Noncomputable Functions in the Blum-Shub-Smale Model

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(Joint work with Wesley Calvert, Murray State University, and Ken Kramer, CUNY.)

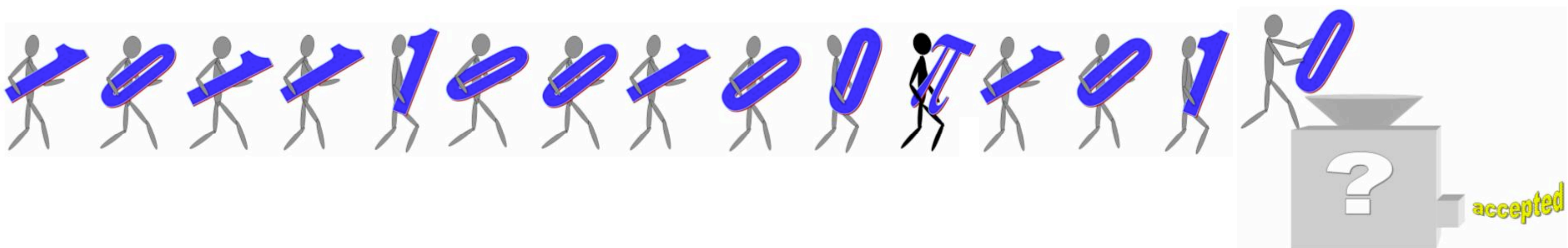
Slides available at
`qc.edu/~rmiller/slides.html`

BSS Computation on $\mathbb{R}$

Roughly, a BSS machine M on $\mathbb{R}$ operates like a Turing machine, but with a real number in each cell, rather than a bit.

- M can compute full-precision $+$. $-$. $\cdot$, and $\div$ on numbers in its cells.
- M can compare 0 to the number in any cell, using $=$ or $<$, and fork according to the answer.
- M is allowed finitely many real numbers $z_0, \ldots, z_m$ as *parameters* in its program. The input and output (if M halts) are tuples $\vec{y} \in \mathbb{R}^\infty = \{$ finite tuples from $\mathbb{R}$ $\}$.

A subset $S \subseteq \mathbb{R}^\infty$ is BSS-*decidable* iff its characteristic function χ_S is computable by a BSS machine, and BSS-*semidecidable* iff S is the domain of some BSS-computable function.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Basic Facts about BSS Computation

For a machine M with parameters $\vec{z}$, running on input $\vec{y}$, only elements of the field $\mathbb{Q}(\vec{z}, \vec{y})$ can ever appear in the cells of M.

Cell: 0	$\cdots$	m	$m+1$	$\cdots$	$m+n$	$m+n+1$	$\cdots$
z_0	$\cdots$	z_m	y_1	$\cdots$	y_n		
z_0	$\cdots$	z_m	y_1	$\cdots$	y_n	$z_m + y_n$	
$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	
$f_{0,s}(\vec{y})$	$\cdots$	$f_{m,s}(\vec{y})$	$f_{m+1,s}(\vec{y})$	$\cdots$	$f_{m+n,s}(\vec{y})$	$f_{m+n+1,s}(\vec{y})$	$\cdots$
$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	

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$\vdots$		$\vdots$	$\vdots$		$\vdots$	$\vdots$	

For each input $\vec{y}$, every $f_{i,s}(Y_1,\ldots,Y_n)$ is a rational function with coefficients from the field $\mathbb{Q}(\vec{z})$. If the input $\{y_1,\ldots,y_n\}$ is algebraically independent over $\mathbb{Q}(\vec{z})$, then each $f_{i,s}(\vec{Y})$ is uniquely defined.

Restrictions on BSS Computation

Given a machine M with parameters $\vec{z}$, choose any input $\vec{y}$ algebraically independent over $\mathbb{Q}(\vec{z})$. If $M(\vec{y})$ halts after t steps, then only finitely many functions $f_{i,s}$ appear. So there is an $\epsilon > 0$ such that for all inputs $\vec{x}$ within ϵ of $\vec{y}$, M at stage s contains:

$f_{0,s}(\vec{x})$	$\cdots$	$f_{m,s}(\vec{x})$	$f_{m+1,s}(\vec{x})$	$\cdots$	$f_{m+n,s}(\vec{x})$	$f_{m+n+1,s}(\vec{x})$	$\cdots$

with the same functions $f_{i,s}$ as for $\vec{y}$.

Therefore, on any $\vec{x} \in \mathbb{R}^n$ in an ϵ-ball around $\vec{y}$, M always halts after t steps, and computes the function $\langle f_{0,t}(\vec{x}), \ldots, f_{m+n+t,t}(\vec{x})\rangle$.

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Corollary: No BSS-decidable set can be dense and codense within any nonempty open subset of $\mathbb{R}^n$.

