Local Computability

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Local Descriptions of Structures

Defn.: A *simple cover* $\mathfrak{A}$ of a structure $\mathcal{S}$ is a set $\{\mathcal{A}_i : i \in I\}$ which contains the finitely generated substructures of $\mathcal{S}$, up to isomorphism.

$\mathfrak{A}$ is *computable* if every $\mathcal{A} \in \mathfrak{A}$ is.

$\mathfrak{A}$ is *uniformly computable* if there is a single algorithm listing out all $\mathcal{A}_i$ in $\mathfrak{A}$. In this case $\mathcal{S}$ is *locally computable.*

Examples:

- All fields, and all relational structures, have computable simple covers.
- The ordered field $(\mathbb{R}, <)$ does not.
- The ordered field of computable real numbers is not locally computable, but has a computable simple cover.

Embeddings

Let $\mathcal{S}$ be locally computable via $\{\mathcal{A}_0, \mathcal{A}_1, \ldots\}$. Suppose $\mathcal{B} \subseteq \mathcal{C} \subseteq \mathcal{S}$ are finitely generated. If

$$\begin{array}{ccc}
\mathcal{B} & \xrightarrow[\subseteq]{} & \mathcal{C} \\
\beta \uparrow \cong & & \gamma \uparrow \cong \\
\mathcal{A}_i & \xrightarrow{f} & \mathcal{A}_j
\end{array}$$

commutes, we say that $f : \mathcal{A}_i \hookrightarrow \mathcal{A}_j$ *lifts to the inclusion* $\mathcal{B} \subseteq \mathcal{C}$ via the isomorphisms β and γ.

Defn.: A *cover* of $\mathcal{S}$ also has sets $I_{ij}^{\mathfrak{A}}$ of embeddings $\mathcal{A}_i \hookrightarrow \mathcal{A}_j$, such that every inclusion in $\mathcal{S}$ is the lift of some f in some $I_{ij}^{\mathfrak{A}}$, and every $f \in I_{ij}^{\mathfrak{A}}$ lifts to an inclusion in $\mathcal{S}$.
The cover is *uniformly computable* if all $I_{ij}^{\mathfrak{A}}$ are c.e. uniformly in i and j.

Notice that f is determined by its values on the generators of $\mathcal{A}_i$.

Examples

- Every infinite linear order has the same uniformly computable cover: $\mathcal{A}_i$ is the linear order on i elements, and $I_{ij}^{\mathfrak{A}}$ contains all embeddings $\mathcal{A}_i \hookrightarrow \mathcal{A}_j$.
- In $\mathbb{C}$, every possible embedding $f : \mathcal{A}_i \hookrightarrow \mathcal{A}_j$ lifts to an inclusion. This works for any ACF.
- $\mathbb{R}$ also has a uniformly computable cover. This follows from:

  Lemma: $\mathcal{S}$ has a uniformly computable cover iff $\mathcal{S}$ has a uniformly computable simple cover.
  Proof: Given a simple cover $\{\mathcal{A}_i\}$, consider the cover containing all f.g. substructures of each $\mathcal{A}_i$, with inclusion maps from these substructures into the original $\mathcal{A}_i$.

1-Extensionality

Defn.: Every embedding from any $\mathcal{A}_i$ into $\mathcal{S}$ is 0-*extensional*. An isomorphism $\beta : \mathcal{A}_i \hookrightarrow \mathcal{B} \subseteq \mathcal{S}$ is 1-*extensional* if

- $(\forall j)(\forall f \in I^{\mathfrak{A}}_{ij})(\exists \mathcal{C} \subseteq \mathcal{S})[f$ lifts to $\mathcal{B} \subseteq \mathcal{C}$ via β and some isomorphism $\gamma]$; and
- $(\forall$ f.g. $\mathcal{C} \supseteq \mathcal{B})(\exists j)(\exists f \in I^{\mathfrak{A}}_{ij})[f$ lifts to $\mathcal{B} \subseteq \mathcal{C}$ via β and some isomorphism $\gamma]$.

