Boolean Subalgebras of the Computable Atomless Boolean Algebra

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Low_n Boolean Algebras

Let $\mathcal{A}$ be a Boolean algebra, with domain ω. The *Turing degree* of $\mathcal{A}$ is the join of the degrees of the operations $\wedge$ and $\vee$ on $\mathcal{A}$ (equivalently, the join of the degrees of $\wedge$ and complementation on $\mathcal{A}$).

Theorems

- Every low Boolean algebra is isomorphic to a computable one. (Downey-Jockusch.)
- Every low_2 Boolean algebra is isomorphic to a computable one. (Thurber.)
- Every low_3 and low_4 Boolean algebra is isomorphic to a computable one. (Knight-Stob.)

It remains open whether this holds for low_5 Boolean algebras. By work of Harris and Montalbán, this problem is quantifiably more difficult.

Reframing the Question

Defns.

The *spectrum* of a countable structure $\mathcal{M}$ is the set

$$\text{Spec}(\mathcal{M}) = \{\text{deg}(\mathcal{N}) : \mathcal{N} \cong \mathcal{M} \ \& \ \text{dom}(\mathcal{N}) = \omega\}.$$

So the question asks whether the spectrum of a Boolean algebra $\mathcal{B}$ can contain a low_5 degree without also containing the degree $\mathbf{0}$. Boolean algebras are the only everyday class of structures for which it is known that the spectrum cannot contain a low_4 degree without also containing $\mathbf{0}$.

Spectra of Linear Orders

For linear orders, several questions about spectra are also open.

Question

Do there exist linear orders $\mathcal{L}_0$ and/or $\mathcal{L}_1$ with

- $\text{Spec}(\mathcal{L}_0) = \{\boldsymbol{d} : \boldsymbol{d} > \boldsymbol{0}\}$?
- $\text{Spec}(\mathcal{L}_1) = \{\boldsymbol{d} : \boldsymbol{d}' > \boldsymbol{0}'\}$?

That is, the spectrum of $\mathcal{L}_1$ should contain exactly the *nonlow* degrees. For each $n > 1$, there does exist a linear order $\mathcal{L}_n$ whose spectrum contains precisely the nonlow$_n$ degrees $\boldsymbol{d}$ (those with $\boldsymbol{d}^{(n)} > \boldsymbol{0}^{(n)}$).

An Approach for Linear Orders

For the linear order questions, we do have a result using a related notion of spectrum.

Defn.

Let R be a relation on a computable structure $\mathcal{M}$. The *spectrum of R* (as a relation on $\mathcal{M}$) is the set

$$\text{DgSp}_{\mathcal{M}}(R) = \{\deg(S) : \exists \text{ computable } \mathcal{N} \text{ with } (\mathcal{N}, S) \cong (\mathcal{M}, R)\}.$$

This measures the amount of information which can/must be coded into the relation R on $\mathcal{M}$. By restricting to computable structures $\mathcal{N}$, we mean to measure only the complexity intrinsic to R, without letting the complexity of the underlying structure confuse the issue.

Theorems on Linear Orders

The computable dense linear order $\mathbb{Q}$ is of interest since it is computably ultrahomogeneous and universal for countable linear orders: every countable LO embeds into $\mathbb{Q}$. Indeed, we have:

Theorem (Harizanov & M. 2007)

For every linear order $\mathcal{A}$, there exists an embedding $f : \mathcal{A} \hookrightarrow \mathbb{Q}$ with $\text{DgSp}_{\mathbb{Q}}(f(A)) = \text{Spec}(\mathcal{A})$.

They also asked whether, for every unary R on $\mathbb{Q}$, there exists a linear order $\mathcal{L}$ with $\text{Spec}(\mathcal{L}) = \text{DgSp}_{\mathbb{Q}}(R)$.

Theorems (FHKKM 2011)

There exist relations R and U on $\mathbb{Q}$ such that $\text{DgSp}_{\mathbb{Q}}(R)$ is not the spectrum of any linear order, and $\text{DgSp}_{\mathbb{Q}}(U) = \{\boldsymbol{d} : \boldsymbol{d}' > \boldsymbol{0}'\}$.

