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BY
RUSSELL GEDDES MILLER

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To all my friends at International House of Chicago.

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ABSTRACT

In Chapter 1, we consider the spectrum of a linear order. Slaman and Wehner have constructed structures which distinguish the computable Turing degree $\mathbf{0}$ from the noncomputable degrees, in the sense that the spectrum of each structure consists precisely of the noncomputable degrees. Downey has asked if this can be done for an ordinary type of structure such as a linear order. We show that there exists a linear order whose spectrum includes every noncomputable Δ_2^0 degree, but not $\mathbf{0}$.

In Chapter 2, we define a property $R(A_0, A_1)$ in the partial order $\mathcal{E}$ of computably enumerable sets under inclusion, and prove that R implies that A_0 is noncomputable and incomplete. Moreover, the property is nonvacuous, and the A_0 and A_1 which we build satisfying R form a Friedberg splitting of their union A, with A_1 prompt and A promptly simple. We conclude that A_0 and A_1 lie in distinct orbits under automorphisms of $\mathcal{E}$, yielding a strong answer to a question previously explored by Downey, Stob, and Soare about whether halves of Friedberg splittings must lie in the same orbit.

In Chapter 3, we prove that no computable tree of height ω is computably categorical, and indeed that all such trees have computable dimension ω. The necessary construction requires us to prove several new versions of Kruskal's Lemma on the embeddability of finite trees.

In Chapter 4, given an arbitrary low c.e. set A and an arbitrary noncomputable c.e. set C, we use the New Extension Theorem of Soare to construct an automorphism of $\mathcal{E}$ mapping A to a set B such that $C \not\leq_T B$. Thus, the orbit in $\mathcal{E}$ of the low set A cannot be contained in the upper cone above C. This complements a result of Harrington, who showed that the orbit of a noncomputable c.e. set cannot be contained in the lower cone below any incomplete c.e. set.

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Acknowledgements in a dissertation characteristically form a tree. The author starts with his family and his advisor at the first level, and proceeds upwards by induction, with each acknowledgement at level n reminding him of several more people to put at level $n+1$.

The main reason why this is so is the cumulative nature of a mathematical education, and this reason applies to me as much as to anyone else. Clearly my advisor Robert Soare and my parents Jack and Shirley Miller have been essential to the creation of this dissertation. As soon as I think of Prof. Soare and all his hard work and advice, though, I remember advisors and mentors from other stages: Paul Sally, whose dedication to all aspects of university mathematics continues to inspire me; Carlos Simpson, my undergraduate thesis advisor; Robert Gunning, who introduced me to college-level mathematics; and Elaine Genkins, my calculus teacher at Collegiate School. Another branch from Prof. Soare leads to the many other computability theorists who contributed advice and guidance to this dissertation: Peter Cholak, Leo Harrington, Carl Jockusch, Bakhadyr Khoussainov, Julia Knight, Steffen Lempp, Andre Nies, and Kevin Wald. Then from each of those nodes, other names come to mind, and the number at each level increases exponentially, so that they become too many to list.

The unusual feature of my own acknowledgement-tree is that the branch through my family also leads to mathematicians. My father was my earliest math teacher, and kept at the job all the way into college. Hence, at the succeeding nodes we find not only other family members and friends, including those to whom this dissertation is dedicated, but also some of the other mathematical influences from that period. Ultimately, that branch includes Robin Kalder and then Elaine Genkins again. Thus my acknowledgement-tree is not a tree at all, but only a partial order, since the set of predecessors of Dr. Genkins is not linearly ordered.

Whether the partial order is finite or infinite is a philosophical question into which I do not wish to delve, but clearly it is far more extensive than can be described here. Let me simply make clear that I am much indebted and extremely grateful to all these individuals, and that I hope to emulate the examples of generosity and wisdom which they have set.

INTRODUCTION

Computability theory is the study of finite algorithms and the mathematical problems which such algorithms can and cannot solve. This dissertation focuses on two areas within the general field of computability theory. Chapters 1 and 3 give solutions to two problems in the area of computable model theory, while Chapters 2 and 4 resolve two questions about automorphisms of the lattice of computably enumerable sets.

The four main branches of mathematical logic are set theory, computability theory, model theory, and proof theory. Computable model theory applies the principles of the second branch to the third. To grasp the reasons for hybridizing these two, it is necessary to understand the basic principles of each.

The *Turing machine*, the basic tool of computability theory, is essentially an idealized computer. It has a two-way infinite memory tape, which is assumed to be blank at the start of its operation except for a finite string of 1's giving the value of the input, and it executes a finitely-defined algorithm. If it ever reaches the instruction "halt," it ceases operating, and its output is considered to be the number of 1's written on the tape at this stage. *Church's Thesis*, which is widely accepted, is the claim that the tasks which can be performed by a Turing machine are precisely the functions which can be computed by a human being using pencil and (unlimited) paper. (Here all functions have subsets of ω for their domain and range.)

A set S of natural numbers is said to be *computable* (or *recursive*) if there is a Turing machine such that for any input n, the machine outputs 1 if n is in S and 0 if not. A weaker condition is computable enumerability: S is *computably enumerable* if there is a Turing machine which, given inputs $0, 1, 2, \ldots$ outputs a list of all the elements of S. It is easily seen that S is computable if and only if both S and its complement $\overline{S}$ are computably enumerable.

More generally, a set A is *computable in a set* B, or *Turing-computable in* B, written $A \leq_T B$, if there is a relativized Turing machine which can compute, using a B-oracle, whether or not any given input is in A. (A relativized Turing machine has a read-only tape containing a countable binary sequence. We think of the information on the tape as coding a set, in this case B, and we refer to the read-only tape as a B-oracle, since the machine can ask the tape at any time whether or not any natural number n is in B.) This gives a partial ordering of all subsets of the natural numbers, and we make it a strict partial ordering by declaring two sets to be of the same *Turing degree* if each is computable in the other. The computable sets are precisely those which are computable in the empty set, so $\deg(\emptyset)$ is the least element. We write $\mathbf{0} = \deg(\emptyset)$, since sets in this degree have no "information content;" the information contained in a computable set is information that we could compute for ourselves if we desired. In the same vein, if the Turing degree $\mathbf{A}$ is computable in the degree $\mathbf{B}$, we think of B as containing more information than A. There are 2^ω-many Turing degrees – that is, the same cardinality as that of $\mathbb{R}$ – and the Turing-computability partial order on them is extremely complex and remains a fertile source of questions in computability theory.

Turning to model theory, we take the linear order as a standard example of an algebraic structure. A linear order $\mathcal{L}$ consists of a set S and a binary relation $<$ on S, satisfying the axioms for a linear order. $\mathcal{L}$ is computable (resp. $\mathbf{B}$-computable) if both S and $<$ are computable (resp. $\mathbf{B}$-computable). The *degree* of the ordering is $\mathbf{B}$ if the ordering is computable in $\mathbf{B}$ but not in any degree below $\mathbf{B}$. A similar, more general definition applies to any model-theoretic structure. The degree is always the supremum of the degree of the universe and the degrees of the functions and relations used in the structure. For a group G, for instance, one takes the supremum of the degree of the underlying set and the degrees of the functions representing multiplication and inversion on this set.

A linear ordering may have one isomorphic copy which is $\mathbf{B}$-computable and another which is not. For instance, the standard ordering of ω is computable and isomorphic to the standard ordering of any infinite subset S of ω, but the degree of the latter is the degree of S. The *spectrum* of the linear order is the set of all those Turing degrees $\mathbf{B}$ such that there exists an isomorphic copy of the order whose degree is $\mathbf{B}$.

The most general question that one asks about spectra is simply which ones are possible, for a given type of structure. For instance, what are the possible spectra of countable linear orders? Julia Knight [28] has shown that the spectrum of a linear order must be closed upwards under the Turing-computability relation, and Linda Jean Richter [40] has shown that if such a spectrum has a least degree, then that degree must be $\mathbf{0}$. Interest has focussed on the *Separation Question*, which asks (in its general form) whether there exists a linear order whose spectrum contains a given collection $\mathbf{P}$ of Turing degrees while excluding entirely another such collection $\mathbf{N}$. (From Knight's result it is clear that no degree in $\mathbf{P}$ may be computable in any degree in $\mathbf{N}$.)

As a specific example, we ask whether there exists a linear order whose spectrum is precisely the noncomputable Turing degrees – that is, every Turing degree except the degree of the empty set. Such an order $\mathcal{L}$ would then allow one to characterize computability in terms of $\mathcal{L}$: a set A is computable if and only if it is impossible to compute a copy of $\mathcal{L}$ from A. Thus, in terms of information content, $\mathcal{L}$ would constitute information common to all noncomputable sets yet unknown to any computable set.

It is known (see [45], [50]) that there are structures more complicated than linear orders whose spectra are the noncomputable Turing degrees. On the other hand, for Boolean algebras, it is impossible for the spectrum to contain certain noncomputable Turing degrees without also containing the computable degree. (See [9], [49], [29].) In Chapter 1, we prove that there is a linear order whose spectrum contains every noncomputable Δ^0_2 degree, yet excludes $\mathbf{0}$. (A set is Δ^0_2 if there exists a computable approximation to it. This is a weaker condition than computable enumerability, since any computable enumeration is also a computable approximation.) This is significant in that it distinguishes the linear order, as a model-theoretic structure, from both the set and the Boolean algebra. (The model-theoretic structure of a set is given in a language with one constant c_0, one unary function s, and one unary relation C, with axioms stating that s is one-to-one and that c_0 is the only element not in the range of s. The set modelled is just the set of those $n \in \omega$ for which $C(s^n(c_0))$ holds.)

These results suggest a possible approach for measuring the complexity of model-theoretic structures. The set appears to be too simple a structure to separate the noncomputable sets from the computable ones; the possible spectra of sets are mainly the so-called upper cones $\{\mathbf{D} : \mathbf{C} \leq_{\mathbf{T}} \mathbf{D}\}$ of degrees in which a given set C is computable. Adding the structure of a linear order allows us at least to separate the noncomputable Δ^0_2 degrees from $\mathbf{0}$, but simultaneously removes the posssibility of getting a nontrivial upper cone as a spectrum (by Richter's result).

Two questions present themselves as we observe this trade-off. First, we ask to what extent similar trade-offs hold for other common mathematical structures. Can we use these trade-offs to rank such structures by their complexity? Second, we ask if we can define more exotic structures which achieve the complexity of the linear order without losing the simplicity of the set. Is there a model-theoretic structure for which every collection of Turing degrees (or at least every upward-closed collection) is a possible spectrum? Or is it the case that one cannot gain complexity in a structure without sacrificing simplicity?

CHAPTER 1

THE Δ_2^0-SPECTRUM OF A LINEAR ORDER

1.1 Introduction

Definition 1.1.1 The *spectrum* $\mathrm{Spec}(\mathcal{A})$ of a structure $\mathcal{A}$ is the class of Turing degrees of presentations of $\mathcal{A}$,

$$\mathrm{Spec}(\mathcal{A}) = \{\deg(\mathcal{B}) : \mathcal{B} \cong \mathcal{A}\}.$$

(Here the degree of a structure $\mathcal{B}$ is the supremum of the degree of its universe and the degree of its open diagram. For our purposes, the universe will generally be ω.)

Slaman [45] and Wehner [50] have recently each constructed a countable first-order structure $\mathcal{A}$ such that $\mathrm{Spec}(\mathcal{A}) = \mathbf{D} - \{\mathbf{0}\}$, where $\mathbf{D}$ is the class of all Turing degrees and $\mathbf{0}$ is the degree of the computable sets. This answers a question in [7] from Lempp, who had asked whether it was possible to distinguish the noncomputable degrees from the degree $\mathbf{0}$ in such a way. Slaman remarks that the open diagram of each of these models contains information which is common to all noncomputable real numbers, yet which is not itself computable. (In contrast, a single subset of ω with no algebraic structure cannot contain such information; the existence of a minimal pair of Turing degrees ensures that any set which is computable in every noncomputable real must itself be computable.)

The structures constructed by Slaman and Wehner were built specifically for this purpose and are not readily recognizable to most mathematicians. Downey [7] has asked whether one could do the same for better-known types of mathematical objects, particularly for linear orders. Indeed, he posed a series of questions:

Question 1.1.2 (Downey) *Is there a linear order whose spectrum contains every computably enumerable Turing degree except* $\mathbf{0}$*?*

Question 1.1.3 (Downey) *Is there a linear order whose spectrum contains every* Δ^0_2 *degree except* $\mathbf{0}$*?*

Question 1.1.4 (Downey) *Is there a linear order whose spectrum contains every degree except* $\mathbf{0}$*?*

We can rephrase these questions using the following terminology.

Definition 1.1.5 If $\mathbf{C}$ is a class of Turing degrees, the $\mathbf{C}$-*spectrum* of $\mathcal{A}$, written $\mathrm{Spec}^{\mathbf{C}}(\mathcal{A})$, is the intersection of $\mathbf{C}$ with $\mathrm{Spec}(\mathcal{A})$.

We will consider Σ^0_1 and Δ^0_2 as classes of degrees, not classes of sets. Thus, Question 1.1.2 asks whether the Σ^0_1-spectrum of a linear order $\mathcal{A}$ can be precisely the noncomputable Σ^0_1 degrees, and Questions 1.1.3 and 1.1.4 are the corresponding questions for $\mathrm{Spec}^{\Delta^0_2}(\mathcal{A})$ and $\mathrm{Spec}(\mathcal{A})$.

For certain common mathematical structures, the answers to such questions are negative. For instance, Downey and Jockusch have shown in [9] that any Boolean algebra $\mathcal{B}$ of low degree is isomorphic to a computable Boolean algebra,

$$\mathrm{Spec}(\mathcal{B}) \cap \mathbf{L_1} \neq \emptyset \implies \mathbf{0} \in \mathrm{Spec}(\mathcal{B}).$$

Hence the Σ^0_1-spectrum of a Boolean algebra cannot contain every noncomputable computably enumerable (c.e.) degree without also containing $\mathbf{0}$. (This result was extended to the low_2 degrees by Thurber [49] and then as far as the low_4 degrees by Knight and Stob [29], who proved that any Boolean algebra of low_4 degree is isomorphic to a computable Boolean algebra.)

However, it is known that for every noncomputable Turing degree, there exists a linear order of that degree which is not isomorphic to any computable linear order. Jockusch and Soare [25] proved this statement for noncomputable c.e. degrees, by creating a linear order which could be "separated" into countably many components, which are used to diagonalize against all possible computable linear orders. Later, Downey and Seetapun (both unpublished) independently extended this result to the noncomputable Δ^0_2 degrees. Finally, Knight proved the result for an arbitrary noncomputable Turing degree (see [7], p. 179), suggesting that a positive answer to Downey's most general question might be possible.

The argument by Jockusch and Soare is uniform in the given noncomputable c.e. set C in whose degree we wish to build a linear order with no computable copy. It does give different results, namely non-isomorphic linear orders, for different sets

C. The same is true of Downey and Seetapun's results, which use the same basic module. Therefore these results do not answer any of Downey's questions.

In this chapter we modify the Jockusch-Soare basic module so that for any two noncomputable c.e. sets C and D, it produces isomorphic copies of the same linear order. Also, we modify and develop the method of Δ^0_2-permitting so that the basic module can handle any noncomputable Δ^0_2 set C, while still producing isomorphic linear orders regardless of the choice of C. We use this new basic module in Section 1.4 to prove:

Theorem 1.1.6 *There exists a linear order $\mathcal{A}$ which has a copy in every noncomputable Δ^0_2 degree, but no computable copy,*

$$\mathrm{Spec}^{\Delta^0_2}(\mathcal{A}) = \Delta^0_2 - \{\mathbf{0}\}.$$

Furthermore, this order may be taken to be of the form

$$\mathcal{A} = \sum_{i\in\omega}(\mathcal{S}_i + \mathcal{A}_i),$$

where each $\mathcal{S}_i \cong 1+\nu+i+\nu+1$ and each $\mathcal{A}_i$ is either ω or of the form $c_i+\omega^*+\omega$ for some $c_i \in \omega$.

