Perfect Local Computability and Computable Simulations

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Covers of Structures

Defn.: A *cover* $\mathfrak{A}$ of a structure $\mathcal{S}$ consists of a set $\{\mathcal{A}_i\}$ containing all finitely generated substructures of $\mathcal{S}$, up to isomorphism (and nothing else!), along with sets $I_{ij}^{\mathfrak{A}}$ of embeddings $\mathcal{A}_i \hookrightarrow \mathcal{A}_j$, such that every $f \in I_{ij}^{\mathfrak{A}}$ *lifts* to an inclusion $\mathcal{B} \subseteq \mathcal{C}$ in $\mathcal{S}$, and every such inclusion is the lift of some embedding:

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow[\subseteq]{} & \mathcal{C} \\ \beta \uparrow \cong & & \gamma \uparrow \cong \\ \mathcal{A}_i & \xrightarrow{f} & \mathcal{A}_j . \end{array}$$

Local Computability

Definition: A cover $\mathfrak{A}$ is *computable* if every $\mathcal{A} \in \mathfrak{A}$ is.

$\mathfrak{A}$ is *uniformly computable* if there is a single algorithm listing out all $\mathcal{A}_i$ in $\mathfrak{A}$ and all f in each $I_{ij}^{\mathfrak{A}}$. In this case $\mathcal{S}$ is *locally computable.*

Examples: The fields $\mathbb{R}$ and $\mathbb{C}$ are locally computable. The ordered field $(\mathbb{R}, <)$ is not, because it has finitely generated substructures which are not computably presentable. The ordered field of computable real numbers has no computable listing $\{\mathcal{A}_i\}$ of its f.g. substructures.

Extensionality

Defn.: Every embedding from any $\mathcal{A}_i$ into $\mathcal{S}$ is 0-*extensional*. An isomorphism $\beta : \mathcal{A}_i \to \mathcal{B} \subseteq \mathcal{S}$ is $(m+1)$-*extensional* if

- $(\forall j)(\forall f \in I_{ij}^{\mathfrak{A}})(\exists \mathcal{C} \subseteq \mathcal{S})[f$ lifts to $\mathcal{B} \subseteq \mathcal{C}$ via β and some m-extensional $\gamma]$; and
- $(\forall$ f.g. $\mathcal{C} \supseteq \mathcal{B})(\exists j)(\exists f \in I_{ij}^{\mathfrak{A}})[f$ lifts to $\mathcal{B} \subseteq \mathcal{C}$ via β and some m-extensional $\gamma]$.

$\mathfrak{A}$ is an m-*extensional cover* if every $\mathcal{A}_i \in \mathfrak{A}$ is the domain of an m-extensional embedding and every f.g. $\mathcal{B} \subseteq \mathcal{S}$ is the range of one.

∞-Extensionality

Defn.: A set M of embeddings $\beta : \mathcal{A}_i \hookrightarrow \mathcal{S}$ is a *correspondence system* if:

- $(\forall i)(\exists \beta \in M)\mathcal{A}_i = \mathrm{dom}(\beta)$; and
- $(\forall$ f.g. $\mathcal{B} \subseteq \mathcal{S})(\exists \beta \in M)\mathcal{B} = \mathrm{range}(\beta)$; and

and for all maps $\beta : \mathcal{A}_i \to \mathcal{B}$ in M:

- $(\forall j \forall f \in I_{ij}^{\mathfrak{A}})(\exists \mathcal{C} \supseteq \mathcal{B})[f$ lifts to the inclusion $\mathcal{B} \subseteq \mathcal{C}$ via β and some $\gamma \in M]$; and
- $(\forall$ f.g. $\mathcal{C} \supseteq \mathcal{B})(\exists j \exists f \in I_{ij}^{\mathfrak{A}})[f$ lifts to the inclusion $\mathcal{B} \subseteq \mathcal{C}$ via β and some $\gamma \in M]$.

An $\mathcal{S}$ with such an $\mathcal{M}$ is ∞-*extensional.*

Perfect Local Computability

If $\mathcal{S}$ is ∞-extensional, and $\vec{a} \in \mathcal{S}$, then the atomic type satisfied by $\vec{a}$ is computable, and in general, for $\theta < \omega_1^{CK}$, the Σ_θ-*type* of $\vec{a}$,

$$\{\Sigma_\theta \text{ formulas } \varphi(\vec{x}) \mid \ \mathcal{S} \models \varphi(\vec{a})\}$$

is arithmetically a Σ_θ set.

• This is also true of every (globally) computable structure!

M is *perfect*, and $\mathcal{S}$ is *perfectly locally computable*, if for all $\beta, \gamma \in M$ with $\text{range}(\beta) = \text{range}(\gamma)$, we have $(\gamma^{-1} \circ \beta) \in I_{ij}^{\mathfrak{A}}$, where $\mathcal{A}_i = \text{dom}(\beta)$ and $\mathcal{A}_j = \text{dom}(\gamma)$.

Theorem (Miller, 2007): A countable structure $\mathcal{S}$ is computably presentable iff $\mathcal{S}$ is perfectly locally computable.

Back and Forth with Covers

Lemma: Let structures $\mathcal{C}$ and $\mathcal{S}$ have correspondence systems over the same cover. Suppose that $\mathcal{C}$ is countable, and that P is a countable subset of $\mathcal{S}$. Then there exists an elementary embedding of $\mathcal{C}$ into $\mathcal{S}$ whose image contains P.