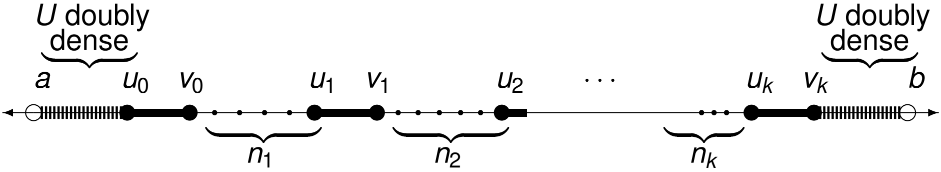
It is unknown whether $\{\boldsymbol{d} : \boldsymbol{d}' > \boldsymbol{0}'\}$ can be the spectrum of a LO.

Construction for the FHKKM Thm.

By a result of Wehner, for each set $C \subseteq \omega$, there exists a family $\mathbb{F}$ of finite sets such that for all D:

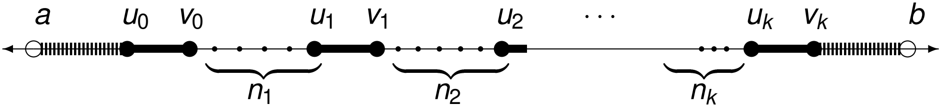
$$\mathbb{F} \text{ has an enumeration uniformly computable in } D \iff D >_T C.$$

For a single finite set $F = \{n_1, n_2, \ldots, n_k\}$, we code F into a relation $U = U_{F,a,b}$ on the interval $[a, b]$ of $\mathbb{Q}$:



"Doubly dense" means that both U and its complement are dense in that subinterval.

Enumerating F from a U' Oracle



With a U' oracle, but without knowing a or b, we can:

- Recognize an open interval with end points in U and interior $\subseteq \overline{U}$;
- Decide whether each end point is some u_i or v_i, or whether it is one of the n_i points inside (v_{i-1}, u_i);
- Eventually determine n_i and enumerate it into F;
- Having found u_i, identify v_i, provided $i < k$; and
- Having found v_{i-1}, identify u_{i-1}, provided $i > 1$.

We will not ever be able to identify u_0 or v_k, nor will we ever compute k. But the above is sufficient for us to *enumerate* F from our U'-oracle.

And for $([a, b], U) \cong ([\tilde{a}, \tilde{b}], \tilde{U})$, a $\tilde{U}'$-oracle will also enumerate F.

The Entire Relation U

The relation U on $\mathbb{Q}$ consists of the union of intervals of this form $U_{F_i,a,b}$, over densely many disjoint intervals $[a, b]$ whose union is $\mathbb{Q}$. For each F_i, we consider each possible order of its (finitely many) elements, and make sure that densely many intervals $[a, b]$ have $U_{F_i,a,b}$, with that order on F_i, attached to them.

Then, for any $(\tilde{\mathbb{Q}}, \tilde{U}) \cong (\mathbb{Q}, R)$, we can enumerate $\mathbb{F}$ uniformly in an oracle for $\tilde{U}'$. Hence $\tilde{U}' >_T C$. If $C = \emptyset'$, this means that $\tilde{U}$ is nonlow, so that $\text{DgSp}_{\mathbb{Q}}(U)$ contains only nonlow degrees.

All Nonlow Degrees Lie in $\text{DgSp}_{\mathbb{Q}}(U)$

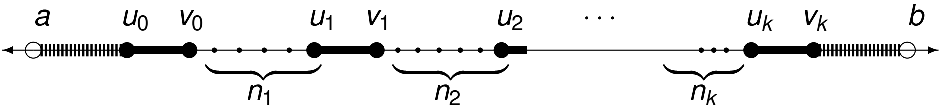
Conversely, if D has $D' >_T \emptyset'$, then D' can enumerate $\mathbb{F}$. So with a D-oracle, we use a D-computable approximation of D' to try to enumerate F and use it to build $\tilde{U}$ on $\mathbb{Q}$.

If the approximation changes, we can always take the finitely much of $\tilde{U}$ and its complement so far built and blend them into a future construction.