(Here ν represents the countable dense linear order with end points.)

This answers Downey's Questions 1.1.2 and 1.1.3. Question 1.1.4 is still open, and is discussed in the final section.

Although the method of Δ^0_2-*permitting* has been occasionally used in computability theory, the literature on it is far less complete than the literature on Σ^0_1-permitting. Perhaps the most useful reference for Δ^0_2-permitting has been the twenty-year-old paper of Posner [37]. Therefore, we devote Section 1.2 to a revision, updating, and expansion of Posner's presentation. This includes an explanation of the intuition behind the method, with examples, and a general lemma, omitted from Posner's paper, explaining why one must receive permission infinitely often.

The rest of the chapter serves the dual purpose of answering Downey's question and providing a full example of Δ^0_2-permitting. In Section 1.3 we give the basic module for the construction, with Δ^0_2-permitting prominently used and explained, and in Section 1.4 we present the complete construction.

We use the notation of Soare [47] regarding Turing degrees and computability, and that of Rosenstein [43] for linear orders. (Thus ω^* represents the reverse order of ω, i.e. the order type of the negative integers.) When $\{C_s : s \in \omega\}$ is a computable approximation for a set C, we will usually just write $\langle C_s \rangle$ to stand for the entire approximation. Also, we use the symbol $S\upharpoonright\!\!\upharpoonright x$ to denote $S\upharpoonright(x+1)$, the restriction of the subset $S \subseteq \omega$ (viewed as a function) to the elements $0, 1, \ldots x$.

1.2 Δ^0_2 Permitting

Δ^0_2 permitting is not as transparent as c.e. permitting. Posner [37] has succinctly outlined the differences, as well as the tree approach we use to overcome them. In the c.e. case, we can be sure at least that every element that has entered the permitting set C will stay there; for a Δ^0_2 set C, there is no such guarantee for any element. Let $\{C_s\}_{s\in\omega}$ be a computable approximation of the permitting set, and suppose A is the C-computable set we wish to build. The permitting condition is actually the same for both the c.e. case and the Δ^0_2 case, and suffices to ensure that $A \leq_T C$:

Requirement 1.2.1 (Permitting Condition) *If $C_s \upharpoonright m = C_t \upharpoonright m$ and $m \leq \min(s,t)$, then $A_s \upharpoonright m = A_t \upharpoonright m$.*

However, for a c.e. permitting set C, we know that permission, once given, will never be withdrawn. That is, if $C_s \upharpoonright m \neq C_{s+1} \upharpoonright m$, then we must also have $C_s \upharpoonright m \neq C_t \upharpoonright m$ for every $t > s$, and therefore we never again have to worry about making $A_t \upharpoonright m$ equal to $A_s \upharpoonright m$. In the Δ^0_2 case, on the other hand, it is perfectly possible to have $C_s \upharpoonright m \neq C_{s+1} \upharpoonright m$ and $C_s \upharpoonright m = C_t \upharpoonright m$ for some $t > s+1$. If so, we must undo everything we have done to $A \upharpoonright m$ since stage s and ensure that $A_t \upharpoonright m = A_s \upharpoonright m$.

The easiest way to visualize our solution to this difficulty is by use of a tree, called the approximation-tree for C, which we define below after setting up some machinery. For $s > 0$, let:

$$x_s = \max\{x : (\exists t < s)[x \leq t \ \&\ C_s \upharpoonright x = C_t \upharpoonright x]\},$$

$$t_s = \min\{t : x_s \leq t < s \ \&\ C_s \upharpoonright x_s = C_t \upharpoonright x_s\}.$$

Thus x_s is the greatest length of agreement of C_s with any preceding stage, and t_s is that preceding stage (or the first such stage, if there is more than one). Notice that we always have $x_s \leq t_s$. (The requirement $x \leq t$ in the definition of x_s

averts the possibility of x_s being infinite, if there should be a stage $t < s$ such that $C_t = C_s$.)

The *approximation tree* $T(\{C_s\})$ for C is a computable tree with an integer at each node. The top node of this tree is 0, and each integer s is added to the tree as an immediate successor of t_s. The precise definition of the approximation tree is as follows.

$$T(\{C_s\}) = \{\sigma \in \omega^{<\omega} : \sigma(0) = 0 \; \& \; (\forall n < (lh(\sigma) - 1)) \; [\sigma(n) = t_{\sigma(n+1)}]\}.$$

(Clearly this depends on the choice of approximation $\{C_s\}$, not just on C.)

For instance, one possible set of approximations is given up to stage 8 in Figure 1.2, along with the corresponding approximation tree restricted to those stages. (It is convenient to write a dash in place of the 0 or 1 for each $C_s(y)$ with $y \geq s$, since this makes it clear when the requirement $x_s \leq t_s$ comes into play. If desired, we could easily ensure that $C_s(y) = 0$ for all $y \geq s$ and still have $\{C_s\}$ be a computable approximation of C.)

Lemma 1.2.2 *If the node t precedes the node s on the approximation-tree, then $x_t < x_s$ and $C_t\restriction x_t = C_s\restriction x_t$.*

Proof. We induct on the number of levels between s and t. If t immediately precedes s, then $t = t_s$, so $C_t \restriction x_s = C_s \restriction x_s$. Now we must have $x_t < x_s$, since otherwise C_s would agree up to x_s with a stage preceding t, contradicting the definition of t_s. Hence $C_t\restriction x_t = C_s\restriction x_t$.

For the inductive step, we simply note that $C_s \restriction x_s = C_{t_s} \restriction x_s$ and apply the inductive hypothesis to t_s. (Once again we have $x_s > x_{t_s} > x_t$.) ■

We now introduce the notion of a *true stage* for the approximation $\{C_s\}$. A true stage for this approximation is a stage s such that the length of agreement of C_s with C is greater than the corresponding length of agreement for every preceding stage,

$$(\exists x \leq s) \; [C_s\restriction x = C\restriction x \; \& \; (\forall t) \; [x \leq t < s \; \Rightarrow \; C_t\restriction x \neq C\restriction x \;]] \, .$$

Figure 1.1: Example of an Approximation Tree

s	$C_s(0)$	$C_s(1)$	$C_s(2)$	$C_s(3)$	$C_s(4)$	$C_s(5)$	$C_s(6)$	$C_s(7)$	t_s	x_s
0	–	–	–	–	–	–	–	–		
1	1	–	–	–	–	–	–	–	0	0
2	0	1	–	–	–	–	–	–	0	0
3	0	1	1	–	–	–	–	–	2	2
4	1	1	1	0	–	–	–	–	1	1
5	1	0	1	0	0	–	–	–	1	1
6	1	0	1	1	0	0	–	–	5	3
7	1	0	1	1	1	0	0	–	6	4
8	0	0	1	1	1	0	0	0	2	1

Approximation Tree

(For our purposes, the "length of agreement" is bounded by the stage number. Thus, we need not worry about stages t with $t < x$.)

For c.e. sets, the true stages are precisely the *nondeficiency stages*, as defined by Dekker [6], namely those such that an element a enters the set at that stage and no element less than a ever enters at any subsequent stage.

Clearly, if s is a true stage, then t_s is precisely the previous true stage. The true stages form an infinite path through the tree, indeed the only infinite path. If this path were computable, then we could compute C. (Notice, however, that the tree need not be computably bounded, so one cannot automatically compute the unique infinite path.)

Ultimately, we only need to know A_s for the true stages s. After all, there are infinitely many true stages, and the Permitting Condition (and the convergence of $\lim_s C_s$) forces $\lim_s A_s$ to converge, so any infinite increasing subsequence $\{A_{s_i} : i \in \omega\}$ of approximations must converge to A as well. Moreover, if s is a true stage, we know that $A_s \restriction x_s = A \restriction x_s$.

The difficulty, of course, is that it is impossible to compute the sequence of true stages, given that C is noncomputable. Our general strategy for Δ^0_2-permitting is to assume at each stage s that the node s lies on the unique infinite path through the tree, i.e. that s is a true stage. We ensure that $A_s \restriction x_s = A_{t_s} \restriction x_s$, thereby satisfying the Permitting Condition for s and all stages preceding it. If it turns out that s is not a true stage, then at some subsequent true stage we will have the opportunity to undo the injury done at stage s to the preceding true stages.

For a c.e. permitting set C, one characteristically uses the noncomputability of C to prove that there will be infinitely many stages at which C "gives permission" to make a change to A. The analogous result for a Δ^0_2 set C is as follows.

Lemma 1.2.3 (Δ^0_2 Permission) *Let $s_0 = 0, s_1, s_2, \ldots$ be the true stages of a computable approximation $\langle C_s \rangle_{s\in\omega}$ of C, with $s_i < s_{i+1}$ for all i. Let $\langle n_s \rangle_{s\in\omega}$ be a non-decreasing unbounded computable sequence. If $\{q : n_{(s_q)} > x_{(s_q)}\}$ is finite, then C is computable.*

(Notice that we conclude that permission is given at infinitely many true stages, not merely at infinitely many stages. Again, the true stages are the stages which we care about for purposes of computing A from a C-oracle.)

Proof. Suppose that there were a number k' such that for all true stages $s_q \geq k'$, we have $x_{s_q} \geq n_{s_q}$. Since $\lim_s n_s = \infty$, we can compute for each stage s the least stage t such that $n_t > s$. Define $g(s)$ to be this stage t, so the function g is computable and total and $n_{g(s)} > s$ for every s.

Let $s_q \geq k'$ be a true stage. Then $s_q = t_{s_{(q+1)}} \geq x_{s_{(q+1)}} \geq n_{s_{(q+1)}}$ (since $s_{q+1} \geq k'$). But $n_{g(s_q)} > s_q$ by definition of g, so $n_{g(s_q)} > n_{s_{(q+1)}}$. Since $\langle n_s \rangle$ is a nondecreasing sequence, we see that $g(s_q) > s_{q+1}$. This holds as long as $s_q \geq k'$, but in fact we could redefine g at the finitely many true stages below k', to yield the following:

Sublemma 1.2.4 *Under the hypotheses of Lemma 1.2.3, there exists a computable function g such that for every true stage s_q we have $s_{q+1} < g(s_q)$.* ■

We remark that this function g does not provide a computable bound on the approximation-tree $T(\{C_s\})$. It is possible that there is a stage s with an immediate successor t such that $t > g(s)$. Sublemma 1.2.4 simply asserts that in this case t cannot be a true stage.

However, this information suffices for us to compute the path of true stages in $T(\{C_s\})$. 0 is always a true stage, of course, and knowing the true stage s_q, we find all immediate successors of s_q which are less than $g(s_q)$. Say that these are $t_0, t_1, \ldots t_p$. One of these must be the next true stage s_{q+1}, and all the others have only finitely many nodes below them (by Konig's Lemma). To determine which one is the next true stage, we simultaneously find all successors of each t_j which are less than $g(t_j)$, and eliminate each t_j which has no such immediate successors. Then we find all immediate successors of those immediate successors, within the bounds provided by g, and eliminate those which have no immediate successors within the bounds. Continuing in this manner, we will eventually eliminate every t_j with only finitely many successors, and once we have only one remaining t_j, we will know that that t_j is the next true stage s_{q+1}.

(Equivalently, let

$$T' = \{\sigma \in T : (\forall n < (lh(\sigma) - 1))\ [\sigma(n+1) < g(\sigma(n))\ ]\ \}.$$

Then T' is a computable subtree of T and contains the path of true stages. But since T' is computably bounded by g, its unique infinite path must be computable.)

Thus the path of true stages is computable, and we use this to show that C is computable. Notice that on the path of true stages, we always have $x_{s_{(q+1)}} > x_{s_q}$, and thus $x_{s_q} \geq q$. Also, for all $p > q$ we have $C_{s_p}\!\upharpoonright x_{s_q} = C_{s_q}\!\upharpoonright x_{s_q}$. To compute whether $c \in C$, therefore, we need only compute the $(c+1)$-st true stage s_{c+1} and evaluate $C_{s_{(c+1)}}(c)$, since $c < x_{s_{(c+1)}}$. ■

1.3 Basic Module for the Construction

Choose an arbitrary noncomputable Δ_2^0 set C with computable approximation $C = \lim_s C_s$. We give the basic module for constructing a linear order $\mathcal{A} = (A, <_{\mathcal{A}})$ of degree $\leq_T C$ which is not isomorphic to the linear order $\mathcal{B}_i$ (if any) computed by the i-th partial computable function φ_i. To achieve this, we choose an element $\hat{b}$ of the universe of $\mathcal{B}_i$ and ensure that no element of A has the same number of predecessors under $<_{\mathcal{A}}$ that $\hat{b}$ does in $\mathcal{B}_i$. This is the same result achieved by the Jockusch-Soare basic module in [25], except that the result of our construction is independent of C.

Proposition 1.3.1 *The basic module described below yields the following outcomes, regardless of the choice of the noncomputable Δ_2^0 set C or the computable approximation to C.*

1. *If $\hat{b}$ has exactly c predecessors in $\mathcal{B}_i$ (or more accurately, if there are exactly c elements x such that $\varphi_i(\langle x, \hat{b}\rangle) \downarrow= 1$), then the basic module constructs a linear order $\mathcal{A}$ of type $c + \omega^*$.*

2. *If $\hat{b}$ has infinitely many predecessors in $\mathcal{B}_i$, then the basic module constructs a linear order $\mathcal{A}$ of type ω.*

(Notice that each outcome ensures that $\mathcal{A} \not\cong \mathcal{B}_i$, since no element of $c+\omega^*$ has exactly c predecessors and no element of ω has infinitely many predecessors.)

The universe A of this order will be $\bigcup_s A_s$, with each $A_s = \{a_0, a_1, \dots a_s\}$. In fact we could just take $a_i = i$ for all i, but this way is clearer, since we can more readily identify the elements of A. On each set A_s we will define a linear order $<_s$, with the final linear order on A being the limit over s of the orders $<_s$.

A_0 is the set $\{a_0\}$, and $<_0$ is the trivial order on it. At stage $s > 0$ we define

$$c_s = |\{x < s : \varphi_{i,s}(\langle x, \hat{b}\rangle) \downarrow= 1\}|.$$

Thus c_s is the number of predecessors of $\hat{b}$ that have appeared within s steps, and the sequence $\langle c_s\rangle_{s\in\omega}$ is computable and non-decreasing. This is the sequence we will use to determine when C "gives permission" to make changes to $\mathcal{A}$. Also, we define x_s as the greatest length of agreement of C_s with any preceding stage, and t_s as that preceding stage (or the first such stage, if there is more than one), exactly as in Section 1.2:

$$x_s = \max\{x : (\exists t < s)[x \leq t \ \& \ C_s \restriction x = C_t \restriction x]\},$$

$$t_s = \min\{t : x_s \leq t < s \ \& \ C_s \restriction x_s = C_t \restriction x_s\}.$$

We let $A_s = A_{s-1} \cup \{a_s\}$ and define the order $<_s$ on A_s, considering two cases:

Case A: $c_s > x_s$. We start by ordering $a_0, a_1, \ldots a_{(x_s-1)}$ according to the order $<_{t_s}$. (This is fully defined, since $x_s \leq t_s$.) Preserving the order $<_{t_s}$ on these elements is necessary in order to obey the permitting condition. Since all the remaining elements have subscripts $\geq x_s$, we have permission to move them wherever we like. We place them above $a_0, \ldots a_{(x_s-1)}$, in order by subscript,

$$\underbrace{a_0, \cdots a_{(x_s-1)}}_{\text{in } <_{t_s}\text{-order}} <_s a_{(x_s)} <_s a_{(x_s+1)} <_s \cdots <_s a_s.$$

The idea is that, if we find ourselves in Case A at infinitely many stages, we will build a copy of ω. No new elements will ever be placed to the left of $a_{(x_s)}$ at any stage which lies below s on the approximation-tree, so if s is a true stage, then each of $a_0, a_1, \ldots a_{(x_s-1)}$ will have only finitely many predecessors. We perform this operation when c_s appears to be getting bigger (namely $c_s > x_s$), since this suggests that $\hat{b}$ will have infinitely many predecessors, and thus cannot map to any of $a_0, a_1, \ldots a_{(x_s-1)}$ under any isomorphism of linear orders.

