Degrees of Categoricity of Algebraic Fields

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Computable Categoricity

Definition

A computable structure $\mathcal{A}$ is *computably categorical* if for each computable $\mathcal{B} \cong \mathcal{A}$ there is a computable isomorphism from $\mathcal{A}$ to $\mathcal{B}$.

Examples: (Dzgoev, Goncharov; Remmel; Lempp, McCoy, M., Solomon)

- A linear order is computably categorical iff it has only finitely many adjacencies.
- A Boolean algebra is computably categorical iff it has only finitely many atoms.
- An ordered Abelian group is computably categorical iff it has finite rank ($\equiv$ basis as $\mathbb{Z}$-module).
- For trees (viewed as partial orders), the known criterion is recursive in the height and not easily stated!

***d*-Computable Categoricity**

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For any Turing degree ***d***, a computable structure $\mathcal{A}$ is ***d****-computably categorical* if for each computable $\mathcal{B} \cong \mathcal{A}$ there is a ***d***-computable isomorphism from $\mathcal{A}$ to $\mathcal{B}$.

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Definition

The *categoricity spectrum* of $\mathcal{A}$ is the set of all ***d*** such that $\mathcal{A}$ is ***d***-computably categorical. The least such degree (if any) is the *degree of categoricity of* $\mathcal{A}$.

Fields

Definition

The *splitting set* of a field F is

$$\{p(X) \in F[X] : \exists \text{ nonconstant } q_0, q_1 \in F[X](q_0 \cdot q_1 = p)\}.$$

Facts:

1. The splitting set is Turing-equivalent to the *root set*

$$\{p(X) \in F[X] : (\exists a \in F)p(a) = 0\}.$$

2. For computable algebraic fields $F_0 \cong F_1$, the splitting sets are Turing-equivalent.

Proofs of these facts use **Rabin's Theorem**: A computable field F has a splitting algorithm iff F has a computable embedding with computable image in a computable presentation of $\overline{F}$.

Negative Results

Theorem

There exists a computable algebraic field F which is not computably categorical, yet has computable splitting set.

First idea: Build computable fields $F \cong \tilde{F}$ with both square roots of each prime p_e. If $\varphi_{e,s}(\sqrt{p_e})\downarrow = y$ with $y^2 = \tilde{p}_e$ in $\tilde{F}$, we adjoin a p-th root of $\sqrt{p_e}$ in F and a p-th root of the square root $\neq y$ in $\tilde{F}$.

- Choose $p > s$ to ensure that F has computable splitting set.
- Always use distinct primes $p > 3$: adjoining a p-th root cannot cause any extraneous q-th roots to appear, for prime $q \neq p$.

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Problem: Adding a p-th root of $\sqrt{p_e}$ puts a p-th root of the other square root of p_e into F as well.

Solution to the Problem

Proposition

Let p and d be odd primes, with $F = \mathbb{Q}[\sqrt{p}]$, and let $\sigma(\sqrt{p}) = -\sqrt{p}$. Then there exists a polynomial $h(X) \in F[X]$ of degree d, with image $h^-(X) \in F[X]$ under σ, such that:

- each of the splitting fields K and K^- of h and h^- over F has Galois group S_d over F; and
- the splitting field of h over K^- also has Galois group S_d, as does the splitting field of h^- over K.

So, when $\varphi_e(\sqrt{p_e})\downarrow = \sqrt{\tilde{p}_e}$, we can adjoin a root of $h(X)$ in F and a root of $\tilde{h}^-(X)$ in $\tilde{F}$.

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- each of the splitting fields K and K^{-} of h and h^{-} over F has Galois group S_d over F; and
- the splitting field of h over K^{-} also has Galois group S_d, as does the splitting field of h^{-} over K.

So, when $\varphi_e(\sqrt{p_e})\downarrow = \sqrt{\tilde{p}_e}$, we can adjoin a root of $h(X)$ in F and a root of $\tilde{h}^{-}(X)$ in $\tilde{F}$. In fact, this gives us more power.

Theorem

There exists a computable algebraic field F which is not even $\emptyset'$-computably categorical.

A field F which is not $\mathbf{0}'$-categorical

Build computable fields $F \cong \tilde{F}$ so that $(\forall e)$

$$f(x) = \lim_s \varphi_e(x, s) \text{ is not an isomorphism.}$$

Basic module for φ_e: Adjoin $\pm\sqrt{p_e}$ to F and $\tilde{F}$.

- While $\varphi_e(\sqrt{p_e}, s) \neq \pm\sqrt{\tilde{p}_e}$, do nothing.
- If $\varphi_e(\sqrt{p_e}, s) = \sqrt{\tilde{p}_e}$, then adjoin a root of an $h(X)$ to F, and a root of $\tilde{h}^-(X)$ to $\tilde{F}$.
- If later $\varphi_e(\sqrt{p_e}, s') = -\sqrt{\tilde{p}_e}$, then adjoin a root of $h^-(X)$ to F, and a root of $\tilde{h}(X)$ to $\tilde{F}$. Find a new $h(X)$ for $\sqrt{p_e}$, and do the reverse.