When eventually the approximation has converged on enough of D' to give an enumeration of some $F_i \in \mathbb{F}$ which never changes again, we wind up building the corresponding $U_{F_i,a,b}$ on densely many $[a, b] \subseteq \mathbb{Q}$. Using the density, we see that this $\tilde{U}$ has $\deg(\tilde{U}) \in \text{DgSp}_{\mathbb{Q}}(U)$ and also $\tilde{U} \leq_T D$. An easy coding makes $D \leq_T \tilde{U}$ too, and this completes the proof.

Back to Boolean Algebras

The key to the FHKKM theorem was the ambient structure $\mathbb{Q}$, and the notion of *double density*: both U and its complement can be dense in the same interval in $\mathbb{Q}$. We used this to divide up $\mathbb{Q}$ so as to enumerate $\mathbb{F}$ below U'. If we only had U itself, as a linear order, the doubly dense intervals $[a, u_0]$ and $v_k, b]$ would blend into $[u_0, v_0]$ and $[u_k, v_k]$, so U' would not enumerate $\mathbb{F}$.



Question

For Boolean algebras, with the computable atomless BA $\mathcal{B}$ as the ambient structure, can we get a Boolean subalgebra $\mathcal{A} \subseteq \mathcal{B}$ for which $\mathrm{DgSp}_{\mathcal{B}}(\mathcal{A})$ contains a low$_n$ degree, but not **0**?

Facts about Boolean Algebras

- The computable atomless Boolean algebra $\mathcal{B}$ is often represented as the BA of (finite unions of) intervals $[a, b)$ in $\mathbb{Q}$ under $\cup$ and $\cap$. (We allow $a = -\infty$ and/or $b = +\infty$.)
- This $\mathcal{B}$ is *spectrally universal* for BA's, just as $\mathbb{Q}$ is for linear orders. (Csima, Harizanov, M., Montalbán.)
- Using results of Jockusch & Soare, H&M showed that there exists a unary relation R on $\mathcal{B}$ whose spectrum contains a low degree, but not $\mathbf{0}$. However, this R is not a Boolean subalgebra. Montalbán asked whether the same can be done for a Boolean subalgebra.

Double Density and $\mathcal{A}$-atoms for Boolean Algebras

Defn.

Let $\mathcal{A}$ be a Boolean subalgebra of $\mathcal{B}$. $\mathcal{A}$ is *doubly dense* within $\mathcal{B}$ if, for every finite Boolean subalgebra $\mathcal{B}_0 \subseteq \mathcal{B}$, $(\mathcal{B}, \mathcal{A})$ realizes every possible finite extension of $(\mathcal{B}_0, \mathcal{A} \cap \mathcal{B}_0)$ to a larger Boolean algebra and Boolean subalgebra.

For a nonempty $x \in \mathcal{B}$, we say that $\mathcal{A}$ is *doubly dense within x* if $x \in \mathcal{A}$ and $\mathcal{A}_x = \{a \in \mathcal{A} : a \subseteq x\}$ is doubly dense within the induced atomless Boolean algebra $\mathcal{B}_x = \{y \in \mathcal{B} : y \subseteq x\}$.

Defn.

An $x \in \mathcal{B}$ is an *$\mathcal{A}$-atom* if $x \in \mathcal{A}$ and $x \neq \emptyset$ and $\mathcal{A}_x = \{\emptyset, x\}$.

It is $\Pi_1^{\mathcal{A}}$ whether a given $x \in \mathcal{B}$ is an $\mathcal{A}$-atom, and $\Pi_2^{\mathcal{A}}$ whether $\mathcal{A}$ is doubly dense within a given $x \in \mathcal{B}$.

