitary Formulas

Question: What about interpretations by arbitrary $L_{\omega_1\omega}$ formulas?

If the interpretation uses X-computable infinitary Σ_1 formulas, then by relativizing, the earlier arguments show that we have an X-computable functor, given by Turing functionals $\Phi^{X\oplus A}$ and $\Phi_*^{X\oplus A\oplus f\oplus\widehat{A}}$. So an $L_{\omega_1\omega}$ interpretation yields a continuous functor.

It is natural to argue that this should be computable "on the cone above X," but this is not the case. Even if $X \leq_T A$, a single functional Φ cannot decide X from A without knowing the index e for which $X = \Phi_e^A$. This index may vary for different copies of A above X.