Spectra of Algebraic Fields and Subfields

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Degree Spectra

Defns: For a countable structure $\mathcal{S}$ with domain ω, the *Turing degree of* $\mathcal{S}$ is the Turing degree of the atomic diagram of $\mathcal{S}$. The *spectrum of* $\mathcal{S}$ is

$$\mathrm{Spec}(\mathcal{S}) = \{\deg(\mathcal{A}) : \mathcal{A} \cong \mathcal{S}\}$$

of all Turing degrees of copies of $\mathcal{S}$.

For a relation R on a computable structure $\mathcal{M}$, the *spectrum of* R, $\mathrm{DgSp}_{\mathcal{M}}(R)$, is

$$\{\deg(f(R)) : f : \mathcal{M} \cong \mathcal{N} \ \&\ \mathcal{N} \text{ is computable}\}.$$

Algebraic Fields

Defn: A field F is *algebraic* if it is an algebraic (but possibly infinite) extension of its prime subfield. Equivalently, F is a subfield of either $\overline{\mathbb{Q}}$ or $\overline{\mathbb{Z}/(p)}$, the algebraic closures of the prime fields.

Thm. (Frolov-Kalimullin-M.): The spectra of algebraic fields of characteristic 0 are precisely the sets of the form

$$\{\boldsymbol{d} : T \text{ is c.e. in } \boldsymbol{d}\}$$

where T ranges over all subsets of ω.

The same holds for infinite algebraic fields of characteristic > 0, and also (by work of Coles, Downey, and Slaman) for torsion-free abelian groups of rank 1.

Normal Extensions of $\mathbb{Q}$

A simple case: let $F \supseteq \mathbb{Q}$ be a normal algebraic extension. Enumerate the irreducible polynomials $p_0(X), p_1(X), \ldots$ in $\mathbb{Q}[X]$. (So for each i, F contains either all roots of p_i, or no roots of p_i.) Define

$$T_F^* = \{i : (\exists a \in F) p_i(a) = 0\}.$$

Claim: $\text{Spec}(F) = \{\boldsymbol{d} : T_F^* \text{ is c.e. in } \boldsymbol{d}\}$.

$\subseteq$ is clear: any presentation of F allows us to enumerate T_F^*.
$\supseteq$: Given a $\boldsymbol{d}$-oracle, start with $E_0 = \mathbb{Q}$. Whenever an i enters T_F^*, check whether E_s yet contains any root of $p_i(X)$. If so, do nothing; if not, enumerate all roots of p_i into E_{s+1}. (Use a computable presentation of $\overline{\mathbb{Q}}$ as a guide.) This builds $E \cong F$ with $E \leq_T \boldsymbol{d}$.

Converse

Problem: Not all $T \subseteq \omega$ can be T_F^*. If $(X^2 - 2)$ and $(X^2 - 3)$ both have roots in F, then so does $(X^2 - 6)$.

Solution: Consider only polynomials $(X^2 - p)$ with p prime. Given T, let F be generated over $\mathbb{Q}$ by $\{\sqrt{p_n} : n \in T\}$. Then

$$\mathrm{Spec}(F) = \{\boldsymbol{d} : T \text{ is c.e. in } \boldsymbol{d}\}.$$

So, for every $T \subseteq \omega$, this spectrum can be realized.

All Algebraic Fields

Defn: Given F, define T_F similarly to T_F^*, but reflecting non-normality:

$$
\begin{array}{ll}
T_F : & \underbrace{1 \quad 0 \quad 0} \quad \underbrace{1 \quad 1 \quad 0 \quad 0} \quad \underbrace{0 \quad 0 \quad 0} \cdots \\
p_i : & \quad X^3 - 7 \qquad X^4 - X^2 + 1 \quad \cdots
\end{array}
$$

Problem: Suppose that first $(X^2 - 3)$ requires a root $\sqrt{3}$ in F, and later $(X^4 - X^2 + 1)$ requires a root x in F. But

$$X^4 - X^2 + 1 = (X^2 + X\sqrt{3} + 1)(X^2 - X\sqrt{3} + 1),$$

and T_F does not say which factor should have x as a root.

Solution

We follow work of Ershov on computable fields.

Let $\langle q_{j0}(X), q_{j1}(X,Y)\rangle_{j\in\omega}$ list all pairs in $(\mathbb{Q}[X] \times \mathbb{Q}[X,Y])$ s.t.:

- $\mathbb{Q}[X]/(q_{j0})$ is a field, and
- q_{j1}, viewed as a polynomial in Y, is irreducible in $(\mathbb{Q}[X]/(q_{j0}))[Y]$.

