Computable Transformations of Structures

Russell Miller

Queens College & CUNY Graduate Center

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Classes of countable structures

A structure $\mathcal{A}$ with domain ω (in a fixed language) is identified with its atomic diagram $\Delta(\mathcal{A})$, making it an element of 2^ω. We consider classes of such structures, e.g.:

$$Alg = \{D \in 2^\omega : D \text{ is an algebraic field of characteristic } 0\}.$$

$$ACF_0 = \{D \in 2^\omega : D \text{ is an ACF of characteristic } 0\}.$$

$$\mathcal{T} = \{D \in 2^\omega : D \text{ is an infinite finite-branching tree}\}.$$

On each class, we have the equivalence relation $\cong$ of isomorphism. The theory $\mathbf{ACF_0}$ is usually considered to be straightforward, yet $\cong$ is a Π_3 relation on ACF_0, whereas $\cong$ is only Π_2 on Alg and on $\mathcal{T}$.
(For computable structures, it is complete at these levels.)

Topology on *Alg* and $Alg/\cong$

Alg inherits the subspace topology from 2^ω: basic open sets are

$$\mathcal{U}_\sigma = \{D \in Alg : \sigma \subset D\},$$

determined by finite fragments σ of the atomic diagram *D*.

We then endow the quotient space $Alg/\cong$ of $\cong$-classes $[D]$, modulo isomorphism, with the quotient topology:

$$\mathcal{V} \subseteq Alg/\cong \text{ is open } \iff \{D \in Alg : [D] \in \mathcal{V}\} \text{ is open in } Alg.$$

Thus a basic open set in $Alg/\cong$ is determined by a finite set of polynomials in $\mathbb{Q}[X]$ which must each have a root (or several roots) in the field.

Examining this topology

The quotient topology on $Alg/\cong$ is not readily recognizable. The isomorphism class of the algebraic closure $\overline{\mathbb{Q}}$ (which is universal for the class Alg) lies in *every* nonempty open set $\mathcal{U}$, since if $F \in \mathcal{U}$, then some finite piece of the atomic diagram of F suffices for membership in $\mathcal{U}$, and that finite piece can be extended to a copy of $\overline{\mathbb{Q}}$.

In contrast, the prime model $[\mathbb{Q}]$ lies in no open set $\mathcal{U}$ except the entire space $Alg/\cong$. If $\mathbb{Q} \in \mathcal{U}$, then some finite piece of the atomic diagram of $\mathbb{Q}$ suffices for membership in $\mathcal{U}$, and this piece can be extended to a copy of any algebraic field.

This does not noticeably illuminate the situation.

Expanding the language for *Alg*

Classifying $Alg/\cong$ properly requires a jump, or at least a fraction of a jump. For each $d > 1$, add to the language of fields a predicate R_d:

$$\models_F R_d(a_0, \ldots, a_{d-1}) \iff X^d + a_{d-1}X^{d-1} + \cdots + a_0 \text{ has a root in } F.$$

Write Alg^* for the class of atomic diagrams of algebraic fields of characteristic 0 in this expanded language.

Now we have computable reductions in both directions between $Alg^*/\cong$ and Cantor space 2^ω, and these reductions are inverses of each other. Hence $Alg^*/\cong$ is homeomorphic to 2^ω.

2^ω is far more recognizable than the original topological space $Alg/\cong$ (without the root predicates R_d). We consider this computable homeomorphism to be a legitimate classification of the class *Alg*, and therefore view the root predicates (or an equivalent) as essential for effective classification of *Alg*.

What do the R_d add?

We do *not* have the same reductions between $Alg/\cong$ and 2^ω: these are not homeomorphic. This seems strange: all R_d are definable in the smaller language, so how can they change the isomorphism relation?

The answer is that they do not change the underlying set: we have a bijection between Alg and Alg^* which respects $\cong$. However, the relations R_d change the topology on $Alg^*/\cong$ from that on $Alg/\cong$. (These are both the quotient topologies of the subspace topologies inherited from 2^ω.)

We do have a continuous map from $Alg^*/\cong$ onto $Alg/\cong$, by taking reducts, and so $Alg/\cong$ is also compact. This map is bijective, but its inverse is not continuous.

Too much information

Now suppose that, instead of merely adding the dependence relations R_d, we add *all* computable Σ_1^c predicates to the language. That is, instead of the algebraic field F, we now have its jump F'.

Fact

$$F \cong K \iff F' \cong K'.$$

However, the class Alg' of all (atomic diagrams of) jumps of algebraic extensions of $\mathbb{Q}$, modulo $\cong$, is no longer homeomorphic to 2^ω. In particular, the Σ_1^c property

$$(\exists p \in \mathbb{Q}[X])(\exists x \in F)\ [p \text{ irreducible of degree } > 1 \ \&\ p(x) = 0]$$

holds just in those fields $\not\cong \mathbb{Q}$. Therefore, the isomorphism class of $\mathbb{Q}$ forms a singleton open set in the space $Alg'/\cong$.
(Additionally, $Alg'/\cong$ is not compact.)

