Genericity, Infinitary Interpretations, and Automorphism Groups of Structures

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(Joint work with Matthew Harrison-Trainor, and Antonio Montalbán, and in part with Alexander Melnikov.)

Our categories

Definition

For a countable infinite structure $\mathcal{A}$, the category $\text{Iso}(\mathcal{A})$ has as objects all isomorphic copies of $\mathcal{A}$ with domain ω. The morphisms in the category are the isomorphisms between objects, under composition.

So a *functor* from $\text{Iso}(\mathcal{B})$ to $\text{Iso}(\mathcal{A})$ consists of one map G sending each $\widehat{\mathcal{B}} \cong \mathcal{B}$ to some $\widehat{\mathcal{A}} = G(\widehat{\mathcal{B}}) \cong \mathcal{A}$, along with a second map H sending each isomorphism $f : \widehat{\mathcal{B}} \to \widetilde{\mathcal{B}}$ to an isomorphism $H(f) : G(\widehat{\mathcal{B}}) \to G(\widetilde{\mathcal{B}})$.

H must respect composition, and must map the identity map on $\widehat{\mathcal{B}}$ to the identity map on $G(\widehat{\mathcal{B}})$. ($\mathcal{A}$ and $\mathcal{B}$ need not have the same signature.)

Interpretations

Many functors from Iso($\mathcal{B}$) to Iso($\mathcal{A}$) arise as follows. Suppose we have an *interpretation* of $\mathcal{A}$ in $\mathcal{B}$, given by formulas (no parameters):

Interpretation

- $\alpha(\vec{x})$ defines a subset D of B^n in $\mathcal{B}$;
- $\beta(\vec{x}, \vec{y})$ defines an equivalence relation $\sim$ on D; and
- for each m-ary relation R_i on $\mathcal{A}$, γ_i defines a subset $C_i = \{\vec{d} \in D^m : \gamma_i(\vec{d})\}$ of D^m invariant under $\sim$,

with $(D/\sim,\ C_0, C_1, \ldots) \cong \mathcal{A}$.

Then, "inside" every $\widehat{\mathcal{B}} \in$ Iso($\mathcal{A}$), we have a copy $\widehat{\mathcal{A}}$ of $\mathcal{A}$ defined by these formulas. (Use a fixed order on ω^n to identify the domain of $\widehat{\mathcal{A}}$ with ω.) Moreover, each isomorphism $\widehat{\mathcal{B}} \to \widetilde{\mathcal{B}}$ will map the copy $\widehat{\mathcal{A}}$ onto the copy $\widetilde{\mathcal{A}}$ inside $\widetilde{\mathcal{B}}$. So the interpretation of $\mathcal{A}$ in $\mathcal{B}$ yields a functor from Iso($\mathcal{B}$) to Iso($\mathcal{A}$).

Functors given by interpretations: a mixed bag

Example: we have an interpretation of the algebraic closure $\overline{\mathbb{Q}}$ in the real closure R of the field $\mathbb{Q}$, viewing elements $a + bi$ of $\overline{\mathbb{Q}}$ as pairs (a, b) from R. This yields a functor F from $\text{Iso}(R)$ to $\text{Iso}(\overline{\mathbb{Q}})$. However, this functor is not *full*: among all the automorphisms of (a fixed copy of) $\overline{\mathbb{Q}}$, only the identity is in the "range" of F, since R is rigid.

More importantly, not all functors arise from interpretations. For example, we have a very natural functor $F : \text{Iso}(\overline{\mathbb{Q}}) \to \text{Iso}(\overline{\mathbb{Q}}[X])$, with isomorphisms between fields extending to isomorphisms between their polynomial rings. However, there is no interpretation of $\overline{\mathbb{Q}}[X]$ in the field $\overline{\mathbb{Q}}$.

Solution: infinitary interpretations

We wish to broaden the notion of interpretation to allow the use of $L_{\omega_1\omega}$ formulas in defining the domain and $\sim$ and the relations. Notice that, even if we allow arbitrary $L_{\omega_1\omega}$ formulas, each interpretation of $\mathcal{A}$ in $\mathcal{B}$ will still yield a functor from $\text{Iso}(\mathcal{B})$ to $\text{Iso}(\mathcal{A})$. However, this project began with *effective interpretations*.