In the example above, q_{j0} would be $(X^2 - 3)$ and q_{j1} could be either factor of $(X^4 - X^2 + 1)$.

Defn: Given F, let U_F be the set:

$$\{j : (\exists x, y \in F)[q_{j0}(x) = 0 = q_{j1}(x,y)]\}$$

and let $V_F = T_F \oplus U_F$. So every presentation of F can enumerate V_F.

Construction of $E \cong F$

Fix F, and suppose $\boldsymbol{d}$ enumerates V_F. When T_F demands that k roots of some $p_i(X)$ enter E, we find $j \in U_F$ such that q_{j0} is the minimal polynomial of a primitive generator x of E_s over $\mathbb{Q}$ (so that $E_s \cong \mathbb{Q}[X]/(q_{j0})$), and $q_{j1}(Y)$ divides $p_i(Y)$ in $(\mathbb{Q}[X]/(q_{j0}))[Y]$. Extend our E_s to E_{s+1} by adjoining a root of $q_{j1}(Y)$. Since $j \in U_F$, E_{s+1} embeds into F via some f_{s+1}.

Now all f_s agree on $\mathbb{Q}$ ($\subseteq E_s$). The least element $x_0 \in E = \cup_s E_s$ has only finitely many possible images in F, so some infinite subsequence of $\langle f_s \rangle_{s \in \omega}$ agrees on $\mathbb{Q}[x_0]$. Likewise, some infinite subsequence of this subsequence agrees on $\mathbb{Q}[x_0, x_1]$, etc. This embeds E into F. But T_F ensures that E has as many roots of each $p_i(X)$ as F does, so the embedding is an isomorphism.

Corollaries

Thm. (Richter): There exists $A \subseteq \omega$ such that there is no least degree $\boldsymbol{d}$ which enumerates A.
Cor. (Calvert-Harizanov-Shlapentokh): There exists an algebraic field whose spectrum has no least degree.

Thm. (Coles-Downey-Slaman): For every $T \subseteq \omega$ there is a degree $\boldsymbol{b}$ which enumerates T, such that all $\boldsymbol{d}$ enumerating T satisfy $\boldsymbol{b}' \leq \boldsymbol{d}'$.
Cor.: Every algebraic field F has a jump degree, i.e. a degree $\boldsymbol{c}$ such that all $\boldsymbol{d} \in \text{Spec}(F)$ have $\boldsymbol{d}' \leq \boldsymbol{c}$ and some $\boldsymbol{d} \in \text{Spec}(F)$ has $\boldsymbol{d}' = \boldsymbol{c}$. In particular, $\boldsymbol{c}$ is the degree of the enumeration jump of V_F.

Cor.: No algebraic field has spectrum $\{\boldsymbol{d} : \mathbf{0} < \boldsymbol{d}\}$. Indeed, $(\forall \boldsymbol{d}_0)(\exists \boldsymbol{d}_1 \not\leq \boldsymbol{d}_0)$ s.t. every algebraic field F with $\{\boldsymbol{d}_0, \boldsymbol{d}_1\} \subseteq \text{Spec}(F)$ is computably presentable.

Rabin's Theorem

Defn.: A homomorphism $g : F \to K$ of computable fields is a *Rabin embedding* if g is computable and K is algebraically closed and algebraic over the image $g(F)$.

Idea: K is an effective algebraic closure of F.

Rabin's Theorem:
1. Every computable field F is the domain of some Rabin embedding g into some K.
2. $S_F = \{\text{reducible polynomials in } F[X]\}$ is computable iff that Rabin embedding has image $g(F)$ computable within K.

Normal Subfields of $\overline{\mathbb{Q}}$

Now we consider spectra of subfields F of $\overline{\mathbb{Q}}$, viewed as unary relations.

Lemma: If $K \cong \overline{\mathbb{Q}}$ is computable and $L \subseteq K$, then there is a subfield $E \subseteq \overline{\mathbb{Q}}$ with $(\overline{\mathbb{Q}}, E) \cong (K, L)$ and $E \equiv_T L$.

Proof: $\overline{\mathbb{Q}}$ is computably categorical.

Prop.: If F is a normal algebraic extension of $\mathbb{Q}$, then $\text{DgSp}_{\overline{\mathbb{Q}}}(F) = \{\deg(T_F^*)\}$.

Proof: A normal field F has only one possible homomorphic image in $\overline{\mathbb{Q}}$.

$\text{DgSp}_{\overline{\mathbb{Q}}}(F) = \{\deg(T_F^*)\}$ also holds if $\mathbb{Q} \subseteq E \subseteq F$ with E finite over $\mathbb{Q}$ and F normal over E.

