Functors in Computable Model Theory

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(Joint work with many researchers.)

A First Example

Background: a structure A with domain ω is *computable* if all of its functions and relations are computable. Such an A is *computably categorical* if, for every computable structure B which is classically isomorphic to A, there is a computable isomorphism from A onto B.

A *nested equivalence structure* is a structure with equivalence relations $R_1, \ldots, R_n$, such that each $R_{i+1} \subseteq R_i$.

Question

Find a criterion for computable categoricity for computable nested equivalence structures.

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Solution

Leah Marshall (Ph.D. student at GWU, with advice from Harizanov, J.C. Reimann, & M.) showed how to convert nested equivalence structures into trees of finite height, and back, effectively. She used this method, along with the known criterion for computable categoricity for computable trees of finite height, to answer the question.

Marshall's Method

Given a nested equivalence structure E with $R_1 \supseteq R_2 \supseteq \cdots \supseteq R_n$, build a tree $\mathcal{T}(E)$ of height $n+1$, with one node at level i for each R_i-equivalence class in E. Node x_{i+1} at level $i+1 \leq n$ lies above node x_i at level i iff the R_{i+1}-class represented by x_{i+1} is contained in the R_i-class for x_i. Add a root at the bottom (or view R_0 as the ER with just one class), and above each x_n at level n, add one node at level $n+1$ for each element of the R_n-class represented by x_n. (Or treat R_{n+1} as the equality relation.)

Conversely, given a full computable tree T of height $(n+1)$, define an n-nested equivalence structure $\mathcal{E}(T)$. Its elements are the nodes at level $n+1$, and each node x_i at level $i \leq n$ defines an R_i-class containing those level-$(n+1)$ nodes above x_i.

These processes, both completely effective, are inverses of each other. Each is a *Turing-computable reduction*, as studied by Knight et al.

Marshall's Results

Theorem (Lempp, McCoy, M., & Solomon, 2005)

A computable tree of finite height is computably categorical iff it has finite type. (The definition of finite type takes several pages, but is purely structural.)

Theorem (Marshall)

A computable *n*-nested equivalence structure is computably categorical iff the corresponding tree is computably categorical, iff....

The key here is that from every isomorphism $f : E \to E'$ of n-nested equivalence structures, we can compute an isomorphism $\mathcal{T}(f) : \mathcal{T}(E) \to \mathcal{T}(E')$ of the corresponding trees, and vice versa with $\mathcal{E}$.

Theorem (Marshall)

Nested equivalence structures cannot have finite computable dimension > 1.

Proof: Finite-height trees can't.

Further Results by Marshall

Theorem (Marshall)

Every computably categorical nested equivalence structure is *relatively computably categorical*.

Proof: This holds for finite-height trees. The definition concerns noncomputable copies of the tree T as well as computable ones. However, our functors $\mathcal{T}$ and $\mathcal{E}$ deal perfectly well with noncomputable T and E as well.

Theorem (Marshall)

The Turing degree spectra of full finite-height trees are precisely those of nested equivalence structures. Likewise for categoricity spectra.

Proof: Recall that $\text{Spec}(A) = \{\deg(B) : B \cong A \ \& \ \text{dom}(B) = \omega\}$. But $E \equiv_T \mathcal{T}(E)$ and $T \equiv_T \mathcal{E}(T)$, so this is immediate.

Yet Another Result by Marshall

Recall: the *isomorphism problem* for a class $\mathfrak{K}$ of computable structures is the set of pairs of (indices of) structures in $\mathfrak{K}$ which are isomorphic to each other.

Theorem (Marshall)

The isomorphism problem for n-nested equivalence structures is exactly as hard as that for full trees of height $n + 1$.

This doesn't even need the isomorphisms $\mathcal{E}(f)$ and $\mathcal{T}(g)$. In fact, neither did the result on spectra. The functors $\mathcal{E}$ and $\mathcal{T}$ here are in fact Turing-computable reductions between the two classes, which is the traditional method of considering isomorphism problems. However, the effective maps $\mathcal{E}$ and $\mathcal{T}$ on isomorphisms are necessary for all the results on computable categoricity and categoricity spectra.

OK, they're functors

Defn.

Let $\mathfrak{C}$ be a category in which the objects are countable structures with domain ω (in a single computable language) and the morphisms are maps; and let $\mathfrak{D}$ be another such category (possibly with a different language). A (type-2) *computable functor* from $\mathfrak{C}$ into $\mathfrak{D}$ consists of two Turing functionals Φ and Φ_* such that:

- for all $A \in \mathfrak{C}$, $\Phi^A \in \mathfrak{D}$; and
- for all morphisms $f : A \to B$ in $\mathfrak{C}$, $\Phi_*^{A \oplus f \oplus B}$ is a morphism from Φ^A to Φ^B in $\mathfrak{D}$; and
- these define a functor from $\mathfrak{C}$ into $\mathfrak{D}$.