So if $\lim_s \varphi_e(\sqrt{p_e}, s)$ converges, then it chooses the wrong value.

And if $\lim_s \varphi_e(\sqrt{p_e}, s)$ diverges, then we satisfy the requirement and still have $F \cong \tilde{F}$.

Isomorphisms as Paths

Let $F = \{x_0, x_1, \ldots\}$. Find the minimal polynomial $q_i(X_i)$ of x_i over $\mathbb{Q}[x_0, \ldots, x_{i-1}]$. Write $p_i(x_0, \ldots, x_{i-1}, X_i) = q(X_i)$ with $p_i \in \mathbb{Q}[\vec{X}]$.

Definition

The *isomorphism tree* $I_{F,\tilde{F}}$ is

$$\{\sigma \in \tilde{F}^n : (\forall i < n) p_{i-1}(\sigma(0), \ldots, \sigma(i-1)) = 0\}.$$

So each $\sigma \in I_{F,\tilde{F}}$ defines a partial isomorphism $F \to \tilde{F}$. Paths through $I_{F,\tilde{F}}$ correspond to (total) isomorphisms.

Low Basis Theorem

Theorem (Jockusch-Soare)

If T is a computable subset of $\omega^{<\omega}$ which forms a finite-branching infinite subtree, and

$$s(\sigma) = |\{\text{immediate successors of } \sigma \text{ in } T\}|$$

has degree $\boldsymbol{s}$, then there is a path f through T with $f' \leq_T \boldsymbol{s}'$. (Such a path f is said to be *low relative to* $\boldsymbol{s}$.)

Indeed, for any fixed $\boldsymbol{s}$, Jockusch and Soare produced a single degree $\boldsymbol{t}$ with $\boldsymbol{t}' \leq_T \boldsymbol{s}'$ which computes a path through *every* such tree.

$\boldsymbol{d}$-Computable Categoricity

Recall: from the splitting set of F, we can compute the number of roots of $p_i(\sigma(0), \ldots, \sigma(i-1), X_i)$ in $\tilde{F}$.

Theorem

If F is a computable algebraic field with splitting set S, then F is $\boldsymbol{d}$-computably categorical for some Turing degree $\boldsymbol{d}$ with $\boldsymbol{d}' \leq_T S'$.

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Corollary

Every computable algebraic field with computable splitting set is $\boldsymbol{d}$-computably categorical for some low Turing degree $\boldsymbol{d}$, indeed for any PA-degree. (A *PA-degree* is the degree of a complete extension of Peano arithmetic.)

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Every computable algebraic field is $\boldsymbol{d}$-computably categorical for some Turing degree $\boldsymbol{d}$ with $\boldsymbol{d}' \leq_T \mathbf{0}''$, indeed for any PA-degree relative to $\mathbf{0}'$.

Degrees of Categoricity

Fact (Jockusch-Soare)

Every nonempty Π^0_1-class contains paths of degrees $\boldsymbol{c}$, $\boldsymbol{d}$ with $\boldsymbol{c} \wedge \boldsymbol{d} = \mathbf{0}$.

Proposition

A computable algebraic field with splitting set S can only have degree of categoricity $\leq_T \deg(S)$.

Corollary

A computable algebraic field with computable splitting set cannot have nonzero degree of categoricity.

More about Degrees of Categoricity

Theorem

For c.e. degrees $\boldsymbol{c}$ and $\boldsymbol{d}$, we have $\boldsymbol{c} \leq_T \boldsymbol{d}$ iff there exists a computable algebraic field F with degree of categoricity $\boldsymbol{c}$ and splitting set of degree $\boldsymbol{d}$.

Proof: Code a c.e. set $C \in \boldsymbol{c}$ into all isomorphisms between F and $\tilde{F}$, by forcing $\sqrt{p_{2e}} \mapsto \sqrt{\tilde{p}_{2e}}$ iff $e \in C$. Code $D \in \boldsymbol{d}$ into the splitting set by adjoining the square roots of p_{2e+1} when/if e enters D.

Extending the Results

Theorem

All $\boldsymbol{d}$-computable categoricities so far are *uniform*. The same holds for computable fields of characteristic p algebraic over F_p.

- When the field has positive finite transcendence degree over $\mathbb{Q}$, the results still hold, but uniformity fails.
- In characteristic p, the results hold (non-uniformly) for *separable* algebraic extensions of $F_p(X_1, \ldots, X_n)$.
- For non-separable algebraic extensions of $F_p(X_1, \ldots, X_n)$, these questions remain open.

Isomorphism trees can be applied to other computable algebraic structures. Cf. work of Rebecca Steiner on finite-branching trees (under predecessor) and finite-valence connected graphs; also Hirschfeldt-Khoussainov-Soare on such graphs.

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