Related spaces

From the preceding discussion, we infer that the root predicates are exactly the information needed for a nice classification of *Alg*.

(What does "nice" mean here? To be discussed....)

For another example, consider the class $\mathcal{T}$ of all finite-branching infinite trees, under the predecessor function P. As before, we get a topological space $\mathcal{T}/\cong$, which is not readily recognizable. (There is still a prime model, with a single node at each level, but no universal model.)

The obvious predicates to add are the *branching predicates* B_n:

$$\models_T B_n(x) \iff \exists^{=n} y\ (P(y) = x).$$

Which yield...

The enhanced class $\mathcal{T}^*$, in the language with the branching predicates, again has a nice classification. Let $T_{m,0}, T_{m,1}, \ldots$ list all finite trees of height exactly m. Given $T \in \mathcal{T}^*$, we can find the unique number $f(0)$ with $T_{1,f(0)} \cong T^{<2}$, where $T^{<2}$ is just T chopped off after level 1.

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Next consider those trees in $T_{2,0}, T_{2,1}, \ldots$ with $T_{2,i}^{<2} \cong T^{<2}$. Choose $f(1)$ so that $T^{<3}$ is isomorphic to the $f(1)$-th tree on this list. Continue choosing $f(2), f(3), \ldots$ recursively this way.

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This yields a computable reduction of $\mathcal{T}^*/\cong$ to Baire space ω^ω, whose inverse is also a computable reduction.

So $\mathcal{T}^*/\cong$ and $Alg^*/\cong$ are *not homeomorphic*. In fact, there are computable reductions in both directions between these spaces, but none is bijective.

What constitutes a nice classification?

With both *Alg* and $\mathcal{T}$, we found very satisfactory classifications, by adding just the right predicates to the language. But it is not always so simple.

Let $\mathbf{TFAb}_1$ be the class of torsion-free abelian groups G of rank exactly 1. We usually view these as being classified by tuples $(\alpha_0, \alpha_1, \ldots)$ from $(\omega + 1)^\omega$, saying that an arbitrary nonzero $x \in G$ is divisible by p_n exactly $f(n)$ times. To account for the arbitrariness of x, we must identify tuples $\vec{\alpha}$ and $\vec{\beta}$ with only finite differences:

$$\exists k[(\forall j > k\ \alpha_j = \beta_j)\ \&\ (\forall j\ |\alpha_j - \beta_j| < k)].$$

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The space $\mathbf{TFAb}_1 / \cong$ has the indiscrete topology: no finite piece of an atomic diagram rules out any isomorphism type. More info needed!

If, for all primes p, we add $D_p(x)$ and $D_{p^\infty}(x)$, saying that x is divisible by p and infinitely divisible by p, then we get the classification above. However, it is not homeomorphic to Baire space itself.

Reducibility on equivalence relations

To broaden our notion of classification, we apply descriptive set theory.

Definition

Let E and F be equivalence relations on 2^ω (or on ω^ω, or other spaces). A *reduction* of E to F is a function $g : 2^\omega \to 2^\omega$ satisfying:

$$(\forall x_0, x_1 \in 2^\omega)\ [x_0\ E\ x_1 \iff g(x_0)\ F\ g(x_1)].$$

Original context: $E \leq_B F$ if there is a reduction which is a Borel function on 2^ω.

Definition

A *continuous reduction* g is given by an oracle Turing functional Φ^S:

$$(\forall A \in 2^\omega)(\forall x \in \omega)\ \Phi^{A \oplus S}(x) = \chi_{g(A)}(x).$$

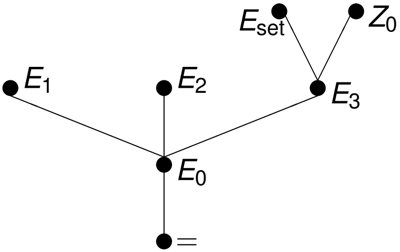
If $S = \emptyset$, then the reduction is *computable*.

Borel reducibility for 2^ω: the basics

Standard Borel ERs are defined using the *columns* A^k of $A \in 2^\omega$:

- $A\ E_0\ B \iff |A \Delta B| < \infty.$
- $A\ E_1\ B \iff \forall^\infty k\ (A^k = B^k).$
- $A\ E_2\ B \iff \sum_{n \in A \triangle B} \frac{1}{n+1} < \infty.$
- $A\ E_3\ B \iff \forall k\ (A^k\ E_0\ B^k).$
- $A\ E_{\text{set}}\ B \iff (\forall j \exists k)\ A^j = B^k\ \&\ (\forall j \exists k)\ B^j = A^k.$
- $A\ Z_0\ B \iff A \triangle B$ has asymptotic density 0.

Picture of $\leq_B$:



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- $A \, E_3 \, B \iff \forall k \, (A^k \, E_0 \, B^k)$.
- $A \, E_{\text{set}} \, B \iff (\forall j \exists k) \, A^j = B^k \ \& \ (\forall j \exists k) \, B^j = A^k$.
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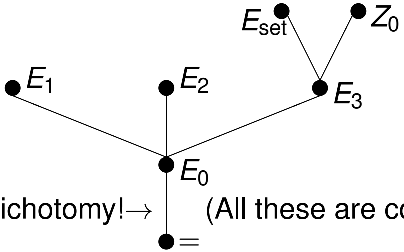
Picture of $\leq_B$:

Glimm-Effros Dichotomy!→



(All these are computable reductions.)