Coding a Fourth Jump $C^{(4)}$ into $\mathcal{A}$

Now we build a specific Boolean subalgebra $\mathcal{A}$ of $\mathcal{B}$.
Let $C^{(4)} = \{n_0 < n_1 < n_2 < \cdots\}$. We first code n_0 into $\mathcal{A}$ as follows:

- Subdivide $[0,1)$ into subintervals $[0,\frac{1}{2}), [\frac{1}{2},\frac{3}{4}), \ldots$, and put all these subintervals (but *not* $[0,1)$ itself) into $\mathcal{A}$.
- Do the same with $[1,2)$, then $[2,3)$, up to $[2^{n_0}-1, 2^{n_0})$.
- Put $[0, 2^{n_0})$ into $\mathcal{A}$.
- Make $\mathcal{A}$ doubly dense within $[2^{n_0}, 2^{n_0}+1)$.
- Go on to n_1, putting $[2^{n_0}+1, 2^{n_0}+1+2^{n_1})$ into $\mathcal{A}$, etc.

We also make $\mathcal{A}$ doubly dense within $(-\infty, 0)$, and close $\mathcal{A}$ under complements and finite unions, so that $\mathcal{A}$ is a Boolean subalgebra of $\mathcal{B}$.

$\mathcal{A}$-suprema

Defn.

An element $x \in \mathcal{B}$ is an *$\mathcal{A}$-supremum* if x is the least upper bound in $\mathcal{B}$ of an infinite set of $\mathcal{A}$-atoms.
Such an x is a *single $\mathcal{A}$-supremum* if x is not the union of two disjoint $\mathcal{A}$-suprema.
Finally, $x \in \mathcal{B}$ is a *k-fold $\mathcal{A}$-supremum* if x is the union of k disjoint single $\mathcal{A}$-suprema.

The property of being a single $\mathcal{A}$-supremum is $\Pi_3^{\mathcal{A}}$: it holds iff:

- $\mathcal{A}$ is not doubly dense within any $y \subseteq x$; and
- x contains infinitely many $\mathcal{A}$-atoms; and
- every $\mathcal{A}$-atom a has either $a \subseteq x$ or $a \cap x = \emptyset$; and
- $(\forall y \in \mathcal{B})$[either $x \cap y$ or $x - y$ is contained in a finite union of $\mathcal{A}$-atoms].

So the property of being a k-fold $\mathcal{A}$-supremum is $\Sigma_4^{\mathcal{A}}$, uniformly in k.

Decoding $C^{(4)}$ from $\mathcal{A}$

The idea is that $n \in C^{(4)}$ iff $\mathcal{A}$ contains a 2^n-fold $\mathcal{A}$-supremum. This property is $\Sigma_4^{\mathcal{A}}$. Therefore, if C is not low_4, then $C^{(4)} \not\leq \emptyset^{(4)}$, and there can be no computable $\tilde{A} \subseteq \mathcal{B}$ with $(\mathcal{B}, \tilde{A}) \cong (\mathcal{B}, \mathcal{A})$.

We claim that, for every C, the process above builds a Boolean subalgebra $\mathcal{A}$ such that $\deg(C) \in \text{DgSp}_{\mathcal{B}}(\mathcal{A})$. By taking C to be low_5 but not low_4, this will prove:

Theorem (M., 2011)

There exists a Boolean subalgebra $\mathcal{A}$ of the computable atomless BA $\mathcal{B}$ such that $\text{DgSp}_{\mathcal{B}}(\mathcal{A})$ contains a low_5 degree, but not $\mathbf{0}$.

Translating C for Coding

To show $\deg(C) \in \mathrm{DgSp}_{\mathcal{B}}(\mathcal{A})$, we build a Boolean subalgebra $\mathcal{D} \equiv_T C$ with $(\mathcal{B}, \mathcal{D}) \cong (\mathcal{B}, \mathcal{A})$. To begin with, we choose a computable function f such that

$$\forall n \left[n \in C^{(4)} \iff (\exists a \forall b) f(n, a, b) \in \mathrm{Fin}^C \right].$$

and for which $\forall n \exists^{\leq 1} a \forall b\, f(n, a, b) \in \mathrm{Fin}^C$.

For each n and a, we choose a distinct interval $I_{n,a}$ within $[0, +\infty)$ in $\mathcal{B}$. These are all separate, going out to $+\infty$, and between one I-interval and the next, we make $\mathcal{D}$ doubly dense. Every I-interval is placed into $\mathcal{D}$. Also, $(-\infty, 0)$ is placed into $\mathcal{D}$, and we make $\mathcal{D}$ doubly dense there. (An easy coding on this part, using our C-oracle, also ensures that $C \leq_T \mathcal{D}$.)