Case B: $c_s \leq x_s$. We preserve the $<_{t_s}$-order on its domain of definition, namely $\{a_j : j \leq t_s\}$, thereby satisfying the permitting condition. Then we insert all new elements, in reverse order of subscript, between the c_s-th and $(c_s + 1)$-st elements of $<_{t_s}$. (Notice that $c_s \leq x_s$ forces $c_s \leq t_s$.) Thus, if we define the subscripts

$i_0, \dots i_{(t_s)}$ so that the $<_{t_s}$-order is given on $a_0, a_1, \dots a_{t_s}$ by

$$a_{i_0} <_{t_s} a_{i_1} <_{t_s} \cdots <_{t_s} a_{i_{(c_s-1)}} <_{t_s} a_{i_{(c_s)}} <_{t_s} \cdots <_{t_s} a_{i_{(t_s)}},$$

then the new elements are inserted between $a_{i_{(c_s-1)}}$ and $a_{i_{(c_s)}}$,

$$\underbrace{a_{i_0} <_s \cdots <_s a_{i_{(c_s-1)}}}_{\substack{\text{first } c_s \text{ elements} \\ \text{from } <_{t_s}}} <_s \underbrace{a_s <_s a_{s-1} <_s \cdots <_s a_{(t_s+1)}}_{\text{new elements}} <_s \underbrace{a_{i_{(c_s)}} <_s \cdots a_{i_{(t_s)}}}_{\substack{\text{final elements} \\ \text{from } <_{t_s}}}.$$

This is the case where it does not appear that $\hat{b}$ has acquired any new predecessors, so we proceed with the process of building a copy of $c_s + \omega^*$, by inserting new elements immediately after the c_s-th existing element. Each of the first c_s elements under $<_s$ has fewer than c_s predecessors, and by building the ω^*-order above them, we attempt to force every other element of A to have infinitely many predecessors. Our guess at this stage is that $\hat{b}$ has exactly c_s predecessors, and if this guess turns out to be correct, then once again, no isomorphism of linear orders will be able to map $\hat{b}$ to any element of A.

This completes the construction.

Lemma 1.3.2 (Permitting Condition) *For all subscripts $i < j$ and all stages $s < t$, if $j < m \leq s$ and $C_s \restriction m = C_t \restriction m$, then*

$$a_i <_s a_j \iff a_i <_t a_j.$$

Proof. Assume $t < s$ and induct on s. Since $C_s \restriction m = C_t \restriction m$, we know that $m \leq x_s$. By our construction, $a_i <_s a_j$ if and only if $a_i <_{t_s} a_j$, and by induction, $a_i <_{t_s} a_j$ if and only if $a_i <_t a_j$. ∎

Lemma 1.3.3 *The orders $<_s$ converge to a linear order $<_{\mathcal{A}} = \lim_s <_s$ on the set $A = \bigcup_s A_s (= \omega)$. Moreover, $<_{\mathcal{A}}$ is Turing-computable in C.*

Proof. Given a_i and a_j, find (using a C-oracle) a stage $s > \max(i,j)$ such that $C_s \Uparrow \max(i,j) = C \Uparrow \max(i,j)$. (Recall that the symbol $S \Uparrow x$ denotes $S \upharpoonright (x+1)$.) Now there exists a stage $t_0 > s$ such that for all $t \geq t_0$, $C_t \Uparrow \max(i,j) = C \Uparrow \max(i,j)$. But then, by the Permitting Condition,

$$a_i <_s a_j \iff (\forall t \geq t_0)[a_i <_t a_j] \iff a_i <_{\mathcal{A}} a_j.$$

Since each $<_s$ is a linear order on A_s, $<_{\mathcal{A}}$ must obey all the axioms for a linear order on A. Moreover, the stage s was computable in C. ■

Notice that the stage s need not be a modulus of convergence (in contrast to the case of c.e. degrees), since there may be a stage $s' > s$ such that $C_{s'} \Uparrow \max(j,k) \neq C_s \Uparrow \max(j,k)$. We simply know that $<_s$ gives a correct evaluation of the order of a_j and a_k in $\mathcal{A}$.

Proof of Proposition 1.3.1. We now turn our attention to the two statements asserted in Proposition 1.3.1. First, suppose that $\hat{b}$ has exactly c predecessors in $\mathcal{B}_i$. Let $\{s_0, s_1, \ldots\}$ be a (noncomputable) enumeration of the true stages in ascending order, and choose k so large that $c_{s_k} = c$ and $x_{s_k} > c$. We write $s = s_k$ to avoid an overabundance of subscripts. Choose subscripts $i_0, i_1, \ldots i_s$ such that the order $<_s$ is given by

$$a_{i_0} <_s a_{i_1} <_s \cdots <_s a_{i_s}.$$

Now Case A will never again apply at any true stage of the approximation, so this order will be preserved at all subsequent true stages. Therefore, at each true stage s_j with $j > k$, the elements $a_{s_{(j-1)}+1}, \ldots a_{s_j}$ are inserted in reverse order of subscript immediately above $a_{i_{(c-1)}}$, as dictated by Case B, with $<_{s_{(j-1)}}$ being preserved on $a_0, a_1, \ldots a_{s_{(j-1)}}$. Thus, if we look only at the true stages, we see the order $c + \omega^*$ being built. But there are infinitely many true stages, so the orders $<_{s_j}$ must converge to $<_{\mathcal{A}}$, and thus $\mathcal{A} \cong c + \omega^*$.

In the other case, when $\hat{b}$ has infinitely many predecessors we claim that $\mathcal{A} \cong \omega$:

Claim 1.3.4 *If $\hat{b}$ has infinitely many predecessors in $\mathcal{B}_i$, then every element a_x of $\mathcal{A}$ has only finitely many predecessors in A.*

Proof of Claim. As before, let $s_0, s_1, s_2, \ldots$ be the true stages in ascending order, and fix x. Since C is not computable, Lemma 1.2.3 of Section 1.2 yields a k so large that $x_{s_k} > x$ and $c_{s_k} > x_{s_k}$. Once again, let $s = s_k$. Let f be the permutation of $\{0, 1, \ldots x_s - 1\}$ such that

$$a_{f(0)} <_s a_{f(1)} <_s \cdots <_s a_{f(x_s-1)}.$$

Pick y such that $f(y) = x$, so a_x has exactly y predecessors under $<_s$.

We claim that for every $j \geq k$, the predecessors of a_x in A_{s_j} are precisely $a_{f(0)}, a_{f(1)}, \ldots a_{f(y-1)}$. For $j = k$ we have the ordering $<_s$ as above on $a_0, \ldots a_{x_s-1}$. Since $c_s > x_s$, we are in Case A of the construction, and all remaining elements are placed above $a_{f(x_s-1)}$, so the only $<_s$-predecessors of a_x are $a_{f(0)}, a_{f(1)}, \ldots a_{f(y-1)}$, as desired. Now assume inductively that these are the only predecessors of a_x under $<_{s_{(j-1)}}$, for $j > k$. Then $<_{s_{(j-1)}}$ is preserved on $a_0, a_1, \ldots a_{(x_{s_j}-1)}$, so by induction, the $<_{s_j}$-predecessors of a_x among these elements are precisely $a_{f(0)}, a_{f(1)}, \ldots a_{f(y-1)}$. If we are in Case A of the construction at stage s_j, then the remaining elements (those with subscripts $\geq x_{s_j}$) are placed above these, yielding no new predecessors to a_x. If we are in Case B, the remaining elements are inserted after the first c_{s_j} of these. But $c_{s_j} \geq c_s$ since $j > k$, and $c_s > x_s > y$, so the new elements are all inserted above a_x, proving the claim. ∎

From Claim 1.3.4 it is clear that $\mathcal{A} \cong \omega$, independent of the choice of C, as stated in Part 2 of Proposition 1.3.1. ∎

We remark that the Jockusch-Soare basic module in [25] also builds $\mathcal{A} \cong \omega$ whenever $\hat{b}$ has infinitely many predecessors. However, if $\hat{b}$ has exactly c predecessors, it builds $\mathcal{A} \cong d + \omega^*$, for some $d \leq c$, and d varies with the choice of the permitting set C. We avoid that difficulty in Case B of our construction, by placing the new elements between the c_s-th and $(c_s + 1)$-st elements of A_{t_s}. The

Jockusch-Soare construction (in their terminology) would place them immediately above the "attached" elements, and the location of the greatest attached element depends on the last permission received, hence depends on C and $\{C_s\}$.

1.4 Full Construction of the Linear Order

Having seen how this basic module works, we now run it simultaneously for each computable linear ordering $\mathcal{B}_i$. To accomplish this we use the method of separators developed by Jockusch and Soare in [25].

Theorem 1.1.6: *There exists a linear order $\mathcal{A}$ which has a copy in every noncomputable Δ^0_2 degree, but no computable copy. Furthermore, this order may be taken to be of the form*

$$\mathcal{A} = \sum_{i \in \omega} (\mathcal{S}_i + \mathcal{A}_i), \tag{1.1}$$

where each $\mathcal{S}_i \cong 1 + \nu + i + \nu + 1$ and the order type of each $\mathcal{A}_i$ is either ω or $c_i + \omega^* + \omega$ for some $c_i \in \omega$. (Again ν represents the countable dense linear order with end points.)

We will construct $\mathcal{A}$ by stringing together linear orders $\mathcal{A}_i$, for each $i \in \omega$. The order $\mathcal{A}_i$ is intended to refute the possibility of $\mathcal{A}$ being isomorphic to the linear order $\mathcal{B}_i$ (if any) computed by the i-th computable partial function φ_i. To keep the orders $\mathcal{A}_i$ separate, we insert the computable linear orders $\mathcal{S}_i$ as separators between them. For this we use the notation $\mathcal{C}(\mathcal{A}_0, \mathcal{A}_1, \ldots)$,

$$\mathcal{A} = \mathcal{C}(\mathcal{A}_0, \mathcal{A}_1, \ldots) = \mathcal{S}_0 + \mathcal{A}_0 + \mathcal{S}_1 + \mathcal{A}_1 + \ldots. \tag{1.2}$$

Since no $\mathcal{A}_i$ will have an interval isomorphic to ν, this will enable us to recognize the beginnings and ends of the different $\mathcal{S}_i$'s, and thus to isolate each $\mathcal{A}_i$.

However, the $\mathcal{S}_i$'s cannot be recognized by any computable process. To pick out the first and last points of an $\mathcal{S}_i$, we follow [25] and define Π^0_2 predicates $R_i(e, x_1, \ldots x_{i+6})$ each of which holds just if, in the linear order (if any) determined by φ_e, the points in the separator $\mathcal{S}_i = 1 + \nu + i + \nu + 1$ which are not in the interior of either copy of ν are $x_1, \ldots x_{i+6}$. Then the predicate

$$S_i(x_1, \ldots x_{i+6}, y_1, \ldots y_{i+7}) = R_i(i, x_1, \ldots x_{i+6}) \wedge R_{i+1}(i, y_1, \ldots y_{i+7})$$

is also Π^0_2 and asserts that if φ_i defines a linear order of the form $\mathcal{C}(\mathcal{B}_0, \mathcal{B}_1, \ldots)$, then

$x_1, \ldots x_{i+6}$ determine the separator $\mathcal{S}_i$ and $y_1, \ldots y_{i+7}$ determine the separator $\mathcal{S}_{i+1}$. Since the set Inf is Π^0_2-complete, there is a computable function ψ_i whose range is the set ω^{2i+13}, such that for each i and each $\alpha \in \omega^{2i+13}$, $S_i(\alpha)$ holds if and only if there are infinitely many $s \in \omega$ such that $\alpha = \psi_i(s)$. Moreover, we may choose these functions ψ_i uniformly in i. (In the terminology of [25], ψ_i *assigns chips* to the $(2i+13)$-tuples α, and $S_i(\alpha)$ holds just if α gets infinitely many chips from ψ_i.)

It will be useful for us to assume that the range of ψ_i is all of ω^{2i+13}. If this does not hold for the original ψ_i, we can simply replace it by $\psi_i \oplus \chi_i$, where χ_i is a computable bijection from ω to ω^{2i+13}. The relevant property of ψ_i, namely that $S_i(\alpha)$ holds precisely for those α with $\psi_i^{-1}(\alpha)$ infinite, is clearly preserved under this substitution.

Let $l(\alpha)$ be the $(i+6)$-th element of the $(2i+13)$-tuple α, and $u(\alpha)$ its $(i+7)$-th element. Then α predicts that, if $\mathcal{B}_i$ is of the form $\mathcal{C}(\mathcal{R}_0, \mathcal{R}_1, \ldots)$, the elements of $\mathcal{R}_i$ will be those x such that x lies between $l(\alpha)$ and $u(\alpha)$ in the ordering determined by φ_i, i.e. such that $\varphi_i(\langle l(\alpha), x\rangle)\downarrow = 1 = \varphi_i(\langle x, u(\alpha)\rangle)\downarrow$.

In our construction we will define elements $\hat{b}^s_\alpha$ in the interval $(l(\alpha), u(\alpha))$ of $\mathcal{B}_i$ (where $2i + 13 = lh(\alpha)$), which approximate the element $\hat{b}$ from the basic module. (Note that $\hat{b}^s_\alpha$ may be undefined for certain s and α.) Also, if $\hat{b}^s_\alpha$ is defined, we will let

$$c^s_\alpha = |\{x \leq s : \varphi_{i,s}(\langle l(\alpha), x\rangle)\downarrow = 1 = \varphi_{i,s}(\langle x, \hat{b}^s_\alpha\rangle)\downarrow\}|.$$

Thus c^s_α is the number of predecessors of $\hat{b}^s_\alpha$ in the interval between $l(\alpha)$ and $u(\alpha)$, under the order $\mathcal{B}_i$, which have appeared by stage s.

For a given noncomputable Δ^0_2 set C, we now fix i and construct the individual order $\mathcal{A}_i$ as follows (uniformly in i). For each j let $a_j = \langle 2i+1, j\rangle$. (The row $\omega^{[2i]}$ is reserved to form the computable separator $\mathcal{S}_i$, built uniformly in i by a straightforward construction.) The universe A_i of $\mathcal{A}_i$ will be $\omega^{[2i+1]}$, namely $\{a_j : j \in \omega\}$. Thus A_i is computable and infinite. A_i will be the union of sets A^s_α, with α ranging over ω^{2i+13} and $s \in \omega$, and we will write A^s_i for $\bigcup\{A^s_\alpha : \alpha \in \omega^{2i+13}\}$. Each A^s_α is a bin into which we place the elements which we manipulate (at stage s) to try to defeat any possible isomorphism between $\mathcal{A}_i$ and $\mathcal{B}_i$, based on the assumption that $S_i(\alpha)$ holds. (Each element of A_i is used in only one such strategy at stage s, so the different bins at stage s are disjoint: $A^s_\alpha \cap A^s_\beta = \emptyset$ for $\alpha \neq \beta$.)

