The Complexity of Computable Categoricity for Algebraic Fields

Russell Miller

Queens College & CUNY Graduate Center
New York, NY

Logic Colloquium & ASL European Summer Meeting

Barcelona, 11 July 2011

(Joint work with Denis Hirschfeldt, University of Chicago; Ken Kramer, CUNY; & Alexandra Shlapentokh, East Carolina University)

Slides available at
qc.edu/~rmiller/slides.html

Computable Categoricity

Defn.

A computable structure $\mathcal{A}$ is *computably categorical* if for each computable $\mathcal{B} \cong \mathcal{A}$ there is a computable isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.

Examples: (Dzgoev, Goncharov; Remmel; Lempp, McCoy, M., Solomon)

- A linear order is computably categorical iff it has only finitely many adjacencies.
- A Boolean algebra is computably categorical iff it has only finitely many atoms.
- An ordered Abelian group is computably categorical iff it has finite rank ($\equiv$ basis as $\mathbb{Z}$-module).
- For trees, the known criterion is recursive in the height and not easily stated!

In all these examples, computable categoricity is a Σ_3^0 property.

Previous Result

Definitions

A field is *algebraic* if it is an algebraic extension of its prime subfield (either $\mathbb{Q}$ or $\mathbb{F}_p$).
A computable field F has a *splitting algorithm* if its *splitting set* $\{p \in F[X] : p \text{ factors properly in } F[X]\}$ is computable.

Previous Result

Definitions

A field is *algebraic* if it is an algebraic extension of its prime subfield (either $\mathbb{Q}$ or $\mathbb{F}_p$).
A computable field F has a *splitting algorithm* if its *splitting set* $\{p \in F[X] : p \text{ factors properly in } F[X]\}$ is computable.

Theorem (Miller-Shlapentokh 2010)

For a computable algebraic field F with a splitting algorithm. TFAE:

- F is computably categorical.
- F is relatively computably categorical.
- The *orbit relation* of F is computable:

$$\{\langle a; b\rangle \in F^2 : (\exists \sigma \in \mathrm{Aut}(F))\ \sigma(a) = b\}.$$

So computable categoricity for such fields is a Σ^0_3 property.

Isomorphism Trees for Algebraic Fields

Fix a computable algebraic field F with domain $\{x_0, x_1, \ldots\}$, and any field $\tilde{F}$. The finite partial isomorphisms $h : \mathbb{Q}(x_0, \ldots, x_n) \to \tilde{F}$ form an $\tilde{F}$-computable, finite-branching tree $I_{F\tilde{F}}$ under $\subseteq$.

Isomorphism Trees for Algebraic Fields

Fix a computable algebraic field F with domain $\{x_0, x_1, \ldots\}$, and any field $\tilde{F}$. The finite partial isomorphisms $h : \mathbb{Q}(x_0, \ldots, x_n) \to \tilde{F}$ form an $\tilde{F}$-computable, finite-branching tree $I_{F\tilde{F}}$ under $\subseteq$.

Paths through $I_{F\tilde{F}}$ correspond to embeddings $F \to \tilde{F}$. By König's Lemma, such an embedding exists iff $I_{F\tilde{F}}$ is infinite, i.e. iff every finitely generated subfield of F embeds into $\tilde{F}$.

Definition

If $\tilde{F} \cong F$, then we call $I_{F\tilde{F}}$ the *isomorphism tree* for F and $\tilde{F}$, since its paths are precisely the isomorphisms from F onto $\tilde{F}$.

For computable algebraic fields F and $\tilde{F}$, being isomorphic is Π^0_2: both $I_{F\tilde{F}}$ and $I_{\tilde{F}F}$ must be infinite.

Computable Dimension

In work with Khoussainov and Soare, Hirschfeldt extended an earlier theorem of Goncharov:

Theorem

Goncharov: if $\mathcal{A}$ and $\mathcal{B}$ are computable structures which are not computably isomorphic, but have a $\mathbf{0}'$-computable isomorphism $\mathcal{A} \to \mathcal{B}$, then $\mathcal{A}$ has computable dimension ω.

Extension: if $\mathcal{A}$ and $\mathcal{B}$ are computable structures which are not computably isomorphic, but have an isomorphism $\mathcal{A} \to \mathcal{B}$ which is *leftmost-path approximable* in a computable tree, then $\mathcal{A}$ has computable dimension ω.

Computable Dimension

In work with Khoussainov and Soare, Hirschfeldt extended an earlier theorem of Goncharov:

Theorem

Goncharov: if $\mathcal{A}$ and $\mathcal{B}$ are computable structures which are not computably isomorphic, but have a $\mathbf{0}'$-computable isomorphism $\mathcal{A} \to \mathcal{B}$, then $\mathcal{A}$ has computable dimension ω.