In the case of nested equivalence structures and trees, the two functors $\mathcal{E}$ and $\mathcal{T}$ were actually inverses of each other.

Other possible functors

Another example is given by Victor Ocasio Gonzalez (PhD student of Knight), using ideas of Dave Marker.

Theorem (Ocasio)

There is a computable functor (Φ, Φ_*) from the category of countable linear orders L into that of countable real closed fields F. Moreover, there is a computable functor (Ψ, Ψ_*) which is a left inverse of (Φ, Φ_*).

Given L, Φ builds the real closure F of the ordered field $\mathbb{Q}(a_0, a_1, \ldots)$, where $(\forall i)(\forall n)\ n < a_i$ in F and

$$i < j \text{ in } L \iff a_i < a_j \text{ in } F \iff (\forall m) a_i^m < a_j \text{ in } L.$$

So L is the linear order of the positive nonstandard elements of F, modulo the equivalence $a \sim b \iff (\exists m \in \omega)[a < b^m \ \&\ b < a^m]$.

Inverse of Ocasio's functor?

For each L, the field $F = \Phi^L$ is built in a straightforward way, with the odd numbers in $\omega = \text{dom}(F)$ serving as the elements a_i in F. Therefore, there is a computable functor (Ψ, Ψ_*) which is a left inverse of (Φ, Φ_*).

However, this Ψ does *not* extend to all countable real closed fields, nor even to those F isomorphic to fields of the form Φ^L. In general, picking out representatives $a_0, a_1, \ldots$ in such an F requires the jump of the atomic diagram of F. If we allow (Ψ, Ψ_*) to be *jump-computable*, with oracles $\Psi^{F'}$ and $\Psi_*^{F' \oplus f \oplus K'}$, then we can get an inverse to (Φ, Φ_*) whose domain is closed under isomorphism.

Ocasio uses this (with a stronger version of Φ) to show that, for every (infinite) L, there is an RCF F such that

$$\text{Spec}(F) = \{\boldsymbol{d} : \boldsymbol{d}' \in \text{Spec}(L)\}.$$

More Marker ideas

A similar process uses the ENI-DOP for the theory $\mathbf{DCF}_0$ to show that, for every countable, automorphically nontrivial graph G, there is a countable differentially closed field K such that

$$\text{Spec}(K) = \{\boldsymbol{d} : \boldsymbol{d}' \in \text{Spec}(G)\}.$$

Indeed, we have a converse, established by a priority construction:

Theorem (Marker-M.)

The spectra of differentially closed fields of characteristic 0 are exactly the preimages, under the jump operation, of the spectra of graphs.

Once again, this can be seen as a construction of a computable functor from graphs to models of $\mathbf{DCF}_0$, which has an inverse functor (on a subclass, closed under isomorphism, of models of $\mathbf{DCF}_0$) that is only jump-computable. The priority construction extends the theorem (but not the inverse functor) to all models of $\mathbf{DCF}_0$.

How nice should functors be?

Theorem (Hirschfeldt-Khoussainov-Shore-Slinko 2002)

For every automorphically nontrivial, countable structure A, there exists a countable graph G which has the same spectrum as A, the same $\boldsymbol{d}$-computable dimension as A (for each $\boldsymbol{d}$), and the same categoricity properties as A under expansion by a constant, and which realizes every $\text{DgSp}_{\mathcal{A}}(R)$ (for every relation R on A) as the spectrum of some relation on G.

Given A, they built a graph $G = \mathscr{G}(A)$ such that the isomorphisms from A onto any B correspond bijectively with the isomorphisms from $\mathscr{G}(A)$ onto $\mathscr{G}(B)$, by a map $f \mapsto \mathscr{G}(f)$ which preserves the Turing degree of f.

Translating HKSS into functors

In terms of functors, the HKSS proof builds:

- a computable functor (Φ, Φ_*) from the category $\mathfrak{C}$ of all countable, automorphically nontrivial structures in a given language into the category $\mathfrak{G}$ of countable, automorphically nontrivial graphs,
- which is *full* and *faithful*: every isomorphism from Φ^A onto Φ^B in $\mathfrak{G}$ is the image $\Phi_*^{A \oplus f \oplus B}$ of a unique isomorphism $f : A \to B$ in $\mathfrak{C}$,
- with another computable functor (Ψ, Ψ_*) back into $\mathfrak{C}$, whose domain contains the range of (Φ, Φ_*) and is closed under isomorphism,
- such that (Ψ, Ψ_*) is a left inverse of (Φ, Φ_*).