Corollary: Any two countable structures with correspondence systems over the same cover are isomorphic.

Simulations

Defn.: A *simulation* of a structure $\mathcal{S}$ is an elementary substructure of $\mathcal{S}$ which realizes the same n-types as $\mathcal{S}$ (for all n).

Example: The algebraic closure of the field $\mathbb{Q}(X_0, X_1, \ldots)$ is a computably presentable simulation of $\mathbb{C}$.

Lemma: Every ∞-extensionally locally computable structure $\mathcal{S}$ has a countable simulation $\mathcal{C}$ with a correspondence system over the cover of $\mathcal{S}$.
Proof: For each i, enumerate *one* image $\alpha(\mathcal{A}_i)$ into $\mathcal{C}$, with α in the correspondence system M for $\mathcal{S}$. Then close $\mathcal{C}$ under the $\forall\exists$ conditions for a correspondence system.

Notice that if M is perfect for $\mathcal{S}$, then the new system is perfect for $\mathcal{C}$.

Computable Simulations

Thm. (Mulcahey-Miller): Every perfectly locally computable structure $\mathcal{S}$ has a computably presentable simulation $\mathcal{C}$.

Moreover, if we fix a computable $\mathcal{D} \cong \mathcal{C}$, then for any countable parameter set $P \subseteq \mathcal{S}$, there exists an embedding $f_P : \mathcal{D} \hookrightarrow \mathcal{S}$ such that $P \subseteq \text{range}(f_P)$ and $\mathcal{S}$ and $f_P(\mathcal{D})$ realize exactly the same finitary types over every finite subset $P_0 \subseteq P$. (We say that $f_P(\mathcal{D})$ *simulates* $\mathcal{S}$ *over* P.)

Towards a Converse

Prop.: The computable simulation $\mathcal{C}$ built above for the PLC structure $\mathcal{S}$, satisfies:
$\exists$ a set of elementary embeddings $\psi_p : \mathcal{C} \hookrightarrow \mathcal{S}$, for every $p : \omega \to \text{dom}(\mathcal{S})$, such that

- $\text{range}(p) \subseteq \psi_p(\mathcal{C})$; and
- $\psi_p(\mathcal{C})$ is a simulation of $\mathcal{S}$ over $\text{range}(p)$; and
- if $p{\upharpoonright}n = p'{\upharpoonright}n$, then $(\forall k < n)$ $\psi_p^{-1}(p(k)) = \psi_{p'}^{-1}(p'(k))$.

Also, every structure which has a computable simulation $\mathcal{C}$ with embeddings ψ_p satisfying these properties is ∞-extensional over a uniformly computable cover.

Covers as Categories

Defn.: For a sturcture $\mathcal{S}$, $\mathbf{FGSub}(\mathcal{S})$ is the category of all finitely generated substructures of $\mathcal{S}$, with inclusion maps as morphisms. $\mathcal{S}$ is the inverse limit of $\mathbf{FGSub}(\mathcal{S})$.
An ∞-extensional cover can be made into a category by closing under composition of morphisms and adding identity morphisms. This is the *derived cover* $\mathfrak{A}$, and it is uniformly computable if the original cover was.

Prop.: If $\mathfrak{A}$ is this derived perfect cover for $\mathcal{S}$, then there exists a faithful functor $\boldsymbol{R}$ mapping $\mathbf{FGSub}(\mathcal{S})$ into $\mathfrak{A}$, and there exists a natural isomorphism

$$\beta : (I_{\mathfrak{A}} \circ \boldsymbol{R}) \to I_{\mathbf{FGSub}(\mathcal{S})}.$$

(Here $I_{\mathfrak{C}}$ denotes the inclusion functor from any category $\mathfrak{C}$ of $\mathcal{L}$-structures into the category of all $\mathcal{L}$-structures under embeddings.)

Proof of Proposition

We may define $\boldsymbol{R}$ by choosing $\boldsymbol{R}(\mathcal{B})$ to be any $\mathcal{A}_i \in \mathfrak{A}$ such that there exists an $\alpha : \mathcal{A}_i \to \mathcal{B}$ in the correspondence system. Let $\beta_{\mathcal{B}}$ be this α.

For an inclusion $\mathcal{B} \subseteq \mathcal{C}$ within $\mathcal{S}$, we have $\beta_{\mathcal{B}} : \mathcal{A}_i \to \mathcal{B}$ and $\beta_{\mathcal{C}} : \mathcal{A}_k \to \mathcal{C}$. There must exist j and $f \in I^{\mathfrak{A}}_{ij}$ and $\gamma : \mathcal{A}_j \to \mathcal{C}$ in M with $\gamma \circ f = \beta_{\mathcal{B}}$. But since γ and $\beta_{\mathcal{C}}$ both have image $\mathcal{C}$, perfection of the cover shows that $(\beta_{\mathcal{C}}^{-1} \circ \gamma) \in I^{\mathfrak{A}}_{jk}$. We define

$$\boldsymbol{R}(\mathcal{B} \subseteq \mathcal{C}) = \beta_{\mathcal{C}}^{-1} \circ \beta_{\mathcal{B}} = \beta_{\mathcal{C}}^{-1} \circ (\gamma \circ f) \in I^{\mathfrak{A}}_{ik}.$$

It follows that $\boldsymbol{R}$ is a functor, since this respects composition of morphisms, and that β is a natural isomorphism.