Intuition: A 1-extensional β is a strong pairing between $\mathcal{A}_i$ and $\mathcal{B}$, in that $\mathfrak{A}$'s ways to extend $\mathcal{A}_i$ are exactly the ways of extending $\mathcal{B}$ within $\mathcal{S}$.

$\mathfrak{A}$ is a 1-*extensional cover* if every $\mathcal{A}_i \in \mathfrak{A}$ is the domain of a 1-extensional embedding and every f.g. $\mathcal{B} \subseteq \mathcal{S}$ is the range of one.

Example

Cantor Space: The linear order on 2^ω has a 1-extensional cover. The objects are all finite linear orders $a_0 \prec \cdots \prec a_n$ under the following specifications. a_0 may or may not be designated as the left end point; likewise a_n as the right end point. Each a_m not so designated may be called either a *left gap point* or a *right gap point* (but not both). If a_m is a LGP and a_{m+1} a RGP, then we must specify whether they belong to the same gap or not.

An embedding $f : \mathcal{A}_i \hookrightarrow \mathcal{A}_j$ belongs to $I_{ij}^{\mathfrak{A}}$ if it respects all these properties: a_m is a left end point iff $f(a_m)$ is, etc.

So, if a_m and a_{m+1} are LGP and RGP for the same gap, then there can be no element between $f(a_m)$ and $f(a_{m+1})$ in $\mathcal{A}_j$.

θ-Extensionality

Defn.: Let θ be an ordinal. An isomorphism $\beta : \mathcal{A}_i \hookrightarrow \mathcal{B} \subseteq \mathcal{S}$ is θ-*extensional* if

- $(\forall \text{ f.g. } \mathcal{C} \supseteq \mathcal{B})(\forall \zeta < \theta)(\exists j)(\exists f \in I_{ij}^{\mathfrak{A}})$ [f lifts to $\mathcal{B} \subseteq \mathcal{C}$ via β and a ζ-extensional γ].
- and $(\forall j)(\forall f \in I_{ij}^{\mathfrak{A}})(\forall \zeta < \theta)(\exists \mathcal{C} \subseteq \mathcal{S})$ [f lifts to $\mathcal{B} \subseteq \mathcal{C}$ via β and a ζ-extensional γ];

Intuition: A θ-extensional β is a strong pairing between $\mathcal{A}_i$ and $\mathcal{B}$, in that $\mathfrak{A}$'s ways to extend $\mathcal{A}_i$ are exactly the ways of extending $\mathcal{B}$ within $\mathcal{S}$ while preserving the Σ_ζ-theory over $\mathcal{B}$.

$\mathfrak{A}$ is a θ-*extensional cover* if every $\mathcal{A}_i \in \mathfrak{A}$ is the domain of an θ-extensional embedding and every f.g. $\mathcal{B} \subseteq \mathcal{S}$ is the range of one.

Bad Example

Lemma: $\mathbb{R}$ has no 1-extensional cover.

Proof: If $\mathfrak{A}$ were such a cover, fix a noncomputable $x \in \mathbb{R}$ and a 1-extensional $\beta : \mathcal{A}_i \hookrightarrow \mathbb{Q}(x) \subseteq \mathbb{R}$. Then for $q \in \mathbb{Q}$:

$$\begin{aligned} q < x &\iff (\exists y \in \mathbb{R})\ y^2 = x - q \\ &\iff (\exists j\ \exists f \in I^{\mathfrak{A}}_{ij}\ \exists a \in \mathcal{A}_j) \\ &\qquad [a^2 = f(\beta^{-1}(x)) - f(\beta^{-1}(q))] \end{aligned}$$

So the lower cut defined by x would be computably enumerable, and similarly for the upper cut.

Σ_θ-Theory of $\mathcal{S}$

Theorem (Miller): Suppose $\mathcal{S}$ has a θ-extensional cover.
Then $(\forall \zeta \leq \theta)$, and for any finite set $\vec{p}$ of parameters in $\mathcal{S}$, the Σ_ζ-theory of $(\mathcal{S}, \vec{p})$ is arithmetically Σ^0_ζ, uniformly in i and $\alpha^{-1}(\vec{p})$, where $\alpha : \mathcal{A}_i \hookrightarrow \langle \vec{p} \rangle$ is θ-extensional.

