Algebraic Fields and Computable Categoricity

Russell Miller & Alexandra Shlapentokh

Queens College & CUNY Graduate Center
New York, NY

East Carolina University
Greenville, NC.

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`qc.edu/~rmiller/slides.html`

Computable Categoricity

Defn.

A computable structure $\mathcal{A}$ is *computably categorical* if for each computable $\mathcal{B} \cong \mathcal{A}$ there is a computable isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.

Examples: (Dzgoev, Goncharov; Remmel; Lempp, McCoy, M., Solomon)

- A linear order is computably categorical iff it has only finitely many adjacencies.
- A Boolean algebra is computably categorical iff it has only finitely many atoms.
- An ordered Abelian group is computably categorical iff it has finite rank ($\equiv$ basis as $\mathbb{Z}$-module).
- For trees, the known criterion is recursive in the height and not easily stated!

Computably Categorical Fields

The following fields are all computably categorical:

- $\mathbb{Q}$.
- All finitely generated extensions of $\mathbb{Q}$ or $\mathbb{F}_p$.
- Every algebraically closed field of finite transcendence degree over $\mathbb{Q}$ or $\mathbb{F}_p$.
- All normal algebraic extensions of $\mathbb{Q}$ or $\mathbb{F}_p$.
- Some (but not all) non-normal algebraic extensions of $\mathbb{Q}$ or $\mathbb{F}_p$.
- Certain fields (but not very many!) of infinite transcendence degree over $\mathbb{Q}$. (Miller-Schoutens.)

Relative Computable Categoricity

Defn.

A computable structure $\mathcal{A}$ is *relatively computably categorical* if for each $\mathcal{B} \cong \mathcal{A}$ with domain ω, there is an isomorphism from $\mathcal{A}$ onto $\mathcal{B}$ which is computable from an oracle for $\mathcal{B}$.

Clearly this implies computable categoricity – but the converse is false! Certain computably categorical structures are not relatively computably categorical.

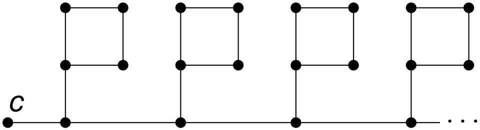
Scott Families

Defn.

A *Scott family* for a structure $\mathcal{A}$ is a set Σ of formulas $\psi(x_0, \ldots, x_n, \vec{c})$, over a fixed finite tuple $\vec{c}$ of parameters from $\mathcal{A}$, such that

- For all $\vec{a} \in \mathcal{A}^{<\omega}$, some $\psi \in \Sigma$ has $\models_{\mathcal{A}} \psi(\vec{a}, \vec{c})$.
- If $\vec{a}, \vec{b} \in \mathcal{A}^n$ satisfy the same $\psi \in \Sigma$, then some $\alpha \in \mathrm{Aut}(\mathcal{A})$ has $\alpha(a_i) = b_i$ for all $i \leq n$.

Example:



Thm. (Ash-Knight-Manasse-Slaman; Chisholm)

A computable structure $\mathcal{A}$ is relatively computably categorical iff $\mathcal{A}$ has a computably enumerable Scott family of existential formulas.

Algebraic Fields with Splitting Algorithms

Definitions

A field is *algebraic* if it is an algebraic extension of its prime subfield (either $\mathbb{Q}$ or $\mathbb{F}_p$).
A computable field F has a *splitting algorithm* if its *splitting set* S_F (or equivalently its *root set* R_F) is computable:

$$S_F = \{p \in F[X] : p \text{ factors properly in } F[X]\}$$
$$R_F = \{p \in F[X] : (\exists a \in F)\ p(a) = 0\}$$

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Facts:

- All finite algebraic extensions of $\mathbb{Q}$ and $\mathbb{F}_p$ have splitting algorithms, uniformly in their generators.
- An algebraic field F has a splitting algorithm iff all computable fields isomorphic to F have splitting algorithms.

