Degree Spectra of Differentially Closed Fields

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Spectra of Countable Structures

Let $\mathcal{S}$ be a structure with domain ω, in a finite language.

Definition

The *Turing degree* of $\mathcal{S}$ is the join of the Turing degrees of the functions and relations on $\mathcal{S}$. If these are all computable, then $\mathcal{S}$ is a *computable structure*.

Definition

The *spectrum* of $\mathcal{S}$ is the set of all Turing degrees of copies of $\mathcal{S}$:

$$\text{Spec}(\mathcal{S}) = \{\deg(\mathfrak{M}) : \mathfrak{M} \cong \mathcal{S} \ \& \ \text{dom}(\mathfrak{M}) = \omega\}.$$

So the spectrum measures the level of complexity intrinsic to the structure $\mathcal{S}$.

Facts About Spectra

Theorem (Knight 1986)

For all countable structures $\mathcal{S}$ but the automorphically trivial ones, the spectrum of $\mathcal{S}$ is upwards-closed under Turing reducibility.

Many interesting spectra can be built using graphs, including upper cones, α-th jump cones $\{\boldsymbol{d} : \boldsymbol{d}^{(\alpha)} \geq_T \boldsymbol{c}\}$, and more exotic sets of Turing degrees. (Greenberg, Montalbán, and Slaman recently constructed a graph whose spectrum contains exactly the nonhyperarithmetic degrees.) Indeed, graphs are *complete*, in the following sense:

Theorem (Hirschfeldt-Khoussainov-Shore-Slinko 2002)

For every countable structure $\mathcal{S}$ in any finite language, there exists a countable graph G which has the same spectrum as $\mathcal{S}$.

Spectra of Algebraically Closed Fields

Spectra of Algebraically Closed Fields

$\{$ all Turing degrees $\}$.

Differentially Closed Fields

A *differential field* is a field along with a differential operator δ on the field elements, respecting addition ($\delta(x+y) = \delta x + \delta y$) and satisfying the product rule $\delta(x \cdot y) = (x \cdot \delta y) + (y \cdot \delta x)$.
Such a field K is *differentially closed* if it also satisfies the *Blum axioms*: for all differential polynomials $p, q \in K\{Y\}$,

$$\text{ord}(q) < \text{ord}(p) \implies (\exists x \in K)\,[p(x) = 0 \;\&\; q(x) \neq 0],$$

where the order $r = \text{ord}(p)$ is the largest derivative $\delta^r Y$ used in p.
This theory $\mathbf{DCF}_0$ is complete and decidable and has quantifier elimination. Moreover, it has computable models:

Theorem (Harrington, 1974)

For every computable differential field k, there exists a computable model K of $\mathbf{DCF}_0$ and a computable embedding g of k into K such that K is a differential closure of the image $g(k)$.

Noncomputable Differentially Closed Fields

By analogy to $\mathbf{ACF}_0$, one may guess that all countable models of $\mathbf{DCF}_0$ have computable presentations. However, it is known that there exist 2^ω-many (non-isomorphic) countable models of $\mathbf{DCF}_0$. Indeed:

Theorem (Marker-M.)

For every countable graph G, there exists a countable $K \models \mathbf{DCF}_0$ with

$$\text{Spec}(K) = \{\boldsymbol{d} : \boldsymbol{d}' \text{ can enumerate the edges in some } G^* \cong G\}.$$

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It is not difficult to show that, for every G, there is another graph H s.t.

$$\{\boldsymbol{d} : \boldsymbol{d}' \text{ enumerates the edges in some } G^* \cong G\} = \{\boldsymbol{d} : \boldsymbol{d}' \in \text{Spec}(H)\},$$

and that conversely, for each H, there is some such G. So the theorem proves that every countable graph H yields a $K \models \mathbf{DCF}_0$ with

$$\text{Spec}(K) = \{\boldsymbol{d} : \boldsymbol{d}' \in \text{Spec}(H)\}.$$

Coding a Graph G into $K \models \mathbf{DCF}_0$

Start with a copy $\hat{\mathbb{Q}}$ of the differential closure of $\mathbb{Q}$. Let A be the following infinite set of indiscernibles in $\hat{\mathbb{Q}}$:

$$A = \{a_0, a_1, \ldots\} = \{y \in \hat{\mathbb{Q}} : \delta y = y^3 - y^2 \ \& \ y \neq 0 \ \& \ y \neq 1\}.$$

Each $a_m \in A$ will represent the node m from G.