Moreover, this applies even to *infinitary computable* Σ_ζ formulas over P.

Local Constructivizability

Defn. (Ershov): A structure $\mathcal{S}$ is *locally constructivizable* if, for all finite tuples $\vec{p} \in \mathcal{S}$, the $\exists$-theory of $(\mathcal{S}, \vec{p})$ is arithmetically Σ^0_1.

Cor.: Every 1-extensional structure is locally constructivizable.

Local constructivizability may be seen as a non-uniform version of 1-extensional local computability.

The field $\mathbb{R}$ is locally computable, but not locally constructivizable.

The field of computable real numbers is locally constructivizable, and locally computable, but not 1-extensional. (The *ordered* field of computable real numbers is not even locally computable.)

Correspondence Systems

Now we want to be able to extend our diagrams infinitely far to the right.

Defn.: A set M of embeddings $\beta : \mathcal{A}_i \hookrightarrow \mathcal{S}$ is a *correspondence system* if:

- $(\forall i)(\exists \beta \in M)\mathcal{A}_i = \mathrm{dom}(\beta)$; and
- $(\forall \text{ f.g. } \mathcal{B} \subseteq \mathcal{S})(\exists \beta \in M)\mathcal{B} = \mathrm{range}(\beta)$; and

and for all maps $\beta : \mathcal{A}_i \cong \mathcal{B}$ in M:

- $(\forall j \forall f \in I^{\mathfrak{A}}_{ij})(\exists \mathcal{C} \supseteq \mathcal{B})[f$ lifts to the inclusion $\mathcal{B} \subseteq \mathcal{C}$ via β and some $\gamma \in M]$; and
- $(\forall \text{ f.g. } \mathcal{C} \supseteq \mathcal{B})(\exists j \exists f \in I^{\mathfrak{A}}_{ij})[f$ lifts to the inclusion $\mathcal{B} \subseteq \mathcal{C}$ via β and some $\gamma \in M]$.

Defn.: A structure is ∞-*extensionally locally computable* if it has a correspondence system over a uniformly computable cover.

Perfect Local Computability

M is *perfect* if, for all $\beta, \gamma \in M$ with range$(\beta) =$ range(γ), we have $(\gamma^{-1} \circ \beta) \in I_{ij}^{\mathfrak{A}}$, where $\mathcal{A}_i = \text{dom}(\beta)$ and $\mathcal{A}_j = \text{dom}(\gamma)$.

- The uniformly computable cover we built for $\mathbb{C}$ has a perfect correspondence system.
- The uniformly computable cover we built for Cantor space (as a linear order) is perfect.
- It is also possible to view Cantor space as the top level of the tree $2^{<\omega+1}$, as a partial order, and to build a perfect correspondence system for this structure.

Such structures are called *perfectly locally computable.*

Globally Computable Structures

Theorem (Miller): For a countable structure $\mathcal{S}$, TFAE:

1. $\mathcal{S}$ is computably presentable;
2. $\mathcal{S}$ is perfectly locally computable;
3. $\mathcal{S}$ has a uniformly computable cover with a correspondence system, satisfying AP.

Proof: For $(1 \implies 2)$, build the *natural cover* $\mathfrak{A}$ containing all f.g. substructures of $\mathcal{S}$, under inclusion.
For $(2 \implies 3)$, all perfect covers have AP.
For $(3 \implies 1)$, amalgamate the $\mathcal{A}_i$ together over all embeddings in $\mathfrak{A}$, to get a computable presentation of $\mathcal{S}$.

∞-Extensionality

(joint work with Dustin Mulcahey)

Lemma: Let structures $\mathcal{C}$ and $\mathcal{S}$ have correspondence systems over the same cover. Suppose that $\mathcal{C}$ is countable, and that P is a countable subset of $\mathcal{S}$. Then there exists an elementary embedding of $\mathcal{C}$ into $\mathcal{S}$ whose image contains P.

