Turing Degree Spectra of Real Closed Fields

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Spectra of Countable Structures

Let S be a structure with domain ω, in a finite language.

Definition

The *Turing degree* of S is the join of the Turing degrees of the functions and relations on S. If these are all computable, then S is a *computable structure*.

Definition

The *spectrum* of S is the set of all Turing degrees of copies of S:

$$\text{Spec}(S) = \{\deg(M) : M \cong S \text{ \& } \text{dom}(M) = \omega\}.$$

So the spectrum measures the level of complexity intrinsic to the structure S.

Spectra for Different Classes

- Every spectrum of an automorphically non-trivial structure, in a computable language, is the spectrum of a graph, a lattice, a group, a partial order, and a field. (Results by HKSS and MPSS.)
- In particular, every upper cone of degrees, $\{$ all high$_n$ degrees $\}$, $\{$ all non-low$_n$ degrees $\}$, $\{$ all nonzero degrees $\}$, $\{$ all non-hyperarithmetic degrees $\}$ are spectra of graphs.
- A Boolean algebra cannot have a low$_4$ degree in its spectrum unless it also has **0**. (Downey-Jockusch, Thurber, Knight-Stob.)
- BA's, trees, and linear orders *cannot* realize an upper cone as a spectrum (Richter). However, LO's can have a spectrum containing any given $\boldsymbol{d} > \mathbf{0}$ and not containing **0**.
- The spectrum of an ACF always contains all degrees.
- The spectra of models of $\mathbf{DCF}_0$ are precisely the preimages under jump of the spectra of graphs. (Marker-M.)
- Spectra of algebraic fields and rank-1 torsion-free abelian groups are defined by the ability to enumerate some specific subset of ω.

Real Closed Fields

Definition

A *real closed field* F is a model of the theory of the real numbers $(\mathbb{R}, 0, 1, +, \cdot)$. The *positive* field elements are those nonzero elements with square roots: this defines an order on F. The *finite* elements are those x for which some natural number n satisfies $-n < x < n$.

F is *archimedean* if every $x \in F$ is finite. If not, then F has both infinite and *infinitesimal* elements.

Every finite $x \in F$ defines a Dedekind cut in $\mathbb{Q}$, with left side $\{q \in \mathbb{Q} : q < x\}$ and right side $\{q \in \mathbb{Q} : x \leq q\}$.

The *residue field* F_0 of (a nonarchimedean) F consists of one element realizing each Dedekind cut realized in F. If F_0 is just the real closure of $\mathbb{Q}$, then it is canonically a subfield of F. However, if F_0 contains transcendentals, then it has no canonical embedding into F.

Computability and real closures

Theorem (Ershov; Madison)

For every $\boldsymbol{d}$-computable ordered field F, there is a $\boldsymbol{d}$-computable presentation of the real closure of F.

So, to give a $\boldsymbol{d}$-computable presentation of the real closure of F, it suffices to present F itself using a $\boldsymbol{d}$-oracle.

Dedekind cuts

In any computable RCF, we can give a computable enumeration $\langle A_{n,s}, B_{n,s}\rangle_{n,s\in\omega}$ of all Dedekind cuts (A_n, B_n) realized in F. We think of each cut as a decreasing sequence of intervals $(a_{n,s}, b_{n,s}]$, with $a_{n,s} = \max(A_{n,s})$ and $b_{n,s} = \min(B_{n,s})$. It is not difficult to make this enumeration injective.

Theorem

For an archimedean RCF F, the following are equivalent:

- $\boldsymbol{d} \in \text{Spec}(F)$.
- $\boldsymbol{d}$ enumerates the Dedekind cuts realized in F as (A_n, B_n), in such a way that the dependence relation on the realizations of these cuts is $\Sigma_1^{\boldsymbol{d}}$.

Upper Cones as Spectra

Proposition (folklore)

Every upper cone $\{\boldsymbol{d} : \boldsymbol{c} \leq \boldsymbol{d}\}$ of Turing degrees is the spectrum of a RCF.

Proof: given $\boldsymbol{c}$, find a real number x (necessarily transcendental, when $\boldsymbol{c} \neq \boldsymbol{0}$) whose Dedekind cut in $\mathbb{Q}$ has degree $\boldsymbol{c}$. The real closure of $\mathbb{Q}(x)$ is then $\boldsymbol{c}$-presentable, but conversely, each of its presentations must compute the Dedekind cut of (the image of) x, hence computes $\boldsymbol{c}$.