Extension: if $\mathcal{A}$ and $\mathcal{B}$ are computable structures which are not computably isomorphic, but have an isomorphism $\mathcal{A} \to \mathcal{B}$ which is *leftmost-path approximable* in a computable tree, then $\mathcal{A}$ has computable dimension ω.

Corollary (HKMS)

Every computable algebraic field has computable dimension 1 or ω.

Just use the leftmost path in the isomorphism tree!

Relative Computable Categoricity

Theorem (HKMS)

A computable algebraic field F is relatively computably categorical iff there is a computable function g such that:
$(\forall$ levels $m)(\forall$ nodes $\sigma \in I_{FF}$ at level $m)$ σ is extendible to a path through I_{FF} iff I_{FF} contains a node of length $g(m)$ extending σ.

If $\tilde{F} \cong F$, then the same fact about g holds in the tree $I_{F\tilde{F}}$. So we can compute a path through $I_{F\tilde{F}}$: start with the root as σ_0, and always extend σ_s to the first node $\sigma_{s+1} \supset \sigma$ we find which has an extension in $I_{F\tilde{F}}$ of length $\geq g(|\sigma_{s+1}|)$. This computation relativizes easily to $\deg(\tilde{F})$.

Conversely, in a Σ_1^0 Scott family for an r.c.c. algebraic field, the formula satisfied by the element $x_m \in F$ allows us to compute (an upper bound for) such a function g.

Computably Categorical, but Not Relatively So

Kudinov and others produced examples of computable graphs G which are computably categorical, but not relatively c.c. Their tree construction works equally well for algebraic fields F, using a tree construction:

- Half the nodes in the tree are *categoricity nodes*, ensuring (for each e) that if the e-th computable structure F_e is a field $\cong F$, then they are computably isomorphic. The node of this type on the true path builds a computable isomorphism from F_e onto F.
- The other half of the nodes ensure that F has no Σ_1^0 Scott family. Such a node α, of length $(2k)$, puts a single root x_α of a polynomial $p_\alpha(X)$ into F, waits for the k-th possible Scott family $\mathcal{S}_k$ to enumerate a formula satisfied by x_α, then adjoins $\sqrt{x_\alpha}$ to F, and then (when permitted by higher-priority categoricity nodes) adjoins another root y_α of p_α to F, but without any square roots. So x_α and y_α lie in distinct orbits, but satisfy the same formula in $\mathcal{S}_k$.

Complexity of Computable Categoricity

Recall: if F and $\tilde{F}$ are computable algebraic fields, then they are isomorphic iff $I_{F\tilde{F}}$ and $I_{\tilde{F}F}$ are both infinite. This is Π^0_2.

Now F is computably categorical iff, for every index e, either:

- the e-th computable structure F_e is not a field (Σ^0_2); or
- F_e is not an algebraic field (Σ^0_2); or
- $F_e \not\cong F$ (normally Π^1_1, but here Σ^0_2); or
- $(\exists i)\ \varphi_i$ is an isomorphism from F_e onto F (Σ^0_3, including the "$\exists i$").

Thus, computable categoricity for algebraic fields is a Π^0_4 property.

Π^0_4-completeness

Theorem (HKMS)

For computable algebraic fields, the property of computable categoricity is Π^0_4-complete.

Fix a computable f such that for all n:

$$n \in \overline{\emptyset^{(4)}} \iff \forall a \exists b \; f(n, a, b) \in \mathbf{Inf}.$$

We build the field $F(n)$ uniformly in n, using a tree with categoricity nodes at odd levels, similar to before. All nodes α at level $(2a)$ are *non-categoricity nodes*, with outcomes $b \in \omega$. For the least b with $f(n, a, b) \in \mathbf{Inf}$, the node $\alpha\hat{\ }b$ will be eligible infinitely often. If $n \in \emptyset^{(4)}$, then (for some a) no such b exists, and the true path will end at level $(2a)$, at a node α which builds a computable field $F_\alpha \cong F(n)$ which is not computably isomorphic to F. The diagonalization by α against φ_e is similar to before.

Standard References on Computable Fields

- A. Frohlich & J.C. Shepherdson; Effective procedures in field theory, *Phil. Trans. Royal Soc. London, Series A* **248** (1956) 950, 407-432.
- M. Rabin; Computable algebra, general theory, and theory of computable fields, *Transactions of the American Mathematical Society* **95** (1960), 341-360.
- Yu.L. Ershov; Theorie der Numerierungen III, *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik* **23** (1977) 4, 289-371.
- G. Metakides & A. Nerode; Effective content of field theory, *Annals of Mathematical Logic* **17** (1979), 289-320.
- M.D. Fried & M. Jarden, *Field Arithmetic* (Berlin: Springer, 1986).
- V. Stoltenberg-Hansen & J.V. Tucker; Computable rings and fields, in *Handbook of Computability Theory*, ed. E.R. Griffor (Amsterdam: Elsevier, 1999), 363-447.

These slides available at `qc.edu/~rmiller/slides.html`