Orbit Relations for Fields

Definition

For a computable field F, the *full orbit relation* A_F for F is the set:

$$\{\langle a_1, \ldots, a_n; b_1, \ldots, b_n\rangle : (\exists \sigma \in \mathrm{Aut}(F))(\forall i)\ \sigma(a_i) = b_i\} \subseteq \cup_n F^{2n}.$$

For algebraic F, by the Effective Theorem of the Primitive Element, A_F is computably isomorphic to the *orbit relation* B_F of F, defined by the action of $\mathrm{Aut}(F)$:

$$B_F = \{\langle a; b\rangle \in F^2 : (\exists \sigma \in \mathrm{Aut}(F))\ \sigma(a) = b\}.$$

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For algebraic $F \supseteq \mathbb{Q}$ in general, $\mathcal{B}_F$ is Π^0_2:

$$\langle a; b\rangle \in B_F \text{ iff } (\forall q \in \mathbb{Q}[X,Y])\ [q(a,Y) \in R_F \iff q(b,Y) \in R_F].$$

However, when F has a splitting algorithm, B_F becomes Π^0_1. (And when $F \supseteq \mathbb{Q}$ is a *normal* algebraic extension, B_F is computable.)

Computable Categoricity

Theorem (MS 2010)

Let F be a computable algebraic field with a splitting algorithm. Then F is computably categorical iff B_F is computable.

Since F has a splitting algorithm, B_F is Π_1^0, so the complexity of this condition is Σ_3^0 in indices for F and its splitting algorithm.

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Corollary

A computable algebraic field with a splitting algorithm is computably categorical iff it is relatively computably categorical.

(The proof below relativizes easily.)

B_F Computable $\implies$ F Computably Categorical

Sketch of Proof: Suppose we have defined $f_s : F_s = \mathbb{Q}(x_0, \ldots, x_s) \to E$, where $F = \{x_0, x_1, \ldots\}$, and $F_s \subseteq F_{s+1}$ are both normal within F.
Assume f_s extends to an isomorphism $\psi : F \to E$.
Find a primitive generator $a \in F$ of F_{s+1}, and find its minimal polynomial $p(X) \in F_s[X]$. Let $a = y_1, y_2, \ldots, y_d$ be all its roots in F.

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For each $j \leq d$ with $\langle a, y_j \rangle \notin B_F$, find some $q_j \in F_s[X, Y]$ with

$$q_j(a, Y) \in R_F \;\; \& \;\; q_j(y_j, Y) \notin R_F.$$

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Assume f_s extends to an isomorphism $\psi : F \to E$.
Find a primitive generator $a \in F$ of F_{s+1}, and find its minimal polynomial $p(X) \in F_s[X]$. Let $a = y_1, y_2, \ldots, y_d$ be all its roots in F.
For each $j \leq d$ with $\langle a, y_j \rangle \notin B_F$, find some $q_j \in F_s[X, Y]$ with

$$q_j(a, Y) \in R_F \;\;\&\;\; q_j(y_j, Y) \notin R_F.$$

Then find all roots $z_1, \ldots, z_d \in E$ of the image $\overline{p}(X)$ of $p(X)$ under f_s.
Define $f_{s+1}(a)$ to be any z_k for which all the polynomials $\overline{q_j}(z_k, Y)$ have roots in E. Then $f_s \subseteq f_{s+1}$ and $\langle a, \psi^{-1}(z_k) \rangle \in B_F$, so f_{s+1} must extend to the isomorphism $\psi \circ \sigma : F \to E$, where $\sigma \in \mathrm{Aut}(F)$ has $\sigma(a) = \psi^{-1}(z_k)$ and $(\forall i)\sigma(x_i) = x_i$. By iterating, we get a computable isomorphism.

F Computably Categorical $\implies B_F$ Computable

Proof: Here we assume that F is computably categorical, and build a computable $E \cong F$. In doing so, whenever possible, we build E so that φ_e will *not* be an isomorphism. (This uses a priority construction, based on the values e.) For the least e such that φ_e defies all our attempts, the isomorphism φ_e will allow us to compute B_F.

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At each stage $s+1$, we look for the least e such that for some $a, b \in F_s$, $\varphi_{e,s}(a)\downarrow$ and $\langle a, b\rangle \in B_{F_s}$, yet $\langle a, b\rangle \notin B_{F_{s+1}}$. (Essentially we search for $q \in \mathbb{Q}[X, Y]$ such that $q(b, Y)$ has a root in F and $q(a, Y)$ does not.) Then, when building the extension E_{s+1} of E_s, we add a root of $q(\varphi_e(a), Y)$, so that our isomorphism $F_{s+1} \to E_{s+1}$ has $b \mapsto \varphi_e(a)$, and no isomorphism $F \to E$ has $a \mapsto \varphi_e(a)$.

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If $\varphi_e : F \to E$ is an isomorphism, with e minimal, then

$$(\forall s \geq s_0)\ [\varphi_{e,s}(a)\downarrow \implies (\forall b)[\langle a, b\rangle \in B_{F_s} \implies \langle a, b\rangle \in B_F]].$$

So B_F is c.e., as well as Π^0_1.