Definition

An *effective interpretation* of $\mathcal{A}$ in $\mathcal{B}$ is an interpretation in which α, β, and all γ_i are Σ_1^c (i.e., computable infinitary existential) formulas, and in which $(\neg\beta)$ and every $(\neg\gamma_i)$ can also be defined by a Σ_1^c formula in $\mathcal{B}$.

The domain D can now consist of arbitrary finite tuples: $D \subseteq B^{<\omega}$ but possibly $\forall n\ D \not\subseteq B^n$. (Formally, this requires α to be a computable disjunction of Σ_1^c formulas α_n, each with free variables $x_1, \ldots, x_n$.)

Computable infinitary interpretations

With an effective interpretation of $\mathcal{A}$ in $\mathcal{B}$, every copy $\widehat{\mathcal{B}}$ of $\mathcal{B}$ yields an $\widehat{\mathcal{B}}$-computable copy $\widehat{\mathcal{A}}$ of $\mathcal{A}$, in a uniform effective way. So we get a *computable functor* from $\text{Iso}(\mathcal{B})$ to $\text{Iso}(\mathcal{A})$:

$$G(\widehat{\mathcal{B}}) = \Phi^{\Delta(\widehat{\mathcal{B}})} \quad \& \quad H(f) = \Phi_*^{\Delta(\widehat{\mathcal{B}}) \oplus f \oplus \Delta(\widetilde{\mathcal{B}})} : G(\widehat{\mathcal{B}}) \to G(\widetilde{\mathcal{B}}),$$

where Φ and Φ_* are Turing functionals (i.e., oracle Turing machines).

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Theorem (Harrison-Trainor, Melnikov, M, Montalbán, or HTM^3)

Every computable functor arises from an effective interpretation (and vice versa).

Basic examples

To interpret $\overline{\mathbb{Q}}[X]$ in $\overline{\mathbb{Q}}$, we use as our domain

$$\{\text{nonempty } (a_0, \ldots, a_n) \in \overline{\mathbb{Q}}^{<\omega} : a_n = 0 \implies n = 0\}.$$

Another example: for a computable structure $\mathcal{A}$, *every* $\mathcal{B}$ has a computable constant functor into Iso($\mathcal{A}$), with $G(\widehat{\mathcal{B}}) = \mathcal{A}$ and $H(f) = \mathrm{id}_{\mathcal{A}}$. By the theorem, $\mathcal{A}$ must have an effective interpretation in each $\mathcal{B}$. In particular, the domain is $\mathcal{B}^{<\omega}$, and $\sim$ identifies tuples of the same length, so that $n \in \mathcal{A}$ can be represented by the $\sim$-class of tuples of length n. A relation R_i on $\mathcal{A}$ is represented by

$$\bigvee\!\!\!\!\bigvee_{(b_1,\ldots,b_m)\in R_i^{\mathcal{A}}} (|\vec{d}_1| = b_1 \ \& \cdots \& \ |\vec{d}_m| = b_m).$$

Since $R_i^{\mathcal{A}}$ is computable, both this and its negation are Σ_1^c formulas.

Given a computable functor, find the interpretation

We know that $\Phi_*^{\Delta(\widehat{\mathcal{B}})\oplus\text{id}\oplus\Delta(\widehat{\mathcal{B}})}$ is the identity map on $\Phi^{\Delta(\widehat{\mathcal{B}})}$.

Whenever we see σ, n, and i for which $\Phi_*^{\sigma\oplus(\text{id}\restriction n)\oplus\sigma}(i)\!\downarrow= i$, we know that σ, viewed as a possible initial segment of some $\Delta(\widehat{\mathcal{B}})$, is "enough information" for Φ_* to have recognized i. Now σ codes a particular configuration ζ_σ of elements $0, 1, \ldots, n$ of $\widehat{\mathcal{B}}$ (including i). So we define the domain $D \subseteq B^{<\omega} \times \omega$ to be the set of pairs $(\vec{b}, i)$ with