Let $E_{a_m a_n}$ be the elliptic curve defined by the equation

$$y^2 = x(x-1)(x - a_m - a_n).$$

The coordinates of all solutions to this curve in $(\hat{\mathbb{Q}})^2$ are algebraic over $\mathbb{Q}\langle a_m + a_n\rangle$ and $E_{a_m a_n}$ forms an abelian group, with exactly j^2 j-torsion points for every j, and with no non-torsion points. There is a homomorphism of differential algebraic groups from $E_{a_m a_n}$ into a vector group, whose kernel $E^{\sharp}_{a_m a_n}$ is called the *Manin kernel* of $E_{a_m a_n}$.

Coding a Graph G into $K \models \mathbf{DCF}_0$

For each $m < n$ with an edge in G from m to n, add a generic point of $E^\sharp_{a_m+a_n}$ to our differential field. The coordinates of this point will each be transcendental over $\mathbb{Q}\langle a_m + a_n\rangle$. Let K be the differential closure of the resulting differential field.

Thus the coding is:

$$G \text{ has an edge from } m \text{ to } n \iff$$
$$(\exists(x,y) \in E^\sharp_{a_m a_n})[x \text{ is transcendental over } \mathbb{Q}\langle a_m + a_n\rangle].$$

In particular, the points we added do *not* accidentally give rise to any transcendental solutions to any other $E^\sharp_{a_{m'} a_{n'}}$.

Spec$(K) = \{\boldsymbol{d} : \boldsymbol{d}'$ enumerates some $G^* \cong G\}$

Now if $\boldsymbol{d}$ is the degree of a copy $K^* \cong K$, then with a $\boldsymbol{d}'$-oracle, we enumerate the edges in some G^* as follows. Find all elements a^*_m of the set A^* of indiscernibles in K^*, go through all solutions to $E_{a^*_m a^*_n}$ for each $m < n$, and ask whether each is transcendental over $\mathbb{Q}\langle a^*_m, a^*_n\rangle$ and lies in $E^\sharp_{a^*_m a^*_n}$. If we ever get an answer "YES," we enumerate (m, n) into the edge relation of the graph G^*. Thus $G^* \cong G$: the isomorphism comes from restricting the isomorphism $K^* \to K$ to $A^* \to A$.

Conversely, if $D \in \boldsymbol{d}$ and D' enumerates the edges in some $G^* \cong G$, we build $K^* \cong K$ using a $\boldsymbol{d}$-oracle. Start building $\hat{\mathbb{Q}}^*$, finitely much at each step. At stage s, if it appears (from D) that D' has enumerated an edge (m, n) in G^*, add a point $x_{mn} \in E^\sharp_{a^*_m a^*_n}$ which is not (yet) algebraic over $\mathbb{Q}\langle a_m, a_n\rangle$. If D' later changes and wipes out this enumeration, we can still make x_{mn} a t-torsion point for some large t, hence algebraic. Finally, use Harrington's theorem to build a D-computable differential closure K^* of the D-computable differential field defined here.

Low and Nonlow Degrees

For every $\boldsymbol{d}' > \mathbf{0}'$, there exists a graph G such that $\boldsymbol{d}'$ enumerates a copy of G, but $\mathbf{0}'$ does not. Therefore:

Corollary

For every nonlow degree $\boldsymbol{d}$ (i.e., with $\boldsymbol{d}' > \mathbf{0}'$), there exists some $K \models \mathbf{DCF}_0$ of degree $\boldsymbol{d}$ such that K is not computably presentable.

We now prove the converse:

Theorem (Marker-M.)

Every low model of $\mathbf{DCF}_0$ is isomorphic to a computable one.

This recalls the famous theorem of Downey-Jockusch that every low Boolean algebra is isomorphic to a computable one.

Principal Types over k

Over a field E, the principal 1-types are generated by the formulas $p(X) = 0$, where $p \in E[X]$ is irreducible. Over a differential field k, this is not enough! Over $\mathbb{Q}$, the differential polynomial $(\delta Y - Y)$ is irreducible, but only the following formula generates a principal type:

$$\delta Y - Y = 0 \ \& \ Y \neq 0.$$

In general, we need pairs (p, q) from $k\{Y\}$, with $\mathrm{ord}(p) > \mathrm{ord}(q)$. If the formula $p(Y) = 0 \neq q(Y)$ generates a principal type, then (p, q) is a *constrained pair*, and p is *constrainable*. Every principal type is generated by a constrained pair, but not all irreducible $p(Y)$ are constrainable. $p(Y) = \delta Y$ is a simple counterexample.