This suffices to transfer all relevant computable-model-theoretic properties from objects in $\mathfrak{C}$ to graphs. In particular, (Ψ, Ψ_*) need not be a right inverse of (Φ, Φ_*).

More nice functors

Theorem (HKSS 2002)

The completeness described above holds not only of graphs, but also of partial orderings, lattices, rings, integral domains of arbitrary characteristic, commutative semigroups, 2-step nilpotent groups,....

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Theorem (M-Park-Poonen-Schoutens-Shlapentokh 2013)

... and fields (of characteristic 0).

In each case, the strategy is the same, now that we know that $\mathfrak{G}$ is complete: build a computable functor (Φ, Φ_*) from $\mathfrak{G}$ to the relevant category, with a computable left-inverse functor (Ψ, Ψ_*) whose domain is closed under isomorphism.

The functor from graphs $\mathfrak{G}$ to fields $\mathfrak{F}$

Theorem (MPPSS)

For every countable graph G, there exists a countable field $\mathscr{F}(G)$ with the same computable-model-theoretic properties as G, as in the HKSS theorem. Indeed, $\mathscr{F}$ may be viewed as a computable, fully faithful functor from the category of countable graphs (under monomorphisms) into the class of fields, with a computable inverse functor (on its image).

Full faithfulness means that each field homomorphism $\mathscr{F}(G) \to \mathscr{F}(G')$ comes from a unique monomorphism $G \to G'$. Isomorphisms $g : G \to G'$ will map to isomorphisms $\mathscr{F}(g) : \mathscr{F}(G) \to \mathscr{F}(G')$.

We do not claim that every $F' \cong \mathscr{F}(G)$ lies in the image of $\mathscr{F}$. This situation will require attention.

Construction of $\mathscr{F}(G)$

We use two curves X and Y, defined by integer polynomials:

$$X : p(u, v) = u^4 + 16uv^3 + 10v^4 + 16v - 4 = 0$$
$$Y : q(T, x, y) = x^4 + y^4 + 1 + T(x^4 + xy^3 + y + 1) = 0$$

Let $G = (\omega, E)$ be a graph. Set $K = \mathbb{Q}(\Pi_{i \in \omega} X)$ to be the field generated by elements $u_0 < v_0 < u_1 < v_1, \ldots$, with $\{u_i : i \in \omega\}$ algebraically independent over $\mathbb{Q}$, and with $p(u_i, v_i) = 0$ for every i. The element u_i in $K \subseteq \mathscr{F}(G)$ will represent the node i in G.

Next, adjoin to K elements x_{ij} and y_{ij} for all $i > j$, with $\{x_{ij} : i > j\}$ algebraically independent over K, and with

$$q(u_i u_j, x_{ij}, y_{ij}) = 0 \text{ if } (i, j) \in E$$
$$q(u_i + u_j, x_{ij}, y_{ij}) = 0 \text{ if } (i, j) \notin E.$$

We write Y_t for the curve defined by $q(t, x, y) = 0$ over $\mathbb{Q}(t)$. So the process above adjoins the function field of either $Y_{u_i u_j}$ or $Y_{u_i + u_j}$, for each $i > j$. $\mathscr{F}(G)$ is the extension of K generated by all x_{ij} and y_{ij}.

Reconstructing G From $\mathscr{F}(G)$

Lemma

Let $G = (\omega, E)$ be a graph, and build $\mathscr{F}(G)$ as above. Then:

(i) $X(\mathscr{F}(G)) = \{(u_i, v_i) : i \in \omega\}$.

(ii) If $(i,j) \in E$, then $Y_{u_i u_j}(\mathscr{F}(G)) = \{(x_{ij}, y_{ij})\}$ and $Y_{u_i+u_j}(\mathscr{F}(G)) = \emptyset$.

(iii) If $(i,j) \notin E$, then $Y_{u_i u_j}(\mathscr{F}(G)) = \emptyset$ and $Y_{u_i+u_j}(\mathscr{F}(G)) = \{(x_{ij}, y_{ij})\}$.

This is the heart of the proof. (i) says that $p(u,v) = 0$ has no solutions in $\mathscr{F}(G)$ except the ones we put there, so we can enumerate

$$\{u_i : i \in \omega\} = \{u \in \mathscr{F}(G) : (\exists v \in \mathscr{F}(G)) p(u,v) = 0\}.$$

Similarly, (ii) and (iii) say that the equations $q(u_i u_j, x, y) = 0$ and $q(u_i + u_j, x, y) = 0$ have no unintended solutions in $\mathscr{F}(G)$. So, given i and j, we can determine from $\mathscr{F}(G)$ whether $(i,j) \in E$: search for a solution to either $q(u_i u_j, x, y) = 0$ or $q(u_i + u_j, x, y) = 0$.