Algebraic Fields Without Splitting Algorithms

Theorem (easy corollary of Ash-Knight-Manasse-Slaman)

Let F be a computable, relatively computably categorical, algebraic field. Then the orbit relation B_F is computably enumerable.

This generalizes our theorem on fields with splitting algorithms, since for those fields, B_F is automatically Π^0_1.

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This generalizes our theorem on fields with splitting algorithms, since for those fields, B_F is automatically Π^0_1.
However, if F has no splitting algorithm, then B_F can be c.e., or even computable, with F *not* computably categorical.

Example: Begin to build $E = F = \mathbb{Q}(\theta_0)$ with $\theta_0^3 = 2$. If $\varphi_e(\theta_0)\downarrow = \theta_0$, then adjoin to E and F two more cube roots θ_1, θ_2 of 2. Also adjoin to E a square root of θ_0, and to F a square root of θ_1. Then $\varphi_e : E \to F$ is not an isomorphism, yet B_E and B_F remain computable.

Full Construction: $B_F \leq_T \emptyset$, but F not C.C.

Lemma

For every Galois extension $\mathbb{Q} \subseteq E$ and every $d > 1$, there exists a monic $f(X) \in \mathbb{Z}[X]$ of degree d such that $\mathrm{Gal}(K/\mathbb{Q}) \cong S_d$ and the splitting field K of $f(X)$ over $\mathbb{Q}$ is linearly disjoint from E.

Corollary

There is a computable sequence $f_0, f_1, \ldots$ in $\mathbb{Z}[X]$ whose splitting fields K_i each have Galois group S_7 over $\mathbb{Q}$ and such that each K_i is linearly disjoint from the compositum of all K_j ($j \neq i$).

Use this sequence to build F and $\tilde{F}$. For distinct roots r_1, r_2, r_3, r_4 of K_i, first adjoin $(r_1 + r_2)$ to $\mathbb{Q}$ in both F and $\tilde{F}$. If $\varphi_i(r_1 + r_2)\downarrow = (r_1 + r_2)$, then adjoin $(r_3 + r_4)$ to both fields, r_1 to F, and r_3 to $\tilde{F}$. Then $F \cong \tilde{F}$, but not via φ_i. However, F is rigid (except for interchanging r_1 with r_2, if they entered F). Thus B_F is computable.

The Isomorphism Tree

Let $F \cong \tilde{F}$ be computable algebraic fields, and let $z_1, z_2, z_3, \ldots$ be a sequence of elements generating F. For simplicity, assume $\mathbb{Q} = \mathbb{Q}(z_0) \subseteq \mathbb{Q}(z_1) \subseteq \mathbb{Q}(z_2) \subseteq \cdots$, with $z_0 = 1$.
Compute polynomials $f_{i+1} \in \mathbb{Q}[Y, Z]$ s.t. $f_{i+1}(z_i, Z)$ is the minimal polynomial of z_{i+1} over $\mathbb{Q}(z_i)$, for each i.

Defn.

The *isomorphism tree* $I_{F\tilde{F}}$ is the following subtree of $\tilde{F}^{<\omega}$:

$$\{\langle \tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_m \rangle : (\forall i < m)\ \tilde{f}_{i+1}(\tilde{z}_i, \tilde{z}_{i+1}) = 0\}.$$

Here $\tilde{f}_i(Y, Z)$ is the image of $f_i(Y, Z)$ under the isomorphism of the prime subfields.

So $I_{F\tilde{F}}$ is a finite-branching tree, and paths through it correspond to isomorphisms from F onto $\tilde{F}$, with each path Turing-equivalent to its isomorphism.

Scott Families for Algebraic Fields

To enumerate a Scott family Σ for F, we need to give an $\exists$-formula ψ_i for each z_i such that, for all $z \in F$,

$$\psi_i(z) \text{ holds in } F \iff \langle z_i, z \rangle \in B_F.$$

For the finitely many roots of $f_i(z_{i-1}, Z)$ in F, ψ_i needs to know some level m of I_{FF} such that all nodes at level i with extensions to level m are extendible (i.e. lie on paths).

If we have a computable function which gives such a level m for every m, then F has a c.e. Scott family, hence is relatively computably categorical. This function gives a computable bound on the height of the tree I_{FF} above nonextendible nodes.

Back to Computable Categoricity

Theorem (HKMS 2010)

There exists a computable algebraic field F which is computably categorical, but not relatively c.c. In particular, B_F is not Σ^0_2.