Fact

$p \in k\{Y\}$ is constrainable $\iff$ p is the minimal differential polynomial of some x in the differential closure K of k.

It is Π_1^k for (p, q) to be constrained, and Σ_2^k for p to be constrainable.

Low Differentially Closed Fields K

If K is low, then the (computable infinitary) Π^0_1-theory of K has degree $\mathbf{0}'$, hence is computably approximable. This allows us to "guess" effectively at the minimal differential polynomial of any $x \in K$ over the differential subfield $\mathbb{Q}\langle x_{i_0}, \ldots, x_{i_n}\rangle \subseteq K$ generated by an arbitrary finite tuple from K.

Writing $K = \{x_0, x_1, \ldots\}$ and guessing thus, we build a computable differential field $F = \{y_0, y_1, \ldots\}$ and finite partial maps $h_s : K \to F$ such that:

- $(\forall n)\ \lim_s h_s(x_n)$ exists; and
- $(\forall m)\ \lim_s h_s^{-1}(y_m)$ exists; and
- $\forall s\ h_s$ is a partial isomorphism, based on the approximations in K to the minimal differential polynomials of its domain elements.

Thus $h = \lim_s h_s$ will be an isomorphism from K onto F.

Differences from Boolean Algebras

The Downey-Jockusch Theorem has been extended.

Theorem (Downey-Jockusch; Thurber; Knight-Stob)

Every low_4 Boolean algebra is isomorphic to a computable one.

In contrast, the first Marker-M theorem established that every nonlow Turing degree computes some $K \models \mathbf{DCF}_0$ with $\mathbf{0} \notin \text{Spec}(K)$.

Fact

There exists a low Boolean algebra which is not $\mathbf{0}'$-computably isomorphic to any computable Boolean algebra. (Downey-Jockusch always gives a $\mathbf{0}''$-computable isomorphism.)

But the theorem for low differentially closed fields built a Δ_2 isomorphism onto the computable copy.

Relativizing the Result

Relativizing the previous theorem yields:

Corollary

For every $K \models \mathbf{DCF}_0$, $\text{Spec}(K)$ respects the equivalence relation $\boldsymbol{c} \sim_1 \boldsymbol{d}$ defined by $\boldsymbol{c}' = \boldsymbol{d}'$.

Proof: If $\boldsymbol{c} \in \text{Spec}(K)$ and $\boldsymbol{d}' = \boldsymbol{c}'$, then $\boldsymbol{d}$ can guess effectively at the minimal differential polynomials in the $\boldsymbol{c}$-computable copy of K, and the process in the theorem builds a $\boldsymbol{d}$-computable copy of K.

Corollary (cf. Andrews, Montalbán, unpublished, using Richter)

For every $K \models \mathbf{DCF}_0$, $\text{Spec}(K)$ cannot be contained within any upper cone of Turing degrees, except the cone above $\mathbf{0}$.

Proof: no other upper cone respects $\sim_1$.

Why Is This a Converse?

Corollary (Marker-M.)

For a set $\mathcal{S}$ of Turing degrees, TFAE:

1. $\mathcal{S}$ is the spectrum of some $K \models \mathbf{DCF}_0$.
2. $\mathcal{S}$ is the spectrum of some ANT graph and $\mathcal{S}$ respects $\sim_1$.
3. $\mathcal{S}$ is the preimage under jump of the spectrum of some ANT graph.

(ANT: automorphically non-trivial.)

($1 \implies 2$) was the relativized version of the second theorem (plus the HKSS theorem), and ($3 \implies 1$) was the first theorem. For ($2 \implies 3$), if $\mathcal{S} = \text{Spec}(G)$, let H be the jump of the structure G (defined in work of Montalbán and Soskov-Soskova). By HKSS, we may take H to be a graph. Then $\text{Spec}(H) = \{\boldsymbol{c}' : \boldsymbol{c} \in \text{Spec}(G)\}$, and so

$$\text{Spec}(G) \subseteq \{\boldsymbol{d} : \boldsymbol{d}' \in \text{Spec}(H)\}.$$

For $\supseteq$, if $\boldsymbol{d}' \in \text{Spec}(H)$, then $\boldsymbol{d}' = \boldsymbol{c}'$ for some $\boldsymbol{c} \in \text{Spec}(G)$, and $\boldsymbol{d} \in \text{Spec}(G)$ since $\text{Spec}(G) = \mathcal{S}$ respects $\sim_1$.