$$\Phi_*^{\Delta(\vec{b})\oplus(\text{id}\restriction|\vec{b}|)\oplus\Delta(\vec{b})}(i)\!\downarrow= i.$$

and define $(\vec{b}, i) \sim (\vec{c}, j)$ if $\vec{b} \cup \vec{c}$ can be extended to a finite tuple $\vec{d}$ for which some permutation τ of $\vec{d}$ has $\tau(b_i) = c_i$ and $\tau(\vec{c} - \vec{b}) = (\vec{b} - \vec{c})$ and

$$\Phi_*^{\Delta(\vec{d})\oplus\tau\oplus\Delta(\tau(\vec{d}))}(i)\!\downarrow= j \quad \& \quad \Phi_*^{\Delta(\tau(\vec{d}))\oplus\tau^{-1}\oplus\Delta(\vec{d})}(j)\!\downarrow= i.$$

Finishing the interpretation

Finally, for a unary relation R, we define $(\vec{b}, i) \in D$ to satisfy R iff there is some $(\vec{c}, j) \sim (\vec{b}, i)$ for which $\Phi^{\Delta(\vec{c})}$ halts and outputs 1 when we run it on (the code number of) the atomic formula $R(j)$.

All the formulas defining this interpretation are Σ_1^c, so the interpretation is effective.

Beyond effective interpretations

Question: what about more complicated interpretations?

Intepretations using Σ_2^c formulas can readily be viewed as functors into the *jump*. This continues to hold for Σ_α^c formulas, for $\alpha < \omega_1^{CK}$.

Defn. (various researchers), roughly stated

The *jump* $\mathcal{B}'$ of a countable structure $\mathcal{B}$ has the same domain as $\mathcal{B}$ and includes the same predicates, but also has a predicate for every Σ_1^c formula (with free variables $v_1, \ldots, v_n$) in the language of $\mathcal{B}$. That predicate holds of $\vec{b}$ in $\mathcal{B}'$ iff the formula holds of $\vec{b}$ in $\mathcal{B}$.

This includes predicates such as "the length of $\vec{b}$ lies in $\emptyset'$," which are not truly structural. We know $\text{Spec}(\mathcal{B}') = \{\boldsymbol{d}' : \boldsymbol{d} \in \text{Spec}(\mathcal{B})\}$.

What about noncomputable infinitary formulas?

Now we allow interpretations using arbitrary $L_{\omega_1\omega}$ formulas (and still using arbitrarily long finite tuples). It remains true that every such interpretation $\mathcal{I}$ of $\mathcal{A}$ in $\mathcal{B}$ yields a functor $F_{\mathcal{I}}$ from Iso($\mathcal{B}$) into Iso($\mathcal{A}$). If the formulas are Σ_1^∞ (but noncomputable), then the functor can still be expressed using Turing functionals, with $G(\widehat{\mathcal{B}}) = \Phi^{S\oplus\Delta(\widehat{\mathcal{B}})}$ and $H(f) = \Phi_*^{S\oplus\Delta(\widehat{\mathcal{B}})\oplus f\oplus\Delta(\widetilde{\mathcal{B}})}$, where S is a fixed oracle capable of enumerating those formulas. If the formulas are Σ_α^∞, then we need to use jumps of the structures.

Notice that with an extra oracle allowed, we could define α-th jumps even for countable ordinals $\geq \omega_1^{CK}$: just fix an oracle which can compute the ordinal you need!

Main theorem on infinitary interpretation

Theorem (HTM^2)

For each Baire-measurable functor $F : \mathrm{Iso}(\mathcal{B}) \to \mathrm{Iso}(\mathcal{A})$, there is an infinitary interpretation $\mathcal{I}$ of $\mathcal{A}$ within $\mathcal{B}$ such that F is naturally isomorphic to the functor $F_{\mathcal{I}}$. If F is Δ^0_α (in the lightface Borel hierarchy), then the interpretation can be done using Δ^c_α formulas, and the isomorphism between F and $F_{\mathcal{I}}$ can be taken to be Δ^0_α.