This distinguishes RCF's from linear orders, trees, Boolean algebras, algebraic fields, and models of **ACF** and $\mathbf{DCF}_0$, in terms of the spectra they can realize.

High degrees

Question: which families of Turing degrees are defined by the property of being able to realize a specific collection of Dedekind cuts?

Theorem (Jockusch, 1972)

The degrees $\boldsymbol{d}$ which can enumerate the computable sets are precisely the high degrees (i.e., those with $\boldsymbol{d}' \geq \mathbf{0}''$).

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Theorem (Korovina-Kudinov)

The spectrum of the field of all computable real numbers contains precisely the high degrees.

This relativizes: the spectrum of the field of $\boldsymbol{c}$-computable real numbers contains precisely those degrees $\boldsymbol{d}$ with $\boldsymbol{d}' \geq \boldsymbol{c}''$.

Proof: Spec($\mathbb{R}_c$) = { high degrees }

$\Rightarrow$: If $\boldsymbol{d}$ computes a copy of the field $\mathbb{R}_c$ of computable real numbers, then $\boldsymbol{d}$ can list out all the Dedekind cuts realized in $\mathbb{R}_c$. From this list, one quickly gets an enumeration of all computable sets. So, by Jockusch's result, $\boldsymbol{d}$ is high.

$\Leftarrow$: If $\boldsymbol{d}$ is high, then some $\boldsymbol{d}$-computable function can approximate $\mathbf{0}''$. We use this to guess, $\boldsymbol{d}$-computably, whether each pair (W_i, W_j) of c.e. subsets of $\mathbb{Q}$ constitutes a Dedekind cut or not. When it appears to be a cut (and when this cut becomes distinct from all previous cuts), we start building an element x_{ij} in our presentation of $\mathbb{R}_c$ to realize that cut. If the approximation changes its mind, we can always turn x_{ij} into a nearby rational element of our presentation, consistently with the finitely many facts so far defined about this presentation.

Dedekind cuts are not enough

Theorem

There exists an archimedean real closed field F with a computable enumeration of all Dedekind cuts realized in F, yet with Spec(F) containing precisely the high degrees.

The set **Inf** is coded into F in such a way that with any presentation of F and with a transcendence basis for that presentation, one can decide **Inf**. We uniformly enumerate Dedekind cuts $\{(a_{e,s}, b_{e,s}) : e \in \omega\}$ such that, for each e, $a_e = \lim_s a_{e,s}$ is transcendental over $\mathbb{Q}$ iff W_e is infinite). In fact, if W_e is infinite, then a_e will be transcendental over the subfield $\mathbb{Q}(a_0, \ldots, a_{e-1})$.

Given any presentation of F, of degree $\boldsymbol{d}$, a $\boldsymbol{d}'$-oracle allows us to find an element realizing the cut $(a_{e,s}, b_{e,s})$, and to check transcendence of this element (which is $\boldsymbol{d}'$-decidable).

Nonarchimedean real closed fields

In a nonarchimedean RCF, we partition the positive infinite elements into *multiplicative classes*:

$$x \sim y \iff \exists n\, [x < y^n \,\&\, y < x^n].$$

These classes are linearly ordered in F. Write L_F for this derived linear order, which is then presentable from the jump of each copy of F.

An RCF F is *principal* if it is the smallest RCF with a given residue field F_0 and with a given linear order L as L_F.

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Theorem (Ocasio, Ph.D. thesis)

For every L, the principal RCF F with residue field $RC(\mathbb{Q})$ and derived linear order L satisfies

$$\text{Spec}(F) = \{\boldsymbol{d} : \boldsymbol{d}' \in \text{Spec}(L)\}.$$

A distinction on derived orders

Proposition

Suppose that the derived linear order L_F of an RCF F has a left end point. Then the property of being finite in F is relatively intrinsically computable. (Hence so is being infinitesimal.)

Proof: Fix an element y_0 in the least positive infinite multiplicative class. Then x is finite in F iff $(\exists n)[-n < x < n]$; while x is infinite in F iff $(\exists m > 0)\ y_0 < x^m$.

Corollary

If L_F has a left end point, then $\text{Spec}(F) \subseteq \text{Spec}(F_0)$.

Proof: F_0 is defined as the quotient of the ring of finite elements of F, modulo the ideal of infinitesimals in F.

Spectra when L_F has a left end point

Ocasio's theorem shows that the containment in the Proposition does not reverse: we can have $\text{Spec}(F) \neq \text{Spec}(F_0)$.