We now fix i and order the elements α of ω^{2i+13} in order type ω. (Specifically, pick a computable bijection $f_i : \omega^{2i+13} \to \omega$, uniformly in i, and define $\alpha \prec \beta$ if and only if $f_i(\alpha) < f_i(\beta)$.) An α-strategy can only be injured by a β-strategy with $\beta \prec \alpha$, and then only at a stage s such that $\psi_i(s) = \beta$. The strategy which succeeds will be the strategy for that α for which $S_i(\alpha)$ holds, namely the least α such that $\alpha = \psi_i(s)$ for infinitely many s. This strategy will be injured only finitely often by the β-strategies for those $\beta \prec \alpha$, and will not be injured at all by the γ-strategies with $\alpha \prec \gamma$.

The ordering $<_s$ which we define on the elements of A^s_i at stage s will respect the ordering $\prec$, in that for $a_j \in A^s_\beta$ and $a_k \in A^s_\alpha$ with $\beta \prec \alpha$, we will have $a_j <_s a_k$. Also, if $\psi_i(s+1) = \alpha$, the elements from each bin A^s_γ with $\gamma \succ \alpha$ will be taken out of this bin and dumped (all together) into the bin A^{s+1}_α at stage $s+1$. This constitutes an injury to the γ-strategy, which must then start its work anew. We write A_α for the set of elements which reach the α-th bin at some point and stay there forever after,

$$A_\alpha = \bigcup_s \bigcap_{t \geq s} A^t_\alpha.$$

For all $\alpha \in \omega^{2i+13}$, let A^0_α be the empty set, and let $\hat{b}^0_\alpha$ and c^0_α be undefined. At each stage $s > 0$, we let $\alpha = \psi_i(s)$.

Step 1. We let

$$A^s_\alpha = \left(\bigcup_{\gamma \succeq \alpha} A^{s-1}_\gamma \right) \cup \{a_s\}.$$

Also, for each $\gamma \succ \alpha$, set $A^s_\gamma = \emptyset$, and for each $\beta \prec \alpha$, set $A^s_\beta = A^{s-1}_\beta$.

Step 2. Let $\hat{b}^s_\gamma$ be undefined for every $\gamma \succ \alpha$, and let $\hat{b}^s_\beta = \hat{b}^{s-1}_\beta$ for every $\beta \prec \alpha$. If $\hat{b}^{s-1}_\alpha$ is defined, let $\hat{b}^s_\alpha = \hat{b}^{s-1}_\alpha$. Otherwise set $n = |\bigcup_{\beta \prec \alpha} A^s_\beta|$, and check whether there are (at least) $n+1$ distinct elements above $l(\alpha)$ and below $u(\alpha)$

in the ordering given by $\varphi_{i,s}$. If so, take $\hat{b}^s_\alpha$ to be the $(n+1)$-st of these, in the ordering given by $\varphi_{i,s}$, so that $c^s_\alpha = n$; if not, then $\hat{b}^s_\alpha$ is undefined.

Step 3. We now define the ordering on A^s_i, by ordering each A^s_β with $\beta \preceq \alpha$ and respecting the order of the bins. As in Section 1.2, we let

$$x_s = \max\{x : (\exists t < s)[x \leq t \ \&\ C_s \restriction x = C_t \restriction x]\},$$

$$t_s = \min\{t : x_s \leq t < s \ \&\ C_s \restriction x_s = C_t \restriction x_s\}.$$

We will need to preserve the order $<_{t_s}$ on $\{a_j \in A^s_i : j < x_s\}$ in order to obey the permitting condition. Therefore we prove, by induction, that $<_{t_s}$ respects the order of the bins A^s_β. In fact, $<_{t_s}$ respects the order of the bins A^t_β for every $t > t_s$. The inductive step follows from Step 1, for all j, k, t, β, β', γ, and γ',

$$[a_j \in A^t_\beta \cap A^{t+1}_{\beta'} \ \&\ a_k \in A^t_\gamma \cap A^{t+1}_{\gamma'} \ \&\ \beta \preceq \gamma] \implies \beta' \preceq \gamma'.$$

As in the basic module (see page 18), we now ask, for each $\beta \preceq \alpha$, whether $c^s_\beta > x_s$.

Case A. $c^s_\beta > x_s$, or c^s_β is undefined.

In this case we preserve the order $<_{t_s}$ on $\{a_j \in A^s_\beta : j < x_s\}$. (This will satisfy the permitting condition given below.) Above these elements, but below all elements of $\cup_{\gamma \succ \beta} A^s_\gamma$, we then place all remaining elements of A^s_β, ordered in increasing order of subscript.

Case B. $c^s_\beta \leq x_s$.

In this case we preserve the $<_{t_s}$ order on its entire domain of definition, namely $\{a_j \in A^s_\beta : j \leq t_s\}$. Above these elements we place the elements of $\{a_j \in A^s_\beta : j > t_s \ \&\ \psi_i(j) \succ \beta\}$, in increasing order of subscript. We then put the elements of $\{a_j \in A^s_\beta : j > t_s \ \&\ \psi_i(j) = \beta\}$ in reverse order of subscript and place them consecutively so that the leftmost of them is the $(c^s_\beta + 1)$-st element of $\bigcup_{\beta' \preceq \beta} A^s_{\beta'}$. (If there are fewer than c^s_β elements in $\cup_{\beta' \preceq \beta} A^s_{\beta'}$ already ordered by $<_s$, then we simply put these new elements at the right end of A^s_β, again in reverse order of subscript.) This completes the construction.

The ordering $\mathcal{A}$ which is the goal of this paper will be precisely

$$\mathcal{C}(\mathcal{A}_0, \mathcal{A}_1, \ldots) = \mathcal{S}_0 + \mathcal{A}_0 + \mathcal{S}_1 + \mathcal{A}_1 + \ldots \quad .$$

Notice that since the entire construction was uniform in i, we can string the $\mathcal{S}_i$'s and $\mathcal{A}_i$'s together computably. We show below that $\deg(\mathcal{A}_i) \leq_T C$ for each i, so $\mathcal{A}$ will be Turing-reducible to C. (The orders $\mathcal{S}_i$ are all computable, uniformly in i.) Indeed, the $\mathcal{S}_i$ and $\mathcal{A}_i$ were constructed so that the union of all their universes is precisely ω. The ordering $<_{\mathcal{A}}$ respects the rows of ω, and within each row $\omega^{[2i]}$ or $\omega^{[2i+1]}$ it is given by the ordering on $\mathcal{S}_i$ or $\mathcal{A}_i$, respectively.

The proofs of the following two lemmas are identical to those of Lemmas 1.3.2 and 1.3.3 in the basic module.

Lemma 1.4.1 (Permitting Condition) *If $C_s \restriction m = C_t \restriction m$ and $a_j, a_k \in A_i$ with $j, k < m \leq \min(s,t)$, then*

$$a_j <_s a_k \text{ if and only if } a_j <_t a_k.$$

■

Lemma 1.4.2 *For each i, the orders $<_s$ converge to a linear order $<_{\mathcal{A}_i}$ on $A_i = \bigcup_s A_i^s$ $(= \omega^{[2i+1]})$. Moreover, $<_{\mathcal{A}_i}$ is Turing-computable in C, uniformly in i.* ■

Lemma 1.4.3 *For any two noncomputable Δ_2^0 sets C and C', any computable approximations $\{C_s\}$ and $\{C'_s\}$, and any i, the linear orders $\mathcal{A}_i$ and $\mathcal{A}'_i$ built by the above construction are isomorphic.*

Proof. We will show that each order $\mathcal{A}_i$ built by the construction is independent of C. Notice that the only time C is used in the construction is in Step 3, and there it rearranges the order of certain elements but never moves elements from one A_α^s to another A_β^s. The movement of elements from one A_α^s to another A_β^s depends only on the function ψ_i. Therefore, for each α and s, the set A_α^s is independent of

C, although the ordering of the elements of the set may depend on C. Also, the definitions of the elements $\hat{b}^s_\alpha$ in Step 2 depend only on φ_i, ψ_i, and the sizes of the sets A^s_α, all of which are independent of C.

Fix i, and let $\alpha \in \omega^{2i+13}$ be minimal such that $\psi_i^{-1}(\alpha)$ is infinite. (If $\psi_i^{-1}(\alpha)$ is finite for all α, then every A_α is finite, so $\mathcal{A}_i \cong \omega$, independent of choice of C.) Let $s_0, s_1, \ldots$ be the true stages in the approximation $\{C_s\}$ of C, in increasing order.

We deal first with the case in which $\lim_s \hat{b}^s_\alpha$ diverges. Pick the least true stage s_q such that $\psi_i(s) \succeq \alpha$ for all $s \geq s_q$. By Step 2 of the construction, we know that if $s \geq s_q$ and $\hat{b}^s_\alpha$ is defined, then $\hat{b}^{s+1}_\alpha$ is defined and equals $\hat{b}^s_\alpha$. Therefore, $\hat{b}^s_\alpha$ must be undefined for every $s \geq s_q$. But then every corresponding c^s_α is undefined, so in Step 3 after stage s_q, we always are in Case A, which instructs us simply to place the elements with subscripts $\geq x_s$ at the right end of A^s_α, in increasing order of subscript. Finitely many elements lie in $\cup_{\beta \prec \alpha} A_\beta$, and any other element a_j must wind up in A_α. (Initially a_j may go into some A^t_γ with $\gamma \succ \alpha$, but it will be dumped into $A^{t'}_\alpha$, at the next t' with $\psi_i(t') = \alpha$.) Eventually we will reach a true stage s_p with $a_j \in A^{s_p}_\alpha$ and $j < x_{s_p}$, and at all true stages thereafter, no more elements will be placed below a_j. Since the orders $<_s$ converge and the true stages form an infinite subsequence, this means that a_j can have only finitely many predecessors in $<_{\mathcal{A}_i}$. So the order $\mathcal{A}_i$ is isomorphic to ω, independent of choice of C.

Now suppose that the elements $\hat{b}^s_\alpha$ converge to some element $\hat{b}_\alpha$ of $\mathcal{B}_i$. Then the sequence $\{c^s_\alpha\}$ is defined for cofinitely many s and either converges to some $c_\alpha \in \omega$ (if $\hat{b}_\alpha$ has exactly c_α predecessors in the interval $(l(\alpha), u(\alpha))$ of $\mathcal{B}_i$) or goes to infinity (if $\hat{b}_\alpha$ has infinitely many predecessors there).

In the case with only finitely many predecessors, we choose a true stage $s = s_q$ so large that $c^s_\alpha = c_\alpha$ and $x_s > c_\alpha$ and $\psi_i(t) \succeq \alpha$ for all $t \geq s$. Then for each true stage s_p with $p > q$, we have $c^{s_p}_\alpha = c_\alpha < x_s \leq x_{s_p}$ so we are in Case B of Step 3 of the construction. Therefore, at each such s_p, we preserve $<_{s_{(p-1)}}$ on its domain of definition, $A_i^{s_{(p-1)}}$. Define the numbers $i_0, i_1, \ldots i_s \in \{0, 1, \ldots s\}$ so that

$$a_{i_0} <_s a_{i_1} <_s \cdots <_s a_{i_s}.$$

Since $t_{s_p} = s_{(p-1)}$, induction on p yields

$$a_{i_0} <_{s_p} a_{i_1} <_{s_p} \cdots <_{s_p} a_{i_s}.$$

Moreover, since we are in Case B at every such true stage, no element is ever inserted to the left of the c_α-th element $a_{i_{(c\alpha-1)}}$. Thus the order which we build will have initial segment c_α.

We claim that the rest of the order has type $\omega^* + \omega$, so that the entire order has type $c_\alpha + \omega^* + \omega$. The ω^*-chain is built of those elements a_j with $j > s$ and $\psi_i(j) = \alpha$. There are infinitely many such elements still to be added to A_i, and each of them, once added, will be inserted (possibly along with other elements) immediately after $a_{i_{(c\alpha-1)}}$ at the next true stage, building the ω^*-chain above $a_{i_{(c\alpha-1)}}$.

The ω-chain is built of those elements a_j with $j > s$ and $\psi_i(j) \succ \alpha$. (There are infinitely many such, since the range of ψ_i is all of ω^{2i+13}.) For such an element, let t be the first stage such that $a_j \in A^t_\alpha$, and let s_p be the first true stage $\geq t$. If there is no true stage between stage j and stage t, then a_j will be placed (possibly along with other elements) at the right end of $A^{s_p}_\alpha$, by Case B of Step 3. If there was a true stage between j and t, then a_j will be placed at the right end of $A^{s_p}_\alpha$ (possibly along with other elements) by the preservation of the order $<_{s_{(p-1)}}$ at stage s_p. In either case, $t_{s_{(p+1)}} = s_p \geq j$, and since we are in Case B at every true stage after s, the order $<_{s_p}$ is preserved (on its domain of definition) at every subsequent true stage. New elements a_k will be added at subsequent true stages only to the right of a_j (if $\psi_i(k) \succ \alpha$) or immediately after $a_{i_{(c\alpha-1)}}$ (if $\psi_i(k) = \alpha$). Since the true stages form an infinite subsequence, this allows us to deduce the type of the order $\mathcal{A}_i$: it will be of the form $c_\alpha + \omega^* + \omega$. Thus the order type of $\mathcal{A}_i$ is independent of C in this case.

In the case where the interval $(l(\alpha), \hat{b}_\alpha)$ of $\mathcal{B}_i$ is infinite, we claim that $\mathcal{A}_i \cong \omega$.

Claim 1.4.4 *If $\lim_s c^s_\alpha = \infty$, then each $a_j \in A_\alpha$ has only finitely many predecessors in $\mathcal{A}_i$.*

Proof. Fix j. There will be a true stage $s = s_q$ for which $x_s > j$ and $(\forall t \geq s)\psi_i(t) \succeq \alpha$, and by Lemma 1.2.3, we may also assume that $c^s_\alpha > x_s$. Therefore, at stage s we will be in Case A of Step 3, so all elements a_k of A^s_α with $k \geq x_s$ will be placed above the elements of $\{a_m \in A^s_i : m < x_s\}$, and hence above a_j. Thus a_j has fewer than x_s predecessors under $<_s$, and all of those predecessors have subscripts $< x_s$ and therefore will precede a_j at every subsequent true stage s_p.

We now induct on the true stages s_p with $p > q$, to see that the predecessors of a_j under each $<_{s_p}$ are precisely the predecessors of a_j under $<_{s_{(p-1)}}$. Let s_p be a true stage with $p > q$. If we are in Case B of Step 3 at stage s_p, then the ordering $<_{s_{(p-1)}}$ is not injured, and all new elements are placed either after the $c^{s_p}_\alpha$-th element, hence to the right of a_j (since $c^{s_p}_\alpha \geq c^s_\alpha > x_s$ and by induction j has fewer than x_s predecessors under $<_{s_{(p-1)}}$), or else at the right end of $A^{s_p}_\alpha$. Thus a_j receives no new predecessors at such a stage. If we are in Case A of Step 3 at stage s_p, then all elements with subscripts $\geq x_{s_p}$ are moved to the right end of $A^{s_p}_\alpha$, and all other elements, including a_j and all its predecessors, are left alone. Therefore, for each $p > q$, the predecessors of a_j under $<_{s_p}$ are precisely the predecessors of a_j under $<_s$. Since the true stages form an infinite subsequence of ω, we see that indeed a_j has only those (finitely many) predecessors under $<_{\mathcal{A}_i}$, just as we had claimed. ■

This holds for every $a_j \in A_\alpha$, while each A_β $(\beta \prec \alpha)$ is finite and each A_γ $(\gamma \succ \alpha)$ is empty, so clearly $\mathcal{A}_i \cong \omega$, independent of the choice of C. (Notice that we did use the noncomputability of C in applying Lemma 1.2.3.) This completes the proof of Lemma 1.4.3. ■

Corollary 1.4.5 *For each i, the linear order $\mathcal{A} = \mathcal{C}(\mathcal{A}_0, \mathcal{A}_1, \ldots)$ has a unique interval isomorphic to $\mathcal{S}_i$.*

(Here $\mathcal{C}$ is the operator defined in (1.2), so $\mathcal{A}$ is precisely the order given in (1.1).)