Interpretations

One can readily view this construction as a way of *interpreting* the graph G in the field $\mathscr{F}(G)$. The domain of G (within $\mathscr{F}(G)$) is defined by the formula

$$(\exists v)\ p(u,v) = 0,$$

under the relation of equality, and the edge relation on such u_0, u_1 is defined by

$$E(u_0,u_1) \iff (\exists x \exists y)\ q(u_0 u_1, x, y) = 0;$$

$$\neg E(u_0,u_1) \iff (\exists x \exists y)\ q(u_0 + u_1, x, y) = 0.$$

Since the domain, E, and $\neg E$ are all defined by Σ_1 formulas, the interpretation may be considered *effective*.

Effective interpretation

Definition (Montalbán)

Let A be an L-structure, and B be any structure. Let us assume that L is a relational language $L = \{P_0, P_1, P_2, \ldots\}$ where P_i has arity $a(i)$; so $A = (A; P_0^A, P_1^A, \ldots)$ and $P_i^A \subseteq A^{a(i)}$.
We say that A is *effectively interpretable* in B if, in B, there is

- a uniformly r.i.c.e. set $D_A^B \subseteq B^{<\omega}$ (the domain of the interpretation),
- a uniformly r.i. computable relation $\eta \subseteq B^{<\omega} \times B^{<\omega}$ which is an equivalence relation on D_A^B (interpreting equality),
- a uniformly r.i. computable sequence of relations $R_i \subseteq (B^{<\omega})^{a(i)}$, closed under the equivalence η within D_A^B (interpreting P_i),
- and a function $f_A^B \colon D_A^B \to A$ which induces an isomorphism:

$$(D_A^B/\eta; R_0, R_1, \ldots) \cong (A; P_0^A, P_1^A, \ldots).$$

With parameters, Montalbán notes, this is equivalent to Σ-definability.

Interpretation and functors

The MPPSS theorem gives an effective interpretation of graphs in fields, uniformly in the presentation of any countable graph. Indeed, the graph G and the field $\mathscr{F}(G)$ always satisfy:

Definition (Montalbán)

Structures A and B effectively interpretable in each other are *effectively bi-interpretable* if the compositions

$$f_B^A \circ \overline{f}_A^B : D_B^{D_A^B} \to B \text{ and } f_A^B \circ \overline{f}_B^A : D_A^{D_B^A} \to A$$

are uniformly relatively intrinsically computable in B and A.

Current work

Question: is interpretation the only way to build computable functors?

Conjecture (Harrison-Trainor, Melnikov, M., Montalbán)

A category $\mathfrak{C}$ of countable structures, under isomorphisms, is effectively interpretable in another such category $\mathfrak{D}$ $\iff$ there exist computable functors $\Phi : \mathfrak{C} \to \mathfrak{D}$ and Ψ such that:

- $\text{rg}(\Phi) \subseteq \text{dom}(\Psi) \subseteq \mathfrak{D}$ and $\text{rg}(\Psi) \subseteq \mathfrak{C}$;
- $\text{dom}(\Psi)$ is closed under isomorphism inside $\mathfrak{D}$;
- $\Psi \circ \Phi$ is the $\mathfrak{C}$-identity functor, up to A-computable isomorphism: for all $A \in \mathfrak{C}$, $\Psi(\Phi(A))$ is A-computably isomorphic to A; and
- $\Phi \circ \Psi$ is the identity functor on all $B \in \text{dom}(\Psi)$, up to B-computable isomorphism.

For $\mathfrak{C}$ to be effectively interpretable in $\mathfrak{D}$ means that each $A \in \mathfrak{C}$ is effectively bi-interpretable with some $B \in \mathfrak{D}$, using the same set of formulas for the interpretations between each A and its B.

Further questions

As in the examples by Ocasio and Marker-M, we extend these notions:

Definition

In an α*-jump-computable functor* (Φ, Φ_*) from $\mathfrak{C}$ to $\mathfrak{D}$, the outputs are $\Phi^{A^{(\alpha)}}$ and $\Phi_*^{A^{(\alpha)} \oplus f \oplus B^{(\alpha)}}$ (for A, B, and $f : A \to B$ from $\mathfrak{C}$).

The HTM^3 conjecture for $\alpha = 0$ extends naturally to a version of jump-effective bi-interpretability using $\Sigma^c_{\alpha+1}$ formulas in place of Σ^c_1 formulas for the interpretations. This may allow us to compare classes of structures more rigorously.

Examples

- Is there an ω-jump-computable functor from graphs $\mathfrak{G}$ into BA's?
- Is there a 2-jump-computable functor from $\mathfrak{G}$ into linear orders?
- We conjecture that there is no α-jump-computable functor from fields into algebraic fields, with $\alpha < \omega_1^{CK}$.