The proof uses a forcing notion, with $\mathcal{B}^* = \{$finite 1-1 tuples from $\mathcal{B}\}$, so that generics are bijections (by genericity) from ω onto $\mathcal{B}$. We want to build several mutually generic structures (and examine how F acts on the maps between them), so we use product forcing with $(\mathcal{B}^*)^k$. The forcing notion will be definable in $\mathcal{B}$ (at least, for a restricted sublanguage L'), yielding the formulas for the interpretation.

Forcing language

We want to force statements of the form $F(\mathcal{B}_{g_1}, g_2^{-1} \circ g_1, \mathcal{B}_{g_2})(i) = j$. (Here $\mathcal{B}_g$ is the pullback of $\mathcal{B}$ to the domain ω along $g : \omega \to \mathcal{B}$.)

Finitary formulas in the forcing language L and its restriction L':

- $\dot{g}_i^{-1} \circ \dot{g}_j(m) = n$ and its negation;
- $R^{\mathcal{B}_{\dot{g}_i}}(a_1, \ldots, a_n)$ and its negation, for $\vec{a} \in \omega^n$ and R an n-ary relation in the language of $\mathcal{B}$;
- finite conjunctions and disjunctions;
- $\dot{g}_i(m) = n$ and its negation. (These are *not* in L'!)

We then build L and L' by taking infinitary conjunctions and disjunctions.
Now F is a Borel functional, so $F(\mathcal{B}_g)$ computes its atomic diagram using infinitary conjunctions and disjunctions of statements from $\Delta(\mathcal{B}_g)$. So, for P in the signature of $\mathcal{A}$, $F(\mathcal{B}_g) \models P(\vec{j})$ is expressible in L', as is $F(\mathcal{B}_{g_1}, g_2^{-1} \circ g_1, \mathcal{B}_{g_2})(i) = j$, preserving complexities.

Definition of forcing

Let $p = (\overline{b}_1, \ldots, \overline{b}_k) \in (\mathcal{B}^*)^k$.

φ	$p \Vdash_{(\mathcal{B}^*)^k} \varphi$ iff...
$\dot{g}_i^{-1} \circ \dot{g}_j(m) = n$	$\overline{b}_i(n)\downarrow = \overline{b}_j(m)\downarrow$.
$\dot{g}_i^{-1} \circ \dot{g}_j(m) \neq n$	$\overline{b}_i(n)\downarrow \neq \overline{b}_j(m)\downarrow$ or $\exists m' \neq m\ \overline{b}_i(n)\downarrow = \overline{b}_j(m')\downarrow$ or $\exists n' \neq n\ \overline{b}_i(n')\downarrow = \overline{b}_j(m)\downarrow$.
$R^{\mathcal{B}_{\dot{g}_i}}(\vec{a})$	$\mathcal{B} \models R(\overline{b}_i(a_1)\downarrow, \ldots, \overline{b}_i(a_n)\downarrow)$.
$\neg R^{\mathcal{B}_{\dot{g}_i}}(\vec{a})$	$\mathcal{B} \models \neg R(\overline{b}_i(a_1)\downarrow, \ldots, \overline{b}_i(a_n)\downarrow)$.
$\dot{g}_i(m) = n$	$\overline{b}_i(m)\downarrow = n$.
$\dot{g}_i(m) \neq n$	$\overline{b}_i(m)\downarrow \neq n$ or $\exists m' \neq m\ \overline{b}_i(m')\downarrow = n$.
finite disjunction	p forces some disjunct.
finite conjunction	p forces all conjuncts.
$\bigvee_n \psi_n$	$\exists n$ for which $p \Vdash_{(\mathcal{B}^*)^k} \psi_n$.
$\bigwedge_n \psi_n$	$(\forall n)(\forall q \supseteq p)(\exists r \supseteq q)\ r \Vdash_{(\mathcal{B}^*)^k} \psi_n$.

Forcing Lemma

Say that $\mathbf{g} = (g_1, \ldots, g_k) \in (\mathcal{B}^*)^k$ is *S-generic* if the g_i are mutually $(S \oplus \mathcal{B})$-hyperarithmetically generic functions $\omega \to \mathcal{B}$. We say that $\varphi[\mathbf{g}]$ *holds* if φ becomes true when each g_i in $\mathbf{g}$ is substituted for $\dot{g}_i$ in φ.