Proof. From the proof of Lemma 1.4.3, we see that the only possible outcomes of the construction of each $\mathcal{A}_i$ are ω and $n + \omega^* + \omega$, where n is finite. None of these has an interval isomorphic to ν, the countable dense linear order with end points, but every one is infinite, so the only copy of $1+\nu+i+\nu+1$ in $\mathcal{A}$ is $\mathcal{S}_i$ itself. ∎

Corollary 1.4.6 *$\mathcal{A}$ is not isomorphic to any of the computable linear orders $\mathcal{B}_i$.*

Proof. We note first, using the preceding corollary, that if $\mathcal{A} \cong \mathcal{B}_i$ for some i, then $\mathcal{B}_i$ has unique intervals isomorphic to $\mathcal{S}_i$ and $\mathcal{S}_{i+1}$. Hence there is a unique $\alpha \in \omega^{2i+13}$ for which $S_i(\alpha)$ holds, so $\psi_i^{-1}(\alpha)$ is infinite, but $\psi_i^{-1}(\beta)$ is finite for all $\beta \neq \alpha$. Since $\mathcal{A} \cong \mathcal{B}_i$, $\mathcal{A}_i$ must be isomorphic to the interval $(l(\alpha), u(\alpha))$ of $\mathcal{B}_i$.

If the sequence $\langle \hat{b}^s_\alpha \rangle$ diverges, then $\hat{b}^s_\alpha$ is undefined for cofinitely many s, as noted in the proof of Lemma 1.4.3. By Step 2 of the construction, this can only happen if the interval $(l(\alpha), u(\alpha))$ contains at most $|\bigcup_{\beta \prec \alpha} A_\beta|$ elements, But $\mathcal{A}_i \cong \omega$, so $\mathcal{B}_i \ncong \mathcal{A}$.

If $\hat{b}^s_\alpha$ converges to an element $\hat{b}_\alpha$ with only c_α-many elements between $l(\alpha)$ and $\hat{b}_\alpha$, then $\mathcal{A}_i \cong \; c_\alpha + \omega^* + \omega$. Thus every element of $\mathcal{A}_i$ has either fewer than c_α predecessors or infinitely many in $\mathcal{A}_i$, so no isomorphism could take $\hat{b}_\alpha$ to any element of $\mathcal{A}_i$.

Finally, if $\hat{b}^s_\alpha$ converges to an element $\hat{b}_\alpha$ with infinitely many elements between $l(\alpha)$ and $\hat{b}_\alpha$, then $\mathcal{A}_i \cong \omega$, so again there can be no isomorphism taking $\hat{b}_\alpha$ to any element of $\mathcal{A}_i$. ∎

Thus $\mathcal{A}$ is a linear order with no computable copy. However, for every non-computable Δ^0_2 set C, we have seen (in Lemma 1.4.2) that there is a copy of $\mathcal{A}$ computable in C. We discuss Julia Knight's full theorem (from [28]) in the next section, as Theorem 1.5.2, but an easy consequence of it, cited in [25] and [7], implies that for each such C, there is a copy of $\mathcal{A}$ whose Turing degree is exactly the degree of C. This is precisely the property we had promised would hold for $\mathcal{A}$.

1.5 Further Questions

The obvious generalization of Theorem 1.1.6 would be a positive answer to Downey's third question:

Question 1.1.4 (Downey) *Is there a linear order whose spectrum contains every degree except* $\mathbf{0}$*?*

This question remains open, however. It is known that for every noncomputable degree $\mathbf{C}$ there is a linear order whose spectrum includes $\mathbf{C}$ but not $\mathbf{0}$. However, Knight's proof of this result (see [7]) is highly nonuniform: one uses the Downey-Seetapun result for Δ^0_2 degrees, a coding construction for non-low$_2$ degrees, and a combination of these two techniques for the remaining degrees. Therefore, it would be far harder to make Knight's construction yield the same result independent of the choice of $\mathbf{C}$, as we managed to do for the Jockusch-Soare construction.

A more general question, also posed by Downey [7], is simply to ask what spectra are possible for a linear order.

Question 1.5.1 (Downey) *What can be said about Spec*$(\mathcal{L})$ *for a given linear order* $\mathcal{L}$*?*

There are two main results so far. One we have already used in proving Theorem 1.1.6, namely Knight's result that the spectrum must be closed upwards under Turing reducibility. This follows from a stronger theorem of Knight [28].

Theorem 1.5.2 (Knight) *If* $\mathcal{A}$ *is any structure, then exactly one of the following two statements holds:*

(5.1) For all Turing degrees $\mathbf{C} \leq_{\mathbf{T}} \mathbf{D}$*, if there is an isomorphic copy of* $\mathcal{A}$ *of degree* $\mathbf{C}$*, then there is an isomorphic copy of* $\mathcal{A}$ *of degree* $\mathbf{D}$*;*

(5.2) There exists a finite subset S *in the universe* A *of* $\mathcal{A}$ *such that any permutation of* A *fixing* S *is an automorphism of* $\mathcal{A}$*.*

For any infinite linear order $\mathcal{L}$, (5.2) clearly fails, so the upward-closure property (5.1) holds. (If $\mathcal{L}$ is finite, then (5.2) holds, and indeed in this case every copy of $\mathcal{L}$ is computable.)

The second main result about the spectrum of a linear order is due to Richter [40]:

Theorem 1.5.3 (Richter) *If the spectrum of a linear order has a least degree, then that degree is* $\mathbf{0}$.

The least degree of the spectrum of a structure is often simply called the degree of the isomorphism type of that structure. Thus, Richter's result says that $\mathbf{0}$ is the only possible degree for the isomorphism type of a linear order; a linear order with no computable copy cannot have any least degree in its spectrum. This can be viewed as a result on the difficulty of coding sets into linear orders. If we wish to code a noncomputable set S into a linear order, so that S would be computable from every copy of the order, then that order cannot have a copy computable from S. (Otherwise, $\deg(S)$ would be the least degree of the spectrum of the linear order.)

These two results rule out many possible spectra for linear orders. On the other hand, Theorem 1.1.6 is an example of a positive response to Question 1.5.1: the spectrum can contain all Δ^0_2 degrees except $\mathbf{0}$. We can also use Knight's result on noncomputable degrees to show that it is possible to separate any two degrees $\mathbf{C} <_{\mathbf{T}} \mathbf{D}$ via the spectrum of a linear order. That is:

Corollary 1.5.4 *If* $\mathbf{C} <_{\mathbf{T}} \mathbf{D}$*, then there exists a linear order* $\mathcal{L}$ *such that* $\mathbf{D} \in$ *Spec*$(\mathcal{L})$ *and* $\mathbf{C} \notin$ *Spec*$(\mathcal{L})$.

Proof. Simply take Knight's proof for the case $\mathbf{C} = \mathbf{0}$ and relativize it to the degree $\mathbf{C}$. ∎

We might ask if it is possible to separate any two Turing degrees in this way, even if they are incomparable. Also, we can ask if it is possible to separate collections of degrees:

Question 1.5.5 *If* $\mathbf{P}$ *and* $\mathbf{N}$ *are collections of Turing degrees such that no degree in* $\mathbf{P}$ *is reducible to any degree in* $\mathbf{N}$*, is there a linear order* $\mathcal{L}$ *whose spectrum contains all of* $\mathbf{P}$ *but does not intersect* $\mathbf{N}$*?*

This question is intended to be asked for specific choices of $\mathbf{P}$ and $\mathbf{N}$, particularly classes of c.e. sets (or Δ^0_2 sets) whose indices cannot be computably separated. We have seen in the preceding sections that linear orders can contain more information than subsets of integers. There is no set which is computable in every nonzero Δ^0_2 degree but not in $\mathbf{0}$, whereas there is a linear order which is computable in every Δ^0_2 degree except $\mathbf{0}$. What else can linear orders do? For instance, could a linear order contain enough information to separate the high Δ^0_2 sets from the low ones?

Clearly the answer to Question 1.5.5 is not always positive, for otherwise we could contradict Richter's result by taking $\mathbf{P}$ to be the upper cone above a noncomputable degree C, including C itself, and $\mathbf{N}$ to be the complement of $\mathbf{P}$. Indeed, this is an example in which a set (namely C) contains information which a linear order cannot contain (namely, how to compute C). Knight's and Richter's results both clearly restrict the amount of information encoded in a linear order. Perhaps there are other common mathematical structures which escape Richter's restriction, which would entail failing her "Recursive Enumerability Condition" (see [40]). Knight's restriction appears inevitable, since under (5.2) in Theorem 1.5.2, the information contained by the structure is essentially encoded in a single finite set.

CHAPTER 2
DEFINABLE INCOMPLETENESS AND FRIEDBERG SPLITTINGS

2.1 Introduction

The computably enumerable sets form an upper semi-lattice under Turing reducibility. Under set inclusion, they form a lattice $\mathcal{E}$, as first noted by Myhill in [35], and the properties of a c.e. set as an element of $\mathcal{E}$ often help determine its properties under Turing reducibility. Even before Myhill, Post had suggested that there should be a property of c.e. sets, definable using the inclusion relation, which would imply that the Turing degree of such a set must lie strictly between the computable degree $\mathbf{0}$ and the complete c.e. degree $\mathbf{0}'$.

Post's own attempts to find such a property failed. The properties he defined turned out to be extremely useful in computability theory, but each of them – simplicity, hypersimplicity, and hyperhypersimplicity – actually does hold of some complete set. The existence of a Turing degree between $\mathbf{0}$ and $\mathbf{0}'$ was first proven by completely different means, namely the finite injury constructions of Friedberg and Muchnik ([15], [34]).

Post's Program, the search for an $\mathcal{E}$-definable property implying incompleteness, remained unfinished until 1991, when Harrington and Soare ([21]) found a property $Q(A)$ definable in $\mathcal{E}$ such that every A satisfying Q must be both noncomputable and Turing-incomplete. We give their definition of $Q(A)$:

$$Q(A): \ (\exists C)_{A \subset_{\mathrm{m}} C} (\forall B \subseteq C)(\exists D \subseteq C)(\forall S)_{S \sqsubset C} \left(\begin{array}{c} B \cap (S - A) = D \cap (S - A) \implies \\ (\exists T)[\overline{C} \subset T \ \& \ A \cap (S \cap T) = B \cap (S \cap T)] \end{array} \right).$$

Here $S \sqsubset C$ abbreviates $(\exists \hat{S})[S \cup \hat{S} = C \ \& \ S \cap \hat{S} = \emptyset]$. (All variables represent elements of $\mathcal{E}$, namely c.e. sets.) $A \sqcup B$ denotes the union of two disjoint sets A and B. Also, $A \subset_{\mathrm{m}} C$ abbreviates "A is a major subset of C," meaning that $A \subset C$ with $C - A$ infinite such that for every W, if $\overline{C} \subset W$, then $\overline{A} - W$ is finite. Since the property of being finite is $\mathcal{E}$-definable, the statement $A \subset_{\mathrm{m}} C$ is $\mathcal{E}$-definable as well.

In this chapter we generalize the property $Q(A)$ to an $\mathcal{E}$-definable property

$R(A_0, A_1)$ of two c.e. sets. The statement of R is as follows:

$$\begin{aligned} R(A_0, A_1): \ & A_0 \cap A_1 = \emptyset \ \& \\ & (\exists C)(\forall B \subseteq C)(\exists D \subseteq C)(\forall S \sqsubset C)(\exists T)\Big[A_0 \cup A_1 \subset_{\mathrm{m}} C \ \& \\ & \big[(B \cap (S - A_0)) \cup A_1 = (D \cap (S - A_0)) \cup A_1 \implies \\ & [\overline{C} \subset T \ \& \ (A_0 \cap S \cap T) \cup A_1 = (B \cap S \cap T) \cup A_1]\big]\Big]. \end{aligned}$$

This property can be read to say that A_0 satisfies the Q-property on $\overline{A_1}$. Indeed, the statement $R(A_0, \emptyset)$ is equivalent to $Q(A_0)$. In Section 2.2 we prove that just as with the Q-property, $R(A_0, A_1)$ implies that A_0 is not of prompt degree, and hence not Turing complete in Σ^0_1. (A set which is not of prompt degree is said to be *tardy*, and since A_0 satisfies an $\mathcal{E}$-definable property implying tardiness, we say that A_0 is "definably tardy." Since all tardy sets are incomplete, we also say that A_0 is "definably incomplete.")

Alternatively, we can interpret $R(A_0, A_1)$ in the lattice $\mathcal{E}/\mathcal{A}$, where $\mathcal{A}$ is the principal ideal in $\mathcal{E}$ generated by A_1. (See [42], p. 225.) In this lattice, $C \subseteq_{\mathcal{A}} D$ is defined to mean $C \subseteq D \cup A_1$, and $C \approx_{\mathcal{A}} D$ if $C \subseteq_{\mathcal{A}} D$ and $D \subseteq_{\mathcal{A}} C$. Essentially, $R(A_0, A_1)$ says that $Q(A_0)$ holds in $\mathcal{E}/\mathcal{A}$, with containment and equality replaced by $\subseteq_{\mathcal{A}}$ and $\approx_{\mathcal{A}}$. The only differences are that we cannot state the properties $A_0 \cap A_1 = \emptyset$ or $A_1 \subseteq C$ in $\mathcal{E}/\mathcal{A}$, and that we have left the quantifier $(\forall S \sqsubset C)$ in $R(A_0, A_1)$ just as in the original Q-property, rather than restating it to hold on $\overline{A_1}$. Choosing not to restate it makes the R-property slightly stronger, but the stronger version can still be satisfied.

In Section 2.3 we construct c.e. sets A_0 and A_1 satisfying R, to show that the R-property is non-vacuous. A_0 and A_1 will also be noncomputable. Thus, the following $\mathcal{E}$-definable formula is non-vacuous:

$$(\exists A_1)[A_0 >_T \emptyset \ \& \ R(A_0, A_1)]$$

This formula guarantees that A_0 is noncomputable and incomplete, just as the property $Q(A)$ does for A. (Recall that computability is equivalent to the property of having a complement in $\mathcal{E}$.)

We then consider Friedberg splittings. Two disjoint c.e. sets B_0 and B_1 form a *Friedberg splitting* of $B = B_0 \sqcup B_1$ if for every c.e. W:

$$W - B \text{ is not c.e.} \implies \text{neither } W - B_0 \text{ nor } W - B_1 \text{ is c.e.}$$

The sets B_0 and B_1 are each said to be *half* of this Friedberg splitting. The sets A_0 and A_1 which we construct will have the additional property of forming a Friedberg splitting of their union.

We use the R-property to show that A_0 and A_1 cannot lie in the same orbit under automorphisms of $\mathcal{E}$. (In the argot of this topic, we say that A_0 and A_1 are not *automorphic*. Two sets are automorphic if they lie in the same orbit.) This will follow because the A_1 we construct will be of prompt degree, hence automorphic to a complete set, by another result of Harrington and Soare in [21].

The orbits of halves of Friedberg splittings have been a subject of interest for some time, at least since the discovery of the hemimaximal sets. A set is hemimaximal if it is half of a nontrivial splitting of a maximal set. This is $\mathcal{E}$-definable, and Downey and Stob proved that the hemimaximal sets form an orbit (see [11]).