Corollary: Any two countable structures with correspondence systems over the same cover are isomorphic.

Simulations

Defn.: A *simulation* $\mathcal{C}$ of a structure $\mathcal{S}$ is an elementary substructure of $\mathcal{S}$ which realizes the same n-types as $\mathcal{S}$ (for all n).

If for every $\vec{a} \in \mathcal{C}$ there is $\vec{p} \in \mathcal{S}$ such that $\mathcal{C}$ and $\mathcal{S}$ realize the same n-types over $\vec{a}$ and $\vec{p}$, and likewise for every $\vec{p}$ there is an $\vec{a}$, then $\mathcal{C}$ simulates $\mathcal{S}$ *over parameters*.

Examples: The algebraic closure of the field $\mathbb{Q}(X_0, X_1, \ldots)$ is a computably presentable simulation of $\mathbb{C}$ over parameters.

The intersection of $\mathbb{Q}$ with Cantor space ($\subset [0, 1]$, as linear order) is a computably presentable simulation of Cantor space over parameters.

Building Simulations

Lemma: Every ∞-extensionally locally computable structure $\mathcal{S}$ has a countable simulation $\mathcal{C}$ over parameters with a correspondence system over the cover of $\mathcal{S}$.

Proof: For each i, enumerate *one* image $\alpha(\mathcal{A}_i)$ into $\mathcal{C}$, with α in the correspondence system M for $\mathcal{S}$. Then close $\mathcal{C}$ under the $\forall\exists$ conditions for a correspondence system.

Notice that if M is perfect for $\mathcal{S}$, then the new system is perfect for $\mathcal{C}$.

Computable Simulations

Thm. (Mulcahey-Miller): Every perfectly locally computable structure $\mathcal{S}$ has a computably presentable simulation $\mathcal{C}$ over parameters.

Moreover, if we fix a computable $\mathcal{D} \cong \mathcal{C}$, then for any countable parameter set $P \subseteq \mathcal{S}$, there exists an embedding $f_P : \mathcal{D} \hookrightarrow \mathcal{S}$ such that $P \subseteq \text{range}(f_P)$ and $\mathcal{S}$ and $f_P(\mathcal{D})$ realize exactly the same finitary types over every finite subset of the image of f_P. (We call f_P an *elementary embedding over parameters.*)

Computable Simulations

Thm.: A structure $\mathcal{S}$ has an ∞-extensional cover with AP $\iff$ $\mathcal{S}$ has a computable simulation $\mathcal{C}$ over parameters, such that, for all elementary embeddings $f : \mathcal{C} \hookrightarrow \mathcal{S}$ over parameters, all $\vec{a} \in \mathcal{C}$, and all $x \in \mathcal{S}$, there exists an elementary embedding $g : \mathcal{C} \hookrightarrow \mathcal{S}$ over parameters with $g\restriction\vec{a} = f\restriction\vec{a}$ and $x \in \text{range}(g)$.

The cover $\mathfrak{A}$ is the natural cover of $\mathcal{C}$. The correspondence system contains all restrictions (to elements of $\mathfrak{A}$) of elementary embeddings of $\mathcal{C}$ into $\mathcal{S}$ over parameters.

$\mathbb{C}$ and its Simulations

A computable simulation of the field $\mathbb{C}$ must have infinite transcendence degree and be algebraically closed. Hence it must be the field $F = \overline{\mathbb{Q}(X_0, X_1, \ldots)}$. However,

Fact: The natural cover of F is *not* a perfect cover of $\mathbb{C}$. This follows from:
Lemma: A perfect cover of $\mathbb{C}$ must include a set $I_{ij}^{\mathfrak{A}}$ of size > 1.

Still, the natural cover $\mathfrak{A}$ of F is an ∞-extensional cover of $\mathbb{C}$, and has AP. The correspondence system consists of all embeddings of every $\mathcal{A}_i \in \mathfrak{A}$ into $\mathbb{C}$.