Oracle BSS-Machines

To do the same for a machine M with parameters $\vec{z}$ and an *oracle* $A \subseteq \mathbb{R}$, we would have to ensure that $|\vec{x} - \vec{y}| < \epsilon$ and also, for all $f_{i,s}$,

$$f_{i,s}(\vec{x}) \in A \iff f_{i,s}(\vec{y}) \in A.$$

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Theorem: Let

$$\mathbb{A}_{=d} := \{y \in \mathbb{R} : y \text{ is algebraic of degree } d \text{ over } \mathbb{Q}\}.$$

Then for all $d \geq 0$, $\mathbb{A}_{=d+1} \not\leq_{BSS} \mathbb{A}_{=d}$. Indeed $\mathbb{A}_{=d+1} \not\leq_{BSS} \cup_{c \leq d} \mathbb{A}_c$.

Proving the Theorem for $d = 1$: $\mathbb{A}_{=2} \not\leq_{BSS} \mathbb{A}_{=1} = \mathbb{Q}$

For any machine M with parameters $\vec{z}$, fix $y \in \mathbb{R}$ transcendental over $\mathbb{Q}(\vec{z})$. Let F be the finite set of rational functions $f(Y)$ over $\mathbb{Q}(\vec{z})$ such that $f(y)$ appears in a cell during this computation.
We pick $x = b + \sqrt{u}$ for some $b, u \in \mathbb{Q}$ with

- $|x - y| < \epsilon$;
- $f'(b) \neq 0$ for all nonconstant $f \in F$, and $u > 0$ small enough that $f(b + \sqrt{u}) \neq f(b - \sqrt{u})$; and
- $\sqrt{u} \notin \mathbb{Q}(\vec{z})$.

So $x = b + \sqrt{u}$ has minimal polynomial $p(X) = X^2 - 2bX + (b^2 - u)$ over $\mathbb{Q}(\vec{z})$, with conjugate $(b - \sqrt{u})$.

Proving the Theorem for $d = 1$: $\mathbb{A}_{=2} \not\leq_{BSS} \mathbb{A}_{=1} = \mathbb{Q}$

For each $f(Y) = \frac{g(Y)}{h(Y)} \in F$, write

$$f(X) = \frac{g(X)}{h(X)} = \frac{q_g(X) \cdot p(X) + r_g(X)}{q_h(X) \cdot p(X) + r_h(X)}$$

with $r_g(X)$ and $r_h(X)$ both linear polynomials. Then

$$\frac{r_g(x)}{r_h(x)} = f(x) = f(b + \sqrt{u}) \neq f(b - \sqrt{u}) = \frac{r_g(b - \sqrt{u})}{r_h(b - \sqrt{u})},$$

so $r_g(X)$ is not a constant multiple of $r_h(X)$ whenever f is nonconstant.

Proving the Theorem for $d = 1$: $\mathbb{A}_{=2} \not\leq_{BSS} \mathbb{A}_{=1} = \mathbb{Q}$

But if $a = f(x) = \frac{r_g(x)}{r_h(x)} = \frac{g_1 \cdot (b+\sqrt{u})+g_0}{h_1(b+\sqrt{u})+h_0}$, then

$$\sqrt{u} = \frac{g_1 b + g_0 - ah_1 b - ah_0}{-(g_1 - ah_1)} \text{ or } g_1 = ah_1.$$

In the first case, $f(x) = a \notin \mathbb{Q}(\vec{z})$ since $\sqrt{u} \notin \mathbb{Q}(\vec{z})$. In the second case, $a = \frac{ah_1 \cdot (b+\sqrt{u})+g_0}{h_1(b+\sqrt{u})+h_0}$, forcing $g_0 = ah_0$, so that

$$r_g(X) = g_1 X + g_0 = ah_1 X + ah_0 = a \cdot r_h(X)$$

and f must have been constant. So

$$f(x) \in \mathbb{Q}(\vec{z}) \iff f \text{ is constant} \iff f(y) \in \mathbb{Q}(\vec{z})$$

and specifically $f(x) \in \mathbb{Q} \iff f(y) \in \mathbb{Q}$.

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and specifically $f(x) \in \mathbb{Q} \iff f(y) \in \mathbb{Q}$. So the oracle computation on inputs x and y follows the same path and outputs the same answer. But $y \notin \mathbb{A}_{=2}$ and $x = b + \sqrt{u} \in \mathbb{A}_{=2}$.

Other Results

- **Prop.:** Let p and r be any positive integers. Then $\mathbb{A}_{=p} \leq_{BSS} \mathbb{A}_{=r}$ if and only if p divides r.

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- **Thm.:** If $C \subseteq \mathbb{R}^\infty$ is an oracle set of infinite cardinality $\kappa < 2^\omega$, and $S \subseteq \mathbb{R}$ is a set with $S \leq_{BSS} C$, then S must be locally of bicardinality $\leq \kappa$.

Online Help

- Introduction to BSS computation:
  L. Blum, F. Cucker, M. Shub, and S. Smale; *Complexity and Real Computation* (Berlin: Springer-Verlag, 1997).
- Relevant papers:
  C. Gassner; A hierarchy below the halting problem for additive machines, *Theory of Computing Systems* **43** (2008) 3–4, 464–470.
  K. Meer & M. Ziegler; An explicit solution to Post's Problem over the reals, *Journal of Complexity* **24** (2008) 3–15.
- Full version of these results, joint with Calvert & Kramer, available at `qc.edu/~rmiller/BSSfull.pdf`
- These slides available at `qc.edu/~rmiller/slides.html`