Since the maximal sets themselves form an orbit, and since few orbits are known in $\mathcal{E}$, this led to the conjecture that if $\mathcal{O}$ is any orbit in $\mathcal{E}$, then the collection of "hemi-$\mathcal{O}$" sets, i.e. halves of nontrivial splittings of sets in $\mathcal{O}$, might also be an orbit. Alternatively, it was conjectured that halves of Friedberg splittings of sets in $\mathcal{O}$ might form an orbit. (For the orbit of maximal sets, these classes coincide, since any nontrivial splitting of a maximal set is automatically a Friedberg splitting.)

Downey and Stob refuted both conjectures in [13], by producing two Friedberg splittings $B_0 \sqcup B_1 = C_0 \sqcup C_1$ of the same set B, which were definably different in $\mathcal{E}$. Hence B_0 and C_0 satisfy different 1-types in the language of inclusion and cannot be automorphic.

The present result goes a step further. Since A_0 is definably tardy, every set in its orbit must also be tardy, and hence A_1 must lie in a different orbit. This is thus the first example of a single Friedberg splitting with the two halves known to lie in different orbits in $\mathcal{E}$. It is also the first application of Harrington and Soare's Q-property to derive results about Friedberg splittings.

Our notation mostly follows that of [47]. The finite sets form an ideal $\mathcal{F} \subset \mathcal{E}$, and we write $\mathcal{E}^*$ for the lattice $\mathcal{E}/\mathcal{F}$. (Computability is definable in $\mathcal{E}$ as the property of possessing a complement, and then finiteness is definable, since a set is finite if and only if all its subsets are computable.) We write $A \subseteq^* B$ if $B - A$ is finite, and $A =^* B$ if $A \subseteq^* B$ and $B \subseteq^* A$.

We use the standard enumeration $\{W_e\}_{e\in\omega}$ of the computably enumerable sets, with finite approximations $\{W_{e,s}\}_{s\in\omega}$ to each. For the c.e. sets which we construct ourselves, we will also give finite approximations, usually writing $A = \cup_{s\in\omega} A^s$. If A and B are both enumerated this way, we write $A \setminus B = \{x : (\exists s)[x \in A^s - B^s]\}$, and $A \searrow B = \{x \in A \cap B : (\exists s)[x \in A^s - B^s]\}$. Thus when an element not yet in B enters A, we put it into $A \setminus B$, and if it later enters B, then we put it into $A \searrow B$ as well.

2.2 The R-Property

In order to guarantee that the set A_0 is not automorphic to a complete set, we will force it to satisfy the lattice-definable property R defined in Section 2.1, and prove that this implies tardiness of A_0. Tardiness itself does not guarantee that a set cannot be automorphic to a complete set, of course, but satisfaction of R does, since every other set automorphic to A_0 must also satisfy R and therefore must also be tardy, hence incomplete. (A tardy set must be half of a minimal pair under $\leq_T$, as shown in [47], and therefore must be incomplete.) We restate the R-property here:

$$\begin{aligned} R(A_0, A_1): \ & A_0 \cap A_1 = \emptyset \ \& \\ & (\exists C)(\forall B \subseteq C)(\exists D \subseteq C)(\forall S \sqsubset C)(\exists T)\Big[A_0 \cup A_1 \subset_{\mathrm{m}} C \ \& \\ & \big[(B \cap (S - A_0)) \cup A_1 = (D \cap (S - A_0)) \cup A_1 \implies \\ & [\overline{C} \subset T \ \& \ (A_0 \cap S \cap T) \cup A_1 = (B \cap S \cap T) \cup A_1]\big]\Big] \end{aligned}$$

Theorem 2.2.1 *If A_0 and A_1 are two c.e. sets such that $R(A_0, A_1)$ holds, then A_0 is not of prompt degree.*

Proof. The proof is similar to the corresponding result for the Q-property in [21]. Given A_0 and A_1, we pick a set C as specified in $R(A_0, A_1)$ and fix enumerations $\{A_0^s\}_{s\in\omega}$ of A_0 and $\{C^s\}_{s\in\omega}$ of C such that $A_0 \subseteq C \searrow A_0$.

To prove that a given φ_e is not a promptness function for A_0, we need to find an infinite c.e. set W_i with standard enumeration $\{W_{i,s}\}_{s\in\omega}$ satisfying the tardiness requirement $\mathcal{T}_e$:

$$[(\forall s)\varphi_e(s)\!\downarrow\, \geq s] \implies (\forall x)(\forall s)[x \in W_{i,s} - W_{i,s-1} \implies A_0^s \restriction x = A_0^{\varphi_e(s)} \restriction x\ ].$$

We will prove independently for each e that $\mathcal{T}_e$ holds. Having fixed e, we will assume for the rest of this section that φ_e is total with $\varphi_e(s) \geq s$ for every s, since otherwise $\mathcal{T}_e$ is automatically fulfilled. We will build a strong array

$\{V_{\langle\alpha,k\rangle,n}\}_{k,n\in\omega;\alpha\in\omega\times\omega}$ of c.e. sets with enumerations $\{V^s_{\langle\alpha,k\rangle,n}\}_{s\in\omega}$. The Slowdown Lemma then gives a computable function f such that for each $\langle\alpha,k\rangle$ and each n, $W_{f(\langle\alpha,k\rangle,n)} = V_{\langle\alpha,k\rangle,n}$ and $V_{\langle\alpha,k\rangle,n} \searrow W_{f(\langle\alpha,k\rangle,n)} = V_{\langle\alpha,k\rangle,n}$, so that no element of $V_{\langle\alpha,k\rangle,n}$ enters $W_{f(\langle\alpha,k\rangle,n)}$ until it has already entered $V_{\langle\alpha,k\rangle,n}$. Periodically the strategy for a given $\langle\alpha,k\rangle$ may be injured by a higher-priority strategy. If this happens while we are enumerating $V_{\langle\alpha,k\rangle,n}$, then we give up on $V_{\langle\alpha,k\rangle,n}$ and start enumerating $V_{\langle\alpha,k\rangle,n+1}$. There will exist an $\langle\alpha,k\rangle$ which is only injured n times (with $n<\omega$), yet receives attention at infinitely many stages, and the corresponding $V_{\langle\alpha,k\rangle,n}$ will be infinite and will be the set which proves satisfaction of $\mathcal{T}_e$.

We define the function $n(\langle\alpha,k\rangle,s)$ to keep track of which $V_{\langle\alpha,k\rangle,n}$ we are enumerating at stage s. In particular, if the $\langle\alpha,k\rangle$-strategy receives attention at stage $s+1$, then we may add an element to $V^{s+1}_{\langle\alpha,k\rangle,n(\langle\alpha,k\rangle,s+1)}$. To avoid notational chaos, however, we will write $V^{s+1}_{\langle\alpha,k\rangle,n}$ in the construction and understand $V^{s+1}_{\langle\alpha,k\rangle,n(\langle\alpha,k\rangle,s+1)}$ for it.

To ensure that one of these $W_{f(\langle\alpha,k\rangle,n)}$ will satisfy $\mathcal{T}_e$, we build a c.e. set B to which to apply the property R. When we want to preserve $A_0 \upharpoonright x$ from stage s until stage $\varphi_e(s)$ so as to satisfy $\mathcal{T}_e$, we do so by restraining all elements $<x$ from entering B until stage $\varphi_e(s)$. The R-property then prohibits such elements from entering A_0, since if they did, we would then hold them out of B forever after, thereby contradicting $R(A_0,A_1)$.

To apply the R-property, we need to know which c.e. set W_i is the D specified by the property. Of course, we do not have this information, but our strategy is to use S to cover all the possibilities. Specifically, in the construction we will split C into the disjoint union of c.e. sets:

$$C = \bigsqcup_{i\in\omega} S_i.$$

and apply the R-property to each S_i, with S_i in the role of S. (Clearly each $S_i \sqsubset C$.) We use each S_i to handle the possibility that $D = W_i$.

Of course, the R-property states that the restraints we place on elements from entering B only affect A_0 on $S \cap T \cap \overline{A_1}$. Since $R(A_0, A_1)$ also states that $A_0 \cap A_1$ is empty, we do not need to worry about elements of A_1, for they can never enter A_0. We are allowed to choose the S, since the matrix of R applies for all S, and indeed we have already done so above (namely $S = S_i$, for each i in turn). However, we can only guess at the set T.

To determine the index j such that $T = W_j$ corresponds to the set S which we choose, we use a Π^0_2 guessing procedure, since the conclusion in the matrix of R is a Π^0_2 property. The j for which $T = W_j$ will be the least j which receives infinitely many guesses under this procedure. (We ensure that the hypothesis of the matrix holds, by periodically putting all elements of $D^s \cap (S^s - A^s_0)$ into B^s.) Moreover, in the construction, we will subdivide each S_i into the disjoint union of c.e. sets $S_{i,j}$:

$$S_i = \bigsqcup_{j \in \omega} S_{i,j}.$$

$S_{i,j}$ is used to handle the possibility that $T = W_j$, so we pay attention to $S_{i,j}$ each time j is named by the guessing procedure. Thus the $S_{i,j}$ corresponding to the correct T will receive attention infinitely often.

To simplify the notation, we let the variable $\alpha = \langle i, j \rangle$ range over $\omega \times \omega$, and define:

$$\begin{aligned} D_\alpha &= W_i \\ S_\alpha &= S_{i,j} \\ T_\alpha &= W_j. \end{aligned}$$

We order the elements α of $\omega \times \omega$ by pulling back the usual order $<$ on ω to $\omega \times \omega$ via a standard pairing function. Thus each α has only finitely many predecessors under $<$.

For each α, let $F(\alpha)$ be the conjunction of the hypothesis and conclusion in the matrix of the R-property:

$$F(\alpha): \qquad (B \cap (S_\alpha - A_0)) \cup A_1 = (D_\alpha \cap (S_\alpha - A_0)) \cup A_1 \ \& \tag{2.1}$$

$$[\overline{C} \subset T_\alpha \ \& \ (A_0 \cap S_\alpha \cap T_\alpha) \cup A_1 = (B \cap S_\alpha \cap T_\alpha) \cup A_1] \tag{2.2}$$

Then $F(\alpha)$ is a Π^0_2 condition, uniformly in α, so there is a computable total function g such that $F(\alpha)$ holds just if $g^1(\alpha)$ is infinite. We enumerate the c.e. set $Z_\alpha = g^1(\alpha)$ by setting $Z^s_\alpha = \{t \leq s : g(t) = \alpha\}$.

Now we narrow down each T_α to a c.e. subset U_α, enumerated by:

$$U^s_\alpha = U^{s-1}_\alpha \cup \{x \in T^s_\alpha - C^s : x < |Z^s_\alpha|\}$$

Thus, if T_α actually is the T corresponding to S_i, then U_α will contain all of T_α except certain elements of C. Hence $F(\alpha)$ will hold with U_α in place of T_α. On the other hand, if $F(\alpha)$ fails, then Z_α and U_α are both finite.

If $F(\alpha)$ holds, then $\overline{C} \subseteq U_\alpha$, so $\overline{A_0} \subseteq^* U_\alpha \cup A_1$, because $A_0 \cup A_1 \subset_{\mathrm{m}} C$. For the least α such that $F(\alpha)$ holds, our construction of S^{s+1}_α will yield $C - A_0 \subseteq^* S_\alpha \cup A_1$, with S_β finite for all $\beta < \alpha$. Hence there will exist a k such that

$$C - A_0 \subseteq S_\alpha \cup A_1 \cup \{0, 1, \ldots k-1\} \tag{2.3}$$

Line (2.3) is a Π^0_2 statement, uniformly in k and α, since our definition of S_α will be uniform in α. Therefore, there exists a total function h_α such that (2.3) holds if and only if $h^{-1}_\alpha(k)$ is infinite. We define:

$$h(s) = h_{g(s)}(n), \text{ where } n = |\{t < s : g(t) = g(s)\}|.$$

We will enumerate sets $V_{\langle\alpha,k\rangle,n}$ for each α, k and n. For the least α with Z_α infinite and the least k with $h^{-1}_\alpha(k)$ infinite, the set $V_{\langle\alpha,k\rangle,n}$ (for some n) will be the W_i required by $\mathcal{T}_e$. Elements of each $V_{\langle\alpha,k\rangle,n}$ (the "witness elements" for the requirement $\mathcal{T}_e$) will be denoted $v^s_{\langle\alpha,k\rangle}$. Each $v^s_{\langle\alpha,k\rangle}$ will enter $V_{\langle\alpha,k\rangle,n}$ for at most one n.

The Slowdown Lemma (see [47], p. 284) then yields a computable function f such that, for every $\langle\alpha,k\rangle$ and every n, $V_{\langle\alpha,k\rangle,n} = W_{f(\langle\alpha,k\rangle,n)}$, and at every stage s,

$$(V^s_{\langle\alpha,k\rangle,n} - V^{s-1}_{\langle\alpha,k\rangle,n}) \cap W_{f(\langle\alpha,k\rangle,n),s} = \emptyset.$$

When a witness element $v^s_{\langle\alpha,k\rangle}$ enters $V_{\langle\alpha,k\rangle,n}$, we will find the stage $t^s_{\langle\alpha,k\rangle} > s$ at which $v^s_{\langle\alpha,k\rangle}$ enters $W_{f(\langle\alpha,k\rangle,n)}$ and restrain (with priority $\langle\alpha,k\rangle$) elements $\leq v^s_{\langle\alpha,k\rangle}$ from entering A_0 until stage $\varphi_e(t^s_{\langle\alpha,k\rangle})$. (Recall that $\mathcal{T}_e$ assumes φ_e to be total.) Thus we will have $A_0^{t^s_{\langle\alpha,k\rangle}} \restriction v^s_{\langle\alpha,k\rangle} = A_0^{\varphi_e(t^s_{\langle\alpha,k\rangle})} \restriction v^s_{\langle\alpha,k\rangle}$. If we can achieve this for all $v^s_{\langle\alpha,k\rangle}$ in the (infinite) set $V_{\langle\alpha,k\rangle,n}$ for some n, then the set $W_{f(\langle\alpha,k\rangle,n)}$ will be the set required by $\mathcal{T}_e$ to prove that φ_e is not a promptness function for A_0.

At stage 0, for all $\langle\alpha,k\rangle$, we set $n(\langle\alpha,k\rangle,0) = 0$ and $V^0_{\langle\alpha,k\rangle,0} = \emptyset$, with $v^0_{\langle\alpha,k\rangle}\uparrow$ and $t^0_{\langle\alpha,k\rangle}\uparrow$. Also, let every $S^0_\alpha = \emptyset$ and let $B^0 = \emptyset$.

At stage $s+1$, we first define each S^{s+1}_α. For each $x \in C^{s+1} - C^s$, find the least α such that $x \in U^s_\alpha$ and put x into S^{s+1}_α. If there is no such α, put x into S^{s+1}_ω. (The c.e. set S_ω simply collects elements which enter C without entering any S_α. Thus $C = \bigsqcup_{\alpha\leq\omega} S_\alpha$.)

Set $\alpha = g(s)$, and define:

$$B^{s+1} = B^s \cup \left\{ x : \begin{array}{c} x \in C^s - A^s_0 \;\&\; (\exists\beta \leq \alpha)[x \in D^{s+1}_\beta \cap S^{s+1}_\beta \;\&\; \\ (\forall\delta \leq \beta)(\forall k < s)[t^s_{\langle\delta,k\rangle}\downarrow \implies x \geq v^s_{\langle\delta,k\rangle}]] \end{array} \right\}$$

For each strategy which is injured at stage $s+1$, we begin enumerating a new witness set. To this end, set $n(\langle\gamma,k\rangle,s+1) = n(\langle\gamma,k\rangle,s)+1$ and $v^{s+1}_{\langle\gamma,k\rangle}\uparrow$ and $t^{s+1}_{\langle\gamma,k\rangle}\uparrow$ for each $\langle\gamma,k\rangle$ satisfying any of the following conditions:

- $\gamma > \alpha$.
- $\gamma = \alpha$ and $k > h(s)$.
- There exists $x < k$ with $x \in A^{s+1}_0 - A^s_0$.
- There exists $\beta < \gamma$ with $S^{s+1}_\beta \neq S^s_\beta$.