Cardinalities

Fix any countable sequence $\kappa_0 < \kappa_1 < \cdots$ of cardinals. Let T be the tree of height ω with each node at level n having κ_n-many immediate successors.

This T is perfectly locally computable: $\mathfrak{A}$ contains all finite substructures of $\omega^{<\omega}$, under embeddings which preserve levels, and M contains all level-preserving embeddings $\mathcal{A}_i \hookrightarrow T$.

But we can make the κ-sequence arbitrarily complex!

Covers as Categories

Defn.: For a structure $\mathcal{S}$, $\mathbf{FGSub}(\mathcal{S})$ is the category of all finitely generated substructures of $\mathcal{S}$, with inclusion maps as morphisms. $\mathcal{S}$ is the inverse limit of $\mathbf{FGSub}(\mathcal{S})$.
An ∞-extensional cover can be made into a category by closing under composition of morphisms and adding identity morphisms. This is the *derived cover* $\mathfrak{A}$, and it is uniformly computable if the original cover was.

Prop.: If $\mathfrak{A}$ is this derived perfect cover for $\mathcal{S}$, then there exists a functor $\boldsymbol{R}$ mapping $\mathbf{FGSub}(\mathcal{S})$ into $\mathfrak{A}$, and there exists a natural isomorphism

$$\beta : (I_{\mathfrak{A}} \circ \boldsymbol{R}) \to I_{\mathbf{FGSub}(\mathcal{S})}.$$

(Here $I_{\mathfrak{C}}$ denotes the inclusion functor from any category $\mathfrak{C}$ of $\mathcal{L}$-structures into the category of all $\mathcal{L}$-structures under embeddings.)

Proof of Proposition

We may define $\boldsymbol{R}$ by choosing $\boldsymbol{R}(\mathcal{B})$ to be any $\mathcal{A}_i \in \mathfrak{A}$ such that there exists an $\alpha : \mathcal{A}_i \to \mathcal{B}$ in the correspondence system. Let $\beta_{\mathcal{B}}$ be this α.

For an inclusion $\mathcal{B} \subseteq \mathcal{C}$ within $\mathcal{S}$, we have $\beta_{\mathcal{B}} : \mathcal{A}_i \to \mathcal{B}$ and $\beta_{\mathcal{C}} : \mathcal{A}_k \to \mathcal{C}$. There must exist j and $f \in I^{\mathfrak{A}}_{ij}$ and $\gamma : \mathcal{A}_j \to \mathcal{C}$ in M with $\gamma \circ f = \beta_{\mathcal{B}}$. But since γ and $\beta_{\mathcal{C}}$ both have image $\mathcal{C}$, perfection of the cover shows that $(\beta_{\mathcal{C}}^{-1} \circ \gamma) \in I^{\mathfrak{A}}_{jk}$. We define

$$\boldsymbol{R}(\mathcal{B} \subseteq \mathcal{C}) = \beta_{\mathcal{C}}^{-1} \circ \beta_{\mathcal{B}} = \beta_{\mathcal{C}}^{-1} \circ (\gamma \circ f) \in I^{\mathfrak{A}}_{ik}.$$

It follows that $\boldsymbol{R}$ is a functor, since this respects composition of morphisms, and that β is a natural isomorphism.

Questions

1. Can there exist a structure $\mathcal{S}$ with a computable simulation (over parameters?) such that $\mathcal{S}$ is not perfectly locally computable? Or such that $\mathcal{S}$ is not ∞-extensional with AP?
2. Develop a reasonable theory of maps (and computable maps) among covers.
   - Functors?
3. How locally computable is the structure $(\mathbb{C}, +, \cdot, 0, 1, f)$, where $f(z) = e^z$? (Similar questions for other holomorphic functions.)
4. Find θ-extensionally locally computable structures which are not $(\theta + 1)$-extensional, and which have arbitrarily complex $\Sigma_{\theta+1}$-theory over parameters.