Proof: a tree construction of F.
A node ρ at level $2e$ has two outcomes: $\cong$ and $\ncong$. It tries to ensure that if the structure computed by the e-th Turing program is a field K_e isomorphic to F, then some program P_ρ computes an isomorphism between them.
Each time larger initial fragments of F and K_e are found to embed into each other, ρ makes its stronger outcome $\cong$ eligible. This outcome does not allow lower-priority nodes to do anything until F and K_e match up well enough for P_ρ to be sure how to build its isomorphism.

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Each time larger initial fragments of F and K_e are found to embed into each other, ρ makes its stronger outcome $\cong$ eligible. This outcome does not allow lower-priority nodes to do anything until F and K_e match up well enough for P_ρ to be sure how to build its isomorphism.

Suppose ρ is on the true path. If $F \cong K_e$, then $\rho\hat{}\langle\cong\rangle$ will also be on the true path, and P_ρ will compute an isomorphism. (Finitely much information is needed: ρ, and the last stage at which ρ is initialized.)

Computable Categoricity, continued

Nodes τ at levels $2e + 1$ ensure that the e-th partial computable function φ_e does not compute an m-reduction from B_F to $\emptyset''$. (If this holds for every e, then $B_F \not\leq_m \emptyset''$, hence cannot be Σ^0_2.)

τ adds to F two $\mathbb{Q}$-conjugates x_τ and y_τ. At all stages, there will be two distinct z_s and z_t already in F such that the minimal polynomials of each over x_τ have no root over y_τ. Whenever $W_{\varphi_e(\langle x_\tau, y_\tau \rangle)}$ gets a new element, we add a root over y_τ of the minimal polynomial of z_s over x_τ (where $s < t$), but also add a new $u > t$ for which F has no root over y_τ of the minimal polynomial of z_u over x_τ. Therefore, if τ is never injured, then $\langle x_\tau, y_\tau \rangle \in B_F$ iff $W_{\varphi_e(\langle x_\tau, y_\tau \rangle)}$ is infinite.

B_F and the Galois Group of F over L

Extend the definition to field extensions F/L with F computable:

$$B_{F/L} = \{\langle a; b\rangle \in F^2 : (\exists \sigma \in \mathrm{Gal}(F/L))\ \sigma(a) = b\}.$$

For algebraic F/L, if L is a c.e. subfield of F, $B_{F/L}$ is always Π^0_2.

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Let $B_{F/L}$ be c.e. Then given $\langle a, b\rangle \in B_{F/L}$, we can compute some $\sigma \in \mathrm{Gal}(F/L)$ with $\sigma(a) = b$. Let $F = \{x_0, x_1, \ldots\}$, and let $\sigma(x_s)$ be the first x_t with $\langle a, x_0, \ldots, x_s; b, \sigma(x_0), \ldots, \sigma(x_{s-1}), x_t\rangle \in A_{F/L}$.

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Definition

A computable algebraic extension F/L has *computably approximable Galois group* $\mathrm{Gal}(F/L)$ if $B_{F/L}$ is computably enumerable.

$\mathrm{Gal}(F/L)$ is essentially a type-2 computable object, in the sense of computable analysis. (It may have 2^ω-many elements!)

Automorphism Groups in General

Defn.

For any structure $\mathcal{M}$ with domain ω in which all orbits are finite, say that Aut($\mathcal{M}$) is ***d**-computably approximable* if the following set is computably enumerable in the Turing degree ***d***:

$$A_{\mathcal{M}} = \{\langle \vec{a}; \vec{b} \rangle \in \bigcup_n \left(\omega^{2n}\right) : (\exists \sigma \in \mathrm{Aut}(\mathcal{M}))(\forall i < n) \sigma(a_i) = b_i\}.$$

In general $A_{\mathcal{M}}$ is Σ^1_1. For relatively computably categorical structures $\mathcal{M}$, Aut($\mathcal{M}$) is $\mathcal{M}$-computably approximable: enumerate $\langle \vec{a}; \vec{b} \rangle$ into $A_{\mathcal{M}}$ whenever some ψ in a (computably enumerable) Scott family for $\mathcal{M}$ is found to be satisfied by both $\vec{a}$ and $\vec{b}$.

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However, Steiner has found computable structures $\mathcal{M}$ with all orbits finite and $A_{\mathcal{M}}$ computable, such that $\mathcal{M}$ is not computably categorical. For instance, let $\mathcal{M}$ be an equivalence relation with exactly one equivalence class of each finite size. We saw the same above for a computable algebraic field.

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