Additional ER's on 2^ω

$$A \; E_{\text{card}} \; B \iff |A| = |B|.$$
$$A =^e B \iff \pi_1(A) = \pi_1(B), \text{ where } \pi_1(A) = \{x : \langle x, y\rangle \in A\}.$$
$$A =^f B \iff (\forall x)\, |\{y : \langle x, y\rangle \in A\}| = |\{y : \langle x, y\rangle \in B\}|.$$

More ER's can be built from these. For example, let $A \; E^{\forall}_{\text{card}} \; B$ iff

$$|\{x : \forall y \; \langle x, y\rangle \in A\}| = |\{x : \forall y \; \langle x, y\rangle \in B\}|.$$

Then $2^\omega / E^{\forall}_{\text{card}}$ is homeomorphic to the isomorphism space for algebraically closed fields of characteristic 0.

If we adjoin independence predicates to the language of $\mathbf{ACF}_0$, then this isomorphism space becomes homeomorphic to $2^\omega / E_{\text{card}}$.

Classifying other classes of structures

Also, $2^\omega/{=^f}$ effectively classifies the class of (countable or finite) equivalence structures in the language with unary predicates $C_1, C_2, \ldots, C_\infty$ for the size of the equivalence class of an element. Just count the number of classes of each size $\leq \infty$ in the structure. (Equivalence structures can be classified by elements of ω^ω, but this requires Π^0_4 predicates in the language, much stronger than our C_i's and C_∞.)

$2^\omega/{=^e}$ effectively classifies the subrings of $\mathbb{Q}$: given a subring, just enumerate the set of those n such that the n-th prime p_n has a multiplicative inverse in the subring. Thus the subring $\mathbb{Z}\left[\frac{1}{p_{i_0}}, \frac{1}{p_{i_1}}, \ldots\right]$ gives an enumeration of the set $\{i_0, i_1, \ldots\}$.

These two spaces, $2^\omega/{=^e}$ and $2^\omega/{=^f}$, are *not* homeomorphic.

Back to Alg^*

Since $Alg^*/\cong$ is homeomorphic to 2^ω it seems natural to transfer the Lebesgue measure from 2^ω to $Alg/\cong$. But this requires care.

Fix a computable $\overline{\mathbb{Q}}$, and enumerate $\overline{\mathbb{Q}}[X] = \{f_0, f_1, \ldots\}$. Let $F_\lambda = \mathbb{Q}$. Given $F_\sigma \subset \overline{\mathbb{Q}}$, we find the least i, with f_i irreducible in $F_\sigma[X]$ of prime degree, for which it is not yet determined whether f_i has a root in F_σ. Adjoin such a root to $F_{\sigma\hat{}1}$, but not to $F_{\sigma\hat{}0}$. This gives a homeomorphism from 2^ω onto $Alg^*/\cong$, via $h \mapsto \cup_n F_{h\upharpoonright n}$.

If we transfer standard Lebesgue measure to $Alg^*/\cong$, we get a measure in which the odds of 2 having a 1297-th root are $\frac{1}{2}$, but the odds of 2 having a 16-th root are much smaller.

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Better: the odds of F_σ having a root of the next polynomial f_i (of prime degree d) should be $\frac{1}{d}$. This gives the measure on $Alg^*/\cong$ corresponding to the Haar measure on $\mathrm{Aut}(\overline{\mathbb{Q}})$.

Measuring properties of algebraic fields

Using either of these measures, for (the isomorphism type of) an algebraic field, the property of being normal has measure 0. So does the property of having relatively intrinsically computable predicates R_d.

In Alg^*, the property of being relatively computably categorical has measure 1: given two roots x_1, x_2 of the same irreducible polynomial, one can wait for them to become distinct, since with probability 1 there will be an f for which $f(x_1, Y)$ has a root in the field but $f(x_2, Y)$ does not. This allows computation of isomorphisms between copies of the field. The process works uniformly except on a measure-0 set of fields.

Surprisingly, measure-1-many fields (and all random fields) in Alg remain relatively computably categorical even when the root predicates are removed from the language. However, the procedures for computing isomorphisms are not uniform. A single procedure can succeed only for measure-$(1 - \epsilon)$-many fields.

Things to consider

Question

Is there any way to put Haar measure or similar measures on other classes of countable structures? (Most classes do not have universal structures like $\overline{\mathbb{Q}}$ with compact automorphism groups.)

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For Alg^* and $\mathcal{T}^*$, the homeomorphisms onto 2^ω and ω^ω allow one to transfer notions of randomness to structures in these classes: an isomorphism type is random if and only if it maps to a random real in 2^ω or ω^ω. Do these correspond to other notions of random structures?

Question

Are there computable reductions in either direction between classes with Π^0_4 isomorphism problems? E.g., the classes of equivalence structures and of trees which are finite-branching except at the root?