- There exists $\beta < \gamma$ such that U_β^{s+1} contains an element $\geq m$, where $m = \min(B^{s+1} - B^s)$.

For all other $\langle \gamma, k \rangle$, set $n(\langle \gamma, k \rangle, s+1) = n(\langle \gamma, k \rangle, s)$.

We now define the witness sets at stage $s+1$. For each $\langle \beta, k \rangle \leq \langle \alpha, h(s) \rangle$ (in the lexicographic order) which was not injured at stage $s+1$:

1. If $v^s_{\langle \beta,k \rangle} \uparrow$ and $\langle \beta, k \rangle \neq \langle \alpha, h(s) \rangle$, let $v^{s+1}_{\langle \beta,k \rangle}$ and $t^{s+1}_{\langle \beta,k \rangle}$ diverge also, with $V^{s+1}_{\langle \beta,k \rangle,n} = V^s_{\langle \beta,k \rangle,n}$.

2. If $v^s_{\langle \alpha,h(s) \rangle} \uparrow$, let $v^{s+1}_{\langle \alpha,h(s) \rangle} = s+1$, with $V^{s+1}_{\langle \alpha,h(s) \rangle,n} = V^s_{\langle \alpha,h(s) \rangle,n}$ and $t^{s+1}_{\langle \alpha,h(s) \rangle} \uparrow$.

3. If $v^s_{\langle \beta,k \rangle} \downarrow$ but $t^s_{\langle \beta,k \rangle} \uparrow$, let $v^{s+1}_{\langle \beta,k \rangle} = v^s_{\langle \beta,k \rangle}$, and ask whether the following holds:
$$(\forall y)_{k \leq y \leq v^{s+1}_{\langle \beta,k \rangle}} \left[\begin{array}{c} y \in A_0^{s+1} \vee y \in A_1^{s+1} \vee \\ y \in (U_\beta^{s+1} - C^{s+1}) \vee \\ y \in (C^{s+1} - B^{s+1}) \cap S_\beta^{s+1} \cap U_\beta^{s+1} \end{array} \right] \tag{2.4}$$
If (2.4) holds, let $V^{s+1}_{\langle \beta,k \rangle,n} = V^s_{\langle \beta,k \rangle,n} \cup \{v^{s+1}_{\langle \beta,k \rangle}\}$ and
$$t^{s+1}_{\langle \beta,k \rangle} = \mu t[v^{s+1}_{\langle \beta,k \rangle} \in W_{f(\langle \beta,k \rangle,n),t}].$$
(Such a t must exist, since $W_{f(\langle \beta,k \rangle,n)} = V_{\langle \beta,k \rangle,n}$.) If (2.4) fails, then let $V^{s+1}_{\langle \beta,k \rangle,n} = V^s_{\langle \beta,k \rangle,n}$ and $t^{s+1}_{\langle \beta,k \rangle} \uparrow$.

4. If $v^s_{\langle \beta,k \rangle} \downarrow$ and $t^s_{\langle \beta,k \rangle} \downarrow$ and $\varphi_{e,s}(t^s_{\langle \beta,k \rangle}) \downarrow < s$, then let $v^{s+1}_{\langle \beta,k \rangle} \uparrow$ and $t^{s+1}_{\langle \beta,k \rangle} \uparrow$, with $V^{s+1}_{\langle \beta,k \rangle,n} = V^s_{\langle \beta,k \rangle,n}$.

5. If $v^s_{\langle \beta,k \rangle} \downarrow$ and $t^s_{\langle \beta,k \rangle} \downarrow$ but either $\varphi_{e,s}(t^s_{\langle \beta,k \rangle}) \downarrow \geq s$ or $\varphi_{e,s}(t^s_{\langle \beta,k \rangle})$ diverges, then let $V^{s+1}_{\langle \beta,k \rangle,n} = V^s_{\langle \beta,k \rangle,n}$, $v^{s+1}_{\langle \beta,k \rangle} = v^s_{\langle \beta,k \rangle}$, and $t^{s+1}_{\langle \beta,k \rangle} = t^s_{\langle \beta,k \rangle}$.

This completes the construction.

We now use the sets B and S_α to prove that requirement $\mathcal{T}_e$ is satisfied.

Lemma 2.2.2 *If Z_β is finite, then there exists a stage s_1 such that $t^s_{\langle \beta,k \rangle} \uparrow$ for all $s \geq s_1$ and all k.*

Proof. Pick a stage s_0 such that no $s \geq s_0$ satisfies $g(s) = \beta$, and let $k' = \max\{h(s) : g(s) = \beta\}$. Then for all $k > k'$, $v^s_{\langle\beta,k\rangle}\uparrow$ for all s, and hence $t^s_{\langle\beta,k\rangle}\uparrow$ for all s. (The construction makes it clear that for any k and s, $t^s_{\langle\beta,k\rangle}$ can converge only if $v^s_{\langle\beta,k\rangle}$ converges.)

Now suppose $k \leq k'$ and $v^s_{\langle\beta,k\rangle}\downarrow$ for all $s \geq s_0$. This means that we never execute Step (4) in the construction after stage s_0, and that the $\langle\beta, k\rangle$ strategy is never injured after stage s_0. But if $t^s_{\langle\beta,k\rangle}$ ever converges after stage s_0, then eventually we must reach Step (4), since we assumed φ_e to be total. Hence $t^s_{\langle\beta,k\rangle}$ must diverge for all $s \geq s_0$.

Finally, suppose $k \leq k'$ and $v^{s_{1,k}}_{\langle\beta,k\rangle}\uparrow$ for some $s_{1,k} \geq s_0$. Then $v^s_{\langle\beta,k\rangle}$ will diverge for all subsequent s, since it can only be newly defined at a stage s with $g(s) = \beta$. Thus $t^s_{\langle\beta,k\rangle}$ will diverge for all subsequent s as well. Letting $s_1 = \max_{k\leq k'} s_{1,k}$ completes the proof. ∎

Lemma 2.2.3 *$F(\alpha)$ holds for some α, and for the least such α, there exists a k such that $h^{-1}_\alpha(k)$ is infinite.*

Proof. First we claim that some Z_α must be infinite. Suppose not, so Z_α is finite for all α, and $F(\alpha)$ fails for all α. However, the R-property holds, so there must be some α for which line (2.1) fails. Choose the least such α. Then

$$(B \cap (S_\alpha - A_0)) \cup A_1 \neq (D_\alpha \cap (S_\alpha - A_0)) \cup A_1.$$

Suppose $x \in B \cap (S_\alpha - A_0)$. Pick s such that $x \in B^{s+1} - B^s$. Now to go into B^{s+1}, x must have been in $D^{s+1}_\beta \cap S^{s+1}_\beta$ for some β. Since $x \in S_\alpha$, we know $x \notin S_\beta$ for all $\beta \neq \alpha$. Hence $x \in D_\alpha$, and so

$$(B \cap (S_\alpha - A_0)) \cup A_1 \subseteq (D_\alpha \cap (S_\alpha - A_0)) \cup A_1.$$

Therefore, there must be some element $x \in \overline{A_1} \cap \overline{B} \cap D_\alpha \cap (S_\alpha - A_0)$. Assume x is the least such element. Now for every $\beta < \alpha$, line (2.1) must hold and line

(2.2) must fail, since we chose α to be minimal satisfying the R-property. Hence for all $\beta < \alpha$,

$$(B \cap (S_\beta - A_0)) \cup A_1 = (D_\beta \cap (S_\beta - A_0)) \cup A_1.$$

Now since every Z_β with $\beta \leq \alpha$ is finite, there is a stage s_0 such that for all $s \geq s_0$, $g(s) > \alpha$, and we may also assume that s_0 is so large that $x \in S_\alpha^{s_0} \cap D_\alpha^{s_0} \cap C^{s_0}$. (Notice that $x \in S_\alpha$ forces $x \in C$.)

Now use Lemma 2.2.2 to find a stage $s_1 \geq s_0$ such that:

$$(\forall s \geq s_1)(\forall \beta \leq \alpha)(\forall k)[t^{s_1}_{\langle \beta, k\rangle}\uparrow].$$

Since φ_e is total, there must be a stage $s \geq s_1$ such that $t^s_{\langle \alpha, k\rangle}\uparrow$, and once we reach this stage s, x must go into B^{s_1+1}, contradicting our assumption that $x \notin B$.

Thus, there must be some α such that Z_α is infinite. Let α be the least such. Then every U_β with $\beta < \alpha$ is finite. Since $F(\alpha)$ holds, we have $\overline{C} \subseteq T_\alpha$, so by our construction, $\overline{C} \subseteq U_\alpha$, and by the major subset property, $\overline{A_0} \subseteq^* U_\alpha \cup A_1$.

For this α, we claim that $C - A_0 \subseteq^* S_\alpha \cup A_1$. Suppose $x \in C - A_0$. All but finitely many such x lie in $U_\alpha \cup A_1$, as noted above. If $x \in A_1$, we are done. For each sufficiently large $x \in C - A_0 - A_1$, there exists s such that $x \in U_\alpha^s - U_\alpha^{s-1}$. By definition of U_α^s, we must have $x \notin C^s$. But $x \in C$, so $x \in C^{t+1} - C^t$ for some $t \geq s$. Hence $x \in S_\alpha^{t+1}$ by definition of S_α^{t+1}, unless there exists $\beta < \alpha$ with $x \in U_\beta$. But all U_β with $\beta < \alpha$ are finite, by our choice of α, so all but finitely many of these x lie in S_α. Therefore, line (2.3) holds for some k, and $h_\alpha^{-1}(k)$ is infinite. ∎

Use Lemma 2.2.3 to take the lexicographically least $\langle \alpha, k\rangle$ such that $F(\alpha)$ holds and $h_\alpha^{-1}(k)$ is infinite. Then there are infinitely many stages s for which $g(s) = \alpha$ and $h(s) = k$, but only finitely many for which $\langle g(s), h(s)\rangle$ precedes $\langle \alpha, k\rangle$ in the lexicographic ordering. Let s_0 be the least stage with $\langle g(s_0), h(s_0)\rangle = \langle \alpha, k\rangle$ such that:

- $A_0^{s_0}\restriction k = A_0\restriction k$, and

- $B^{s_0}\upharpoonright m = B\upharpoonright m$, where $m = \max \cup_{\beta<\alpha} U_\beta$, and
- for all $s \geq s_0$, $\langle g(s), h(s)\rangle \geq \langle \alpha, k\rangle$ lexicographically, and
- $S_\beta^{s_0} = S_\beta$ for all $\beta < \alpha$.

The final condition is possible since each $S_\beta \subseteq U_\beta$, which is finite for every $\beta < \alpha$. We also let $s_0 < s_1 < s_2 < \cdots$ be all the stages $s \geq s_0$ with $\langle g(s), h(s)\rangle = \langle \alpha, k\rangle$.

Now the $\langle \alpha, k\rangle$-strategy is never injured after stage s_0, so for every $s \geq s_0$, $n(\langle \alpha, k\rangle, s_0) = n(\langle \alpha, k\rangle, s)$, and we write $n = n(\langle \alpha, k\rangle, s_0)$. (Thus n is the number of times the $\langle \alpha, k\rangle$-strategy was injured during the construction.) Moreover, minimality of s_0 implies that this strategy was injured at some stage $s \leq s_0$ such that there is no s_{-1} with $s \leq s_{-1} < s_0$ and $\langle g(s_{-1}), h(s_{-1})\rangle = \langle \alpha, k\rangle$. Therefore, $V^s_{\langle \alpha,k\rangle,n} = V^{s_0}_{\langle \alpha,k\rangle,n}$ is empty.

We claim that the subset $V_{\langle \alpha,k\rangle,n}$ satisfies requirement $\mathcal{T}_e$. For this we need:

Lemma 2.2.4 *For this $\langle \alpha, k\rangle$, and for each $y \geq k$, there exists an s such that the matrix of line (2.4) holds of y, $\langle \alpha, k\rangle$, and s.*

Proof. Let $y \geq k$. If $y \in A_0 \cup A_1$, we are done. If $y \in \overline{C}$, then $y \in T_\alpha$ since $F(\alpha)$ holds. But Z_α is infinite, so $T_\alpha - C \subseteq U_\alpha$, and y is in $U_\alpha - C$, hence in some $U_\alpha^{s+1} - C^{s+1}$.

So suppose $y \in C - A_0 - A_1$. Now since $h_\alpha^{-1}(k)$ is infinite and $y \geq k$, we know by line (2.3) that $y \in S_\alpha$. But $S_\alpha \subseteq U_\alpha \subseteq T_\alpha$ by definition of S_α^{s+1}. Since $y \notin (B \cap S_\alpha \cap T_\alpha) \cup A_1$ by line (2.2), we know $y \notin B$. Thus there is an s with $y \in (C^{s+1} - B^{s+1}) \cap S_\alpha^{s+1} \cap U_\alpha^{s+1}$. This proves the Lemma. ∎

Now $V_{\langle \alpha,k\rangle,n} = W_{f(\langle \alpha,k\rangle,n)}$, and if s' is the stage at which $v^{s'}_{\langle \alpha,k\rangle}$ enters $V_{\langle \alpha,k\rangle,n}$, then $t^{s'}_{\langle \alpha,k\rangle}\downarrow > s'$ by our choice of f from the Slowdown Lemma. Let $s'' = \varphi_e(t^{s'}_{\langle \alpha,k\rangle})$. Then $s' < s''$, since we assumed φ_e to be increasing.

Lemma 2.2.5 *$V_{\langle \alpha,k\rangle,n}$ is infinite. Moreover, for any element $v^{s'}_{\langle \alpha,k\rangle}$ of $V_{\langle \alpha,k\rangle,n}$, with s' and s'' as above, we have:*

$$B^{s'}\upharpoonright v^{s'}_{\langle \alpha,k\rangle} = B^{s''}\upharpoonright v^{s'}_{\langle \alpha,k\rangle} \quad \textit{and} \quad A_0^{s'}\upharpoonright v^{s'}_{\langle \alpha,k\rangle} = A_0^{s''}\upharpoonright v^{s'}_{\langle \alpha,k\rangle}.$$

Proof. For each $v^s_{\langle\alpha,k\rangle}$ with $s \geq s_0$, Lemma 2.2.4 guarantees that there will be a stage at which Step (3) of the construction applies. The first such stage will be s', since at that stage $v^s_{\langle\alpha,k\rangle} = v^{s'}_{\langle\alpha,k\rangle}$ will enter $V_{\langle\alpha,k\rangle,n}$ and $t^{s'}_{\langle\alpha,k\rangle}$ will be defined. But since φ_e is total, we will eventually reach the stage $s'' > s'$ at which Step (4) applies, leaving $v^{s''+1}_{\langle\alpha,k\rangle}$ undefined. Then at the next $s_m > s''$, we will define $v^{s_m+1}_{\langle\alpha,k\rangle} = s_m + 1$, which is not yet in $V^{s_m}_{\langle\alpha,k\rangle,n}$. Thus, $V_{\langle\alpha,k\rangle,n}$ must be infinite.