Arbitrary Subfields of $\overline{\mathbb{Q}}$

Thm. (Frolov-Kalimullin-M.): If there is no finite extension $\mathbb{Q} \subseteq E$ with $E \subseteq F$ normal, then $\mathrm{DgSp}_{\overline{\mathbb{Q}}}(F)$ is the cone of degrees $\geq \deg(V_F)$.

Idea:
$\subseteq$: If $\tilde{F} \cong F$ with $\tilde{F} \subseteq \overline{\mathbb{Q}}$, then from $\tilde{F}$ we can compute T_F and U_F.

Notice how this uses the ambient field $\overline{\mathbb{Q}}$. If $p(X) \in T_F$, there are only finitely many elements of $\overline{\mathbb{Q}}$ which can be roots of $p(X)$, so finitely many questions for the $\tilde{F}$-oracle.

Coding into a Subfield

Recall: there is no finite extension $\mathbb{Q} \subseteq E$ such that $E \subseteq F$ is normal.

$\supseteq$: If $V_F \leq_T D$, we build a subfield $\tilde{F} \subseteq \overline{\mathbb{Q}}$ coding D, and an isomorphism $g : F \to \tilde{F}$. Start with $F_0 = \tilde{F}_0 = \mathbb{Q}$.

- Search for the first irreducible $p_i \in F_s[X]$ such that $p_i(X)$ has a root in F, but not all its roots. (Use V_F-oracle.)
- Let $r \in \overline{\mathbb{Q}}$ be the $<$-least root of $p_i^g \in \tilde{F}_s[X]$. Adjoin to $\tilde{F}_s$ the same number of roots as p_i has in F. Make $s \in D$ iff $r \in \tilde{F}_{s+1}$.
- F_{s+1} contains all roots in F of some irreducible $p_0, \ldots, p_i \in F_s[X]$. Adjoin to $\tilde{F}$ the needed roots of those p_j^g to form $\tilde{F}_{s+1}$.

Infinite Transcendence Degree

If F has finite transcendence basis B over $\mathbb{Q}$, just replace $\mathbb{Q}$ by $\mathbb{Q}(B)$ to get the same results. (In characteristic > 0, F must be separable over $\mathbb{Z}/(p)$.)

New spectra do arise when we allow an infinite transcendence basis.

Example: Fix $r_0 = e$ and $r_{i+1} = e^{r_i}$. Given $S \subseteq \omega$, let F_S be the closure of $\mathbb{Q}(r_t \mid t \in S)$ under square roots of positive elements. We claim that

$$\text{Spec}(F_S) = \{\boldsymbol{d} : S \text{ is } \Sigma_2^0 \text{ in } \boldsymbol{d}\}.$$

Cor.: For any $A \subseteq \omega$, there is a field whose spectrum contains precisely those $\boldsymbol{d}$ with $A \leq \boldsymbol{d}'$.

$$\begin{aligned} \text{Spec}(F_{A'}) &= \{\boldsymbol{d} : (\exists D \in \boldsymbol{d}) A' \leq_1 D''\} \\ &= \{\boldsymbol{d} : (\exists D \in \boldsymbol{d}) A \leq_T D'\} \end{aligned}$$

So the high degrees form the spectrum of a field.

$\mathbf{Spec}(F_S) = \{\boldsymbol{d} : S$ is Σ^0_2 in $\boldsymbol{d}\}$.

$\subseteq$: If $E \cong F_S$, then S is the set

$$\{t \in \omega : (\exists x \in E)(\forall q \in \mathbb{Q})[q < r_t \leftrightarrow q \prec x \text{ in } E]\}.$$

The order $\prec$ on E is E-computable, by the closure of E under square roots of positive elements.

$\supseteq$: If $S \leq_1 \mathrm{Fin}^D$, let $t \in S$ iff $W^D_{h(t)}$ is finite. Start building F_ω (the field containing all r_t). Each time $W^D_{h(t)}$ receives an element, make the old r_t become rational and add a new r_t to replace it.

News Flash

Prop.: If F_S is viewed as a subfield of its computable algebraic closure E, then

$$\mathrm{DgSp}_E(F_S) = \{\boldsymbol{d} : S \text{ is } \Sigma^0_2 \text{ in } \boldsymbol{d}\}.$$

So when this F_S embeds into its algebraic closure, the image has the same spectrum as the original structure.

This can also happen for algebraic fields, such as

$$\mathbb{Q}[\sqrt{p_{2n}} \ : \ n \in S][\sqrt{p_{2n+1}} \ : \ n \notin S],$$

where the relevant set V_F is 1-equivalent to $\overline{V_F}$.