Building the Subalgebra $\mathcal{D} \leq_T C$

$I_{n,a}$ is partitioned into 2^n distinct intervals, each of which stays out of $\mathcal{D}$ and is further partitioned into ω-many intervals $J_0, J_1, \ldots$. Every J_i in each one goes into $\mathcal{D}$.
Whenever $f(n, a, b)$ "gets a chip," we satisfy the next requirement for double-density of $\mathcal{D}$ within $J_b \cup J_{b+1} \cup \cdots$, in each of the 2^n-many distinct intervals within $I_{n,a}$.

- If $n \in C^{(4)}$, then $\exists! a \forall b\, f(n, a, b) \in \text{Fin}^C$. For that a, each of the 2^n intervals is a single $\mathcal{D}$-supremum (not in $\mathcal{D}$), and their union, which lies in $\mathcal{D}$, is a 2^n-fold $\mathcal{D}$-supremum. For each other a, some b has $f(n, a, b) \notin \text{Fin}^C$, and all double-density requirements are satisfied for the union of cofinitely many of the $\mathcal{D}$-atoms in $I_{n,a}$.
- Likewise, if $n \notin C^{(4)}$, then for all a, some b has $f(n, a, b) \notin \text{Fin}^C$, and again, $I_{n,a}$ is the union of finitely many $\mathcal{D}$-atoms and one interval in which $\mathcal{D}$ is doubly dense.

Getting $(\mathcal{B}, \mathcal{D}) \cong (\mathcal{B}, \mathcal{A})$

From the above, it is clear that $\mathcal{D}$ contains k-fold $\mathcal{D}$-suprema for exactly the same k for which $\mathcal{A}$ contained k-fold $\mathcal{A}$-suprema, namely those $k = 2^n$ with $n \in C^{(4)}$. So these $\mathcal{A}$-suprema and $\mathcal{D}$-suprema may be paired up, for all $n \in C^{(4)}$ and also for all other k. we do make sure to leave over an infinite supply of nonadjacent $\mathcal{A}$-atoms.

Those $n \notin C^{(4)}$ each left finitely many $\mathcal{D}$-atoms in their $I_{n,a}$-intervals. We pair these with the leftover $\mathcal{A}$-atoms above.

We pair up $(-\infty, 0)$ with itself, since it is both $\mathcal{A}$-doubly dense and $\mathcal{D}$-doubly dense.

There remain infinitely many nonadjacent intervals in which $\mathcal{A}$ is doubly dense, and infinitely many in which $\mathcal{D}$ is. Pairing these up completes our isomorphism.

$\mathcal{A}$ as a Boolean Algebra

Just as with linear orders, this construction used the ambient structure $\mathcal{B}$ in an essential way. If we regard $\mathcal{A}$ as a BA in its own right, then all k-fold $\mathcal{A}$-suprema turn into single $\mathcal{A}$-suprema, and the coding of $C^{(4)}$ vanishes. Indeed, this $\mathcal{A}$ has a computable copy. So the question remains:

Question

Does there exist a Boolean algebra whose spectrum contains a low_5 degree, but does not contain $\mathbf{0}$?

Further Questions

Another question is the subject of current work by R. Steiner:

Question

Do all Boolean subalgebras $\mathcal{A} \subseteq \mathcal{B}$ for which $\text{DgSp}_{\mathcal{B}}(\mathcal{A})$ contains a low_4 degree also have computable copies? If not, then how about low_3, low_2, and low?

A negative answer to either question would give an example of a set of Turing degrees which is the spectrum of a Boolean subalgebra of $\mathcal{B}$, but not of any Boolean algebra (as a structure), and would thus prove that for BA's, the ambient structure does enable extra information content. For BA's, it remains open whether this is possible. For LO's, the ambient structure $\mathbb{Q}$ does allow extra information to be coded, but for graphs, the random graph as ambient structure does not allow any information which could not already have been coded into some countable graph.

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