Now pick $v^{s'}_{\langle\alpha,k\rangle} \in V_{\langle\alpha,k\rangle,n}$, with s' and s'' as above. Since $V^{s_0}_{\langle\alpha,k\rangle,n}$ is empty, we know that $s' > s_0$. If s is any stage with $s' \leq s < s''$, then we see from the definition of B^{s+1} that an element y can only enter B^{s+1} on behalf of some γ such that $y \in S^{s+1}_\gamma$. But then $y \in U^{s+1}_\gamma$. Since we chose s_0 to let $B^{s_0}\restriction m = B\restriction m$, we must have $\gamma \geq \alpha$. But $t^s_{\langle\alpha,k\rangle}\downarrow$, so $y \geq v^s_{\langle\alpha,k\rangle} = v^{s'}_{\langle\alpha,k\rangle}$ by definition of B^{s+1}. Hence $B^{s'}\restriction v^{s'}_{\langle\alpha,k\rangle} = B^{s''}\restriction v^{s'}_{\langle\alpha,k\rangle}$.

Having seen that no $y < v^{s'}_{\langle\alpha,k\rangle}$ can enter B between stages s' and s'', we prove that no such y can enter A_0 at those stages either. First, we know that $A^{s_0}_0\restriction k = A_0\restriction k$ by choice of s_0. So suppose $k \leq y < v^{s'}_{\langle\alpha,k\rangle}$. Now since $v^{s'}_{\langle\alpha,k\rangle}$ entered $V_{\langle\alpha,k\rangle,n}$ at stage s', we know by line (2.4) that

$$y \in A^{s'}_0 \vee y \in A^{s'}_1 \ \vee y \in (U^{s'}_\alpha - C^{s'}) \ \vee y \in (C^{s'} - B^{s'}) \cap S^{s'}_\alpha \cap U^{s'}_\alpha.$$

If $y \in A^{s'}_0$, then $A^{s'}_0(y) = A^{s''}_0(y)$, and if $y \in A_1$, then $y \notin A_0$ at all. Therefore, we will assume that $y \notin A^{s'}_0 \cup A_1$ and prove that $y \notin A^{s''}_0$.

If the final clause holds, then $y \in (C^{s'} - B^{s'}) \cap S^{s'}_\alpha \cap U^{s'}_\alpha$. Hence $y \notin B^{s''}$, by the first half of the lemma. If $y \in A^{s''}_0$, then $y \notin B$, since no element that has entered A_0 can later enter B. But then

$$(A_0 \cap S_\alpha \cap T_\alpha) \cup A_1 \neq (B \cap S_\alpha \cap T_\alpha) \cup A_1$$

since y is on the left side and not on the right side. (Notice that $y \in U_\alpha$ implies $y \in T_\alpha$.) This contradicts line (2.2), which we knows holds because $F(\alpha)$ holds. Therefore $y \notin A^{s''}_0$.

So suppose the third clause holds, i.e. $y \in (U_\alpha^{s'} - C^{s'})$. Then $y \notin B^{s'}$ since $B^{s'} \subseteq C^{s'}$, and so $y \notin B^{s''}$. If $y \in A_0^{s''}$, then we must have $y \in C^{s''-1}$ since we chose enumerations such that $A_0 \subseteq C \searrow A_0$. Pick s such that $y \in C^s - C^{s-1}$; then $s' < s < s''$ and $y \notin A_0^s$. Now $y \in U_\alpha^{s'} \subseteq T_\alpha^{s'}$, and by definition of S_α^s we will have $y \in S_\alpha^s$. (Recall that s_0 was chosen so large that $S_\beta^{s_0} = S_\beta$ for all $\beta < \alpha$.) But now $y \notin A_0^{s''}$, since otherwise

$$(A_0 \cap S_\alpha \cap T_\alpha) \cup A_1 \neq (B \cap S_\alpha \cap T_\alpha) \cup A_1$$

just as in the preceding paragraph. ■

Hence $V_{\langle\alpha,k\rangle,n} = W_{f(\langle\alpha,k\rangle,n)}$ is an infinite c.e. set which satisfies the tardiness requirement $\mathcal{T}_e$. This completes the proof of Theorem 2.2.1. ■

2.3 Satisfaction of R

We now prove that the R-property defined in Section 2.2 is nontrivial. The theorem establishes several other properties of the sets A_0 and A_1 as well, in order to yield the corollaries.

Theorem 2.3.1 *There exists a c.e. set A with Friedberg splitting $A = A_0 \sqcup A_1$ such that all of the following hold:*

1. *A is promptly simple of high degree.*
2. *A_1 has prompt degree.*
3. *$R(A_0, A_1)$.*

Corollary 2.3.2 *The formula in one free variable A_0:*

$$(\exists A_1)[A_0 >_T \emptyset \;\&\; R(A_0, A_1)]$$

is definable in $\mathcal{E}$ and non-vacuous, and implies that A_0 is a noncomputable incomplete set.

Proof of Corollary. The statement $A_0 >_T \emptyset$ is equivalent to the statement that A_0 has a complement in $\mathcal{E}$, hence is $\mathcal{E}$-definable. The A_0 and A_1 constructed in Theorem 2.3.1 satisfy the matrix, since halves of a Friedberg splitting must be noncomputable. Finally, Theorem 2.2.1 shows that A_0 is tardy, hence incomplete. ■

Corollary 2.3.3 *There exists a Friedberg splitting $A = A_0 \sqcup A_1$ such that A_0 and A_1 are not automorphic in the lattice of c.e. sets.*

Proof of Corollary. Take the splitting given by Theorem 2.3.1. A_1 is prompt, hence automorphic to a complete set (as shown in [21]). If A_0 and A_1 were automorphic, then A_0 would also be automorphic to that complete set, say via an automorphism Φ. But then $R(\Phi(A_0), \Phi(A_1))$ holds, since R is $\mathcal{E}$-definable, so by Theorem 2.2.1, $\Phi(A_0)$ is tardy, hence incomplete. ■

Proof of Theorem. Let C be any promptly simple set, with computable enumeration $C = \{C^s\}_{s\in\omega}$. Then C is also of prompt degree, so let v and w be the prompt-simplicity and promptness functions for this enumeration of C, satisfying for every i:

$$
\begin{aligned}
W_i \text{ infinite } &\implies (\exists^\infty s)(\exists x \in W_{i,s} - W_{i,s-1})[x \in C^{v(s)}] \\
W_i \text{ infinite } &\implies (\exists^\infty s)(\exists x \in W_{i,s} - W_{i,s-1})[C^{w(s)}\restriction x \neq C^s\restriction x]
\end{aligned}
$$

We construct disjoint sets A_0 and A_1 and auxiliary sets D_i and $T_{i,j}$, and set $A = A_0 \sqcup A_1$. The approximations to A, A_0, and A_1 at stage s will be written A^s, A_0^s, and A_1^s, and will be defined so that $A^s = A_0^s \cup A_1^s \subseteq C^s$ for all s. The construction will satisfy the following requirements for all i and j:

$\mathcal{N}_{\langle i,j\rangle}$ *(matrix of R-property)* :
$[W_i \subseteq C \;\&\; W_j \subseteq C \;\&\; C - W_j$ c.e. &
$(W_i \cap (W_j - A_0)) \cup A_1 = (D_i \cap (W_j - A_0)) \cup A_1] \implies$
$(\exists T)[\overline{C} \subseteq T \;\&\; (A_0 \cap W_j \cap T) \cup A_1 =^* (W_i \cap W_j \cap T) \cup A_1]$

$\mathcal{M}_i$ *(major subset requirement)* :
$\overline{C} \subseteq W_i \implies \overline{A} \subseteq^* W_i$

$\mathcal{P}_i$ *(prompt simplicity of A)* :
W_i infinite $\implies (\exists s)(\exists x \in W_{i,s} - W_{i,s-1})[x \in A^{v(s)}]$

$\mathcal{Q}_i$ *(promptness of A_1)* :
W_i infinite $\implies (\exists s)(\exists x \in W_{i,s} - W_{i,s-1})[A_1^{w(s)}\restriction x \neq A_1^s\restriction x]$

$\mathcal{F}_i$ *(Friedberg requirement for A_0)* :
$W_i \searrow A$ infinite $\implies W_i \cap A_0 \neq \emptyset$

$\mathcal{G}_i$ *(Friedberg requirement for A_1)* :
$W_i \searrow A$ infinite $\implies W_i \cap A_1 \neq \emptyset$

In the requirement $\mathcal{N}_{\langle i,j\rangle}$, of course, W_i plays the role of B and W_j the role of S in the matrix of the R-property. We will construct c.e. sets $T_{i,j}$ for each i and j, and then refine them to form the T demanded by each $\mathcal{N}_{\langle i,j\rangle}$ Once again we order

$\omega \times \omega$ in order type ω and write $\alpha = \langle i, j \rangle$, this time with:

$$\begin{aligned} B_\alpha &= W_i \\ D_\alpha &= D_i \\ \left.\begin{aligned} S_\alpha &= W_{j'} \\ \hat{S}_\alpha &= W_{j''} \end{aligned}\right\} &\text{ where } j = \langle j', j'' \rangle \\ T_\alpha &= T_{i,j} \\ \mathcal{N}_\alpha &= \mathcal{N}_{i,j}. \end{aligned}$$

Thus $\mathcal{N}_\alpha$ says:

$$\begin{gathered} [B_\alpha \subseteq C \;\&\; S_\alpha \sqcup \hat{S}_\alpha = C \;\& \\ (B_\alpha \cap (S_\alpha - A_0)) \cup A_1 = (D_\alpha \cap (S_\alpha - A_0)) \cup A_1] \\ \Longrightarrow\; (\exists T)[\overline{C} \subseteq T \;\&\; (A_0 \cap S_\alpha \cap T) \cup A_1 =^* (B_\alpha \cap S_\alpha \cap T) \cup A_1]. \end{gathered}$$

$\mathcal{N}_\alpha$ is a negative requirement, trying to keep elements from entering A_0 until they can do so without harming the R-property (if ever). All the other requirements are positive ones, trying to put elements into A_0 or A_1. There are no negative restraints on elements of C entering A_1, except that they cannot already be in A_0.

Each element which we try to put into A_0 to satisfy some $\mathcal{F}_e$ or $\mathcal{M}_e$ must receive permission to enter A_0 from each $\mathcal{N}_\alpha$ with $\alpha \leq e$. The *restraint function* $q(x, s)$ will give the greatest $\alpha \leq e$ which has not yet given this permission as of stage s. The *priority function* $p(x, s)$ keeps track of which requirement $\mathcal{F}_e$ or $\mathcal{M}_e$ wanted x to enter A_0. This can change from stage to stage, for several reasons. If a higher-priority requirement decides at stage $s + 1$ that it needs x to enter A_0, then $p(x, s + 1) < p(x, s)$. Alternatively, an $\mathcal{F}_e$ could find itself satisfied by another $x' \in A_0^{s+1}$ and no longer need to put x into A_0, although in this case we leave $p(x, s + 1) = p(x, s)$ so as not to disrupt the flow of elements into A_0. Finally, a higher-priority requirement could make x enter A_1^{s+1}, in which case we define $p(x, s + 1)\uparrow$, removing x from the flow of elements into A_0 since we need $A_0 \cap A_1 = \emptyset$.

We use the Recursion Theorem on our construction of A_0, C, and D_α to define the following Π^0_2 statement $F(\alpha)$ for each α:

$$(B_\alpha \cap (S_\alpha - A_0)) \cup A_1 = (D_\alpha \cap (S_\alpha - A_0)) \cup A_1 \ \& \ B_\alpha \subseteq C \ \& \ S_\alpha \sqcup \hat{S}_\alpha = C.$$

Since $F(\alpha)$ is Π^0_2, there is a computable function $g : \omega \to \omega \times \omega$ such that $F(\alpha)$ holds if and only if the set $Z_\alpha = g^{-1}(\alpha)$ is infinite. We let $Z^s_\alpha = g^{-1}(\alpha) \cap \{0, 1, \dots s-1\}$. Monitoring $|Z^s_\alpha|$ will help us determine for which α the hypothesis in the matrix of the R-property is satisfied. For those α for which the hypothesis fails, $|Z_\alpha|$ is finite, and $\mathcal{N}_\alpha$ will only restrain finitely many elements from entering A_0, since we need not satisfy the conclusion of the R-property for such an α.

At stage $s = 0$, we set $A^0_0 = A^0_1 = \emptyset$. Also, let all $p(x, 0)$ and $q(x, 0)$ diverge.

At stage $s+1$, we first define T^{s+1}_α for each α:

$$T^{s+1}_\alpha = T^s_\alpha \cup \{x \in \overline{C^{s+1}} : x < |Z^{s+1}_\alpha|\}.$$

Next we determine which elements of C^{s+1} to add to A^s_0 to create A^{s+1}_0. For this, we need movable markers for elements currently in $C - A$. Write

$$C^{s+1} - A^s = \{d^{s+1}_0, d^{s+1}_1, \dots d^{s+1}_{m_{s+1}}\}$$

preserving the order of the markers from the preceding stage. (That is, if $d^s_i = d^{s+1}_{i'}$ and $d^s_j = d^{s+1}_{j'}$, then $i < j$ iff $i' < j'$; and if $d^{s+1}_i \in C^s$ and $d^{s+1}_j \notin C^s$, then $i < j$.)

For the sake of $\mathcal{M}_e$, we define

$$V^{s+1}_e = V^s_e \cup \{x \in W_{e,s+1} - C^{s+1} : (\forall y \leq x)[y \in W_{e,s+1} \cup C^{s+1}]\}.$$

(For each e, the sets V^s_e enumerate a c.e. set V_e. If $\overline{C} \not\subseteq W_e$, then V_e will be finite, but if $\overline{C} \subseteq W_e$, then $\overline{C} \subseteq V_e \subseteq W_e$.)

For each $e \leq s$, define the e-state of each d^{s+1}_k at stage $s+1$ to be:

$$\sigma(e, d^{s+1}_k, s+1) = \{i < e : d^{s+1}_k \in V^{s+1}_i\}.$$

We order the different possible e-states by viewing them as binary strings.

Find the least $i \leq s$ such that there exist e and j with $e < i < j \leq s$ and $\sigma(e, d_i^{s+1}, s+1) = \sigma(e, d_j^{s+1}, s+1)$ and $d_i^{s+1} \notin V_e^{s+1}$ and $d_j^{s+1} \in V_e^{s+1}$. For the least such e and the least corresponding j, we say that $\mathcal{M}_e$ *wants to put into* A_0 all the elements $d_i^{s+1}, d_{i+1}^{s+1}, \ldots d_{j-1}^{s+1}$, so as to give the marker d_i a higher $(e+1)$-state at subsequent stages.

Now we consider the requirements $\mathcal{F}_e$. For each $e \leq s$ with $W_{e,s} \cap A_0^s = \emptyset$ and for each x such that

$$x \in (W_{e,s} \cap C^{s+1}) - A^s - \{d_0^{s+1}, d_1^{s+1}, \ldots d_e^{s+1}\}$$

we say that $\mathcal{F}_e$ *wants to put* x *into* A_0.

We set $p(x, s+1)\uparrow$ for all $x \notin C - A^s$. Otherwise $x = d_k^{s+1}$ for some k, and $p(x, s+1)$ is the least $e \leq k$ (if any) such that either $p(x,s)\downarrow= e$ or $\mathcal{M}_e$ or $\mathcal{F}_e$ wants to put x into A_0. Thus, the function $p(x, s+1)$ gives the priority currently assigned to putting x into A_0. If there is no such e, let $p(x, s+1)\uparrow$.

We now follow the following steps for each $x \leq s$:

1. If $p(x, s+1)\uparrow$, then $q(x, s+1)\uparrow$ also.

2. If $p(x, s+1)\downarrow$ but $q(x,s)\uparrow$, we ask if every $\alpha \leq p(x, s+1)$ satisfies either $x \in S_\alpha^{s+1} \cup \hat{S}_\alpha^{s+1}$ or $x \notin T_\alpha^{s+1}$. If so, set $q(x, s+1) = p(x, s+1) + 1$. If not, then $q(x, s+1)