Lemma

Let φ be an S-computable sentence of the forcing language for $(\mathcal{B}^*)^k$.

1. For S-generic $\mathbf{g}$, $\varphi[\mathbf{g}]$ holds iff, for some $p \subset \mathbf{g}$, $p \Vdash_{(\mathcal{B}^*)^k} \varphi$.
2. If φ starts with $\bigwedge$, then $p \Vdash_{(\mathcal{B}^*)^k} \varphi$ iff, for every S-generic $\mathbf{g} \supset p$, $\varphi[\mathbf{g}]$ holds.

The induction is mostly straightforward. Suppose φ is $\wedge_n \psi_n$ and some $p \subset \mathbf{g}$ forces φ. Now for each n, some $q \subset \mathbf{g}$ decides ψ_n. WLOG $p \subseteq q$, so some $r \supseteq q$ has $r \Vdash_{(\mathcal{B}^*)^k} \psi_n$, and so does q (since q decides ψ_n). By induction, then $\psi_n[\mathbf{g}]$ holds, and since this works for all n, $\varphi[\mathbf{g}]$ also holds.

The interpretation

We still need to produce our interpretation of $\mathcal{A}$ in $\mathcal{B}$. Its domain D contains those $(\overline{b}, i) \in \mathcal{B}^* \times \omega$ for which

$$(\overline{b}, \overline{b}) \Vdash_{(\mathcal{B}^*)^2} F(\mathcal{B}_{\dot{g}_1}, \dot{g}_2^{-1} \circ \dot{g}_1, \mathcal{B}_{\dot{g}_2})(i) = i.$$

We define $(\overline{b}, i) \sim (\overline{c}, j)$ iff:

$$(\overline{b}, \overline{c}) \Vdash_{(\mathcal{B}^*)^2} F(\mathcal{B}_{\dot{g}_1}, \dot{g}_2^{-1} \circ \dot{g}_1, \mathcal{B}_{\dot{g}_2})(i) = j.$$

If P (in the language of $\mathcal{A}$) has arity p, define the corresponding R in the interpretation to hold of $((\overline{b}_1, i_1), \ldots, (\overline{b}_p, i_p)) \in D^p$ iff:

$$(\exists \overline{c} \in \mathcal{B}^*)(\exists \vec{j} \in \omega^p) \left[\left(\bigwedge_{s \leq p} (\overline{b}_s, i_s) \sim (\overline{c}, j_s) \right) \& \left(\overline{c} \Vdash_{(\mathcal{B}^*)^1} \vec{j} \in P^{F(\mathcal{B}_{\dot{g}})} \right) \right].$$

Definability

To see that the formulas for the interpretation are $L_{\omega_1\omega}$ in the language of $\mathcal{B}$, one shows by induction that for each Σ^c_α formula φ in L', $\{p \in (\mathcal{B}^*)^k : p \Vdash_{(\mathcal{B}^*)^k} \varphi\}$ is Σ^c_α-definable in $\mathcal{B}$, and likewise for Π_α. (This relativizes easily to S-computable formulas.)

For finitary formulas, notice that $\{(\overline{b}, \overline{c}) : \overline{b}(n) = \overline{c}(m)\}$ is definable by atomic formulas in $\mathcal{B}$, as is $\{\overline{b} : \mathcal{B} \models R(\overline{b}(a_1), \ldots, \overline{b}(a_n))\}$.

If φ is $\bigvee_n \psi_n$, then $p \Vdash_{(\mathcal{B}^*)^k} \varphi$ iff some n has $p \Vdash_{(\mathcal{B}^*)^k} \psi_n$, which by induction is Π^c_β-definable in $\mathcal{B}$ with $\beta < \alpha$.

For $\bigwedge_n \psi_n$, one needs to know that for every p and φ, some $q \supseteq p$ decides φ, and that p cannot force both φ and $(\neg\varphi)$.

Now, if φ is $\bigwedge_n \psi_n$, then $p \Vdash_{(\mathcal{B}^*)^k} \varphi$ iff, for all $q \supseteq p$, $q \not\Vdash_{(\mathcal{B}^*)^k} (\neg\psi_n)$. This is Σ^c_β-definable in $\mathcal{B}$ for some $\beta < \alpha$, so $p \Vdash_{(\mathcal{B}^*)^k} \varphi$ is Π^c_α-definable.

$\mathcal{A} \cong (D/\sim, R_0, R_1, \ldots)$

We wish to define an isomorphism $\pi : \mathcal{A} \to D/\sim$ (where $\mathcal{A} = F(\mathcal{B})$). For this we use a generic $g : \omega \to \mathcal{B}$, which yields a map $\pi_g : F(\mathcal{B}_g) \to D/\sim$. The value $\pi_g(i)$ is the least tuple $(\overline{c}, i) \in D$ with $\overline{c} \subset g$ (which exists, by genericity). Then compose π_g with $F(\mathcal{B}, g^{-1}, \mathcal{B}_g)$, which maps $\mathcal{A} = F(\mathcal{B})$ to $F(\mathcal{B}_g)$, since $g^{-1} : \mathcal{B} \to \mathcal{B}_g$:

$$\mathcal{A} \longrightarrow F(\mathcal{B}_g) \longrightarrow D.$$

Of course, $F(\mathcal{B}, g^{-1}, \mathcal{B}_g)$ is known to be an isomorphism. The work here is to prove that π_g is an isomorphism, and the genericity of g is used heavily.

Finally, one shows that the composition $\pi_g \circ F(\mathcal{B}, g^{-1}, \mathcal{B}_g)$ is independent of the choice of the generic g.

Corollaries: automorphism groups

One can prove that, for every continuous homomorphism $h : \mathrm{Aut}(\mathcal{B}) \to \mathrm{Aut}(\mathcal{A})$, there is a Borel functor $G : \mathrm{Iso}(\mathcal{B}) \to \mathrm{Iso}(\mathcal{A})$ with $G(\mathcal{B}) = \mathcal{A}$, whose restriction to $\mathrm{Aut}(\mathcal{B})$ equals h.

Corollary (HTM^2)

Every continuous homomorphism $h : \mathrm{Aut}(\mathcal{B}) \to \mathrm{Aut}(\mathcal{A})$ is induced by an infinitary interpretation of $\mathcal{A}$ in $\mathcal{B}$.

There do exist discontinuous homomorphisms h, which clearly cannot arise from interpretations. (Cf. Evans-Hewitt, 1990.) However, every Baire-measurable homomorphism from $\mathrm{Aut}(\mathcal{B})$ into $\mathrm{Aut}(\mathcal{A})$ is continuous, hence induced by an interpretation.

Corollary (HTM^2)

Every continuous isomorphism $h : \mathrm{Aut}(\mathcal{B}) \to \mathrm{Aut}(\mathcal{A})$ arises from a Borel adjoint equivalence between the categories $\mathrm{Iso}(\mathcal{A})$ and $\mathrm{Iso}(\mathcal{B})$, and every such equivalence is induced by an infinitary bi-interpretation between $\mathcal{A}$ and $\mathcal{B}$.

Corollaries: indiscernibles

Theorem (HTM^2)

Let $\mathcal{A}$ be countable. Then TFAE:

1. There is a continuous homomorphism from Aut($\mathcal{A}$) onto S_ω (the permutation group of ω).
2. There is an n, an $L_{\omega_1\omega}$-definable $D \subseteq A^n$, and an $L_{\omega_1\omega}$-definable equivalence relation $E \subseteq D^2$ with infinitely many equivalence classes, such that these E-classes are absolutely indiscernible (i.e., every permutation of the E-classes extends to an automorphism of $\mathcal{A}$).

In addition, a continuous isomorphism between Aut($\mathcal{A}$) and S_ω exists iff every element of $\mathcal{A}$ is definable from the set of E-classes above. (That is, if we add one unary relation symbol to name each E-class, every element becomes $L_{\omega_1\omega}$-definable.)

Corollaries: order-indiscernibles

Analogous theorems hold for order-indiscernibles, with S_ω replaced by $\text{Aut}(\mathbb{Q}, <)$. $\mathcal{A}$ is not assumed to possess an order relation; the theorem proves the existence of an $L_{\omega_1\omega}$-definable dense order on the E-classes under which they are order-indiscernible.