Theorem

For every L with a left end point, and every archimedean RCF F_0, the principal RCF F with residue field F_0 and derived linear order L satisfies

$$\text{Spec}(F) = \text{Spec}(F_0) \cap \{\boldsymbol{d} : \boldsymbol{d}' \in \text{Spec}(L)\}.$$

The proof is essentially just Ocasio's construction, with $RC(\mathbb{Q})$ replaced by F_0.

Spectra when L_F has no left end point

Theorem

There exists a computable principal RCF F whose residue field F_0 has no computable presentation. (By the previous theorem, L_F has no left end point.)

The construction of F builds a sequence of elements y_e, with

$$e \in \textbf{Fin} \iff (\exists q \in \mathbb{Q})[(y_e - q) \text{ is infinitesimal}].$$

Use the complete binary tree T, guessing at level e whether $e \in \textbf{Inf}$: At each node α we have a $y_\alpha \in F$, which remains fixed from stage to stage. The set $\{y_\alpha : \alpha \in T\}$ is algebraically independent in F.
y_e will equal y_α for that α on the true path at level e.

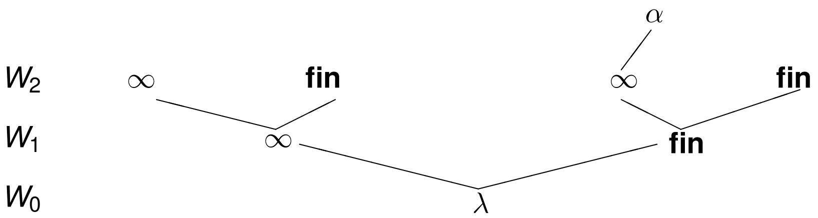
At each stage s, y_α is close to some $q_{\alpha,s} \in \mathbb{Q}$, with $x_{\alpha,s} = q_{\alpha,s} - y_\alpha$ positive and potentially infinitesimal.

$\mathbf{0}''$ Construction

Whenever $W_{e,s+1}$ adds an element, we make the difference $x_{\alpha,s}$ noninfinitesimal, so y_α is not that close to $q_{\alpha,s}$, and choose a new $q_{\alpha,s+1} < q_{\alpha,s}$ for y_α to approximate. Making $x_{\alpha,s}$ noninfinitesimal makes all $x_{\beta,s} > x_{\alpha,s}$ noninfinitesimal as well, injuring those β.

So we choose $x_{\alpha,s} < x_{\beta,s}$ iff $\alpha \prec \beta$ on T:

$$x_{\infty\infty} < x_{\infty} < x_{\infty f} < x_{\lambda} < x_{f\infty} < x_{f} < x_{ff}$$



$\mathbf{Spec}(F_0) \subseteq \{\mathbf{high\ degrees}\}$

Given a $\boldsymbol{d}$-computable copy E_0 of F_0, a $\boldsymbol{d}'$-oracle allows us to find the unique element $z_0 \in E_0$ realizing the same cut as $y_0 = y_\lambda$. Since $\mathbb{Q}$ is $\boldsymbol{d}$-c.e. inside E_0, $\boldsymbol{d}'$ then tells us whether this z_0 is rational in E_0. If so, then $0 \in$ **Fin**; if not, then $0 \in$ **Inf**.

With this info, we know which α_1 at level 1 lies on the true path. Set $y_1 = y_{\alpha_1}$, and find the unique $z_1 \in E_0$ realizing the same cut as y_1. If $z_1 \in \mathbb{Q}$, then $1 \in$ **Fin**; else $1 \in$ **Inf**.

Continuing recursively, we compute **Inf** from the $\boldsymbol{d}'$-oracle.

Conclusions and Questions

It remains open whether RCF's can realize all possible spectra of automorphically nontrivial structures. This seems unlikely, but no counterexample is known.

There appears to be a tight connection between spectra of RCF's and highness properties: such spectra are often defined by the ability of the jump $\boldsymbol{d}'$ to compute some particular degree $\boldsymbol{c}$. Can this be made explicit somehow?

Problem: does the spectrum of an RCF F depend only on:

- Spec(F_0), where F_0 is the residue field of F; and
- Spec(L_F), from the derived linear order L_F of F.

This is false unless we restrict to derived linear orders with no left end point (and allow nonprincipal RCF's, of course).

Problem: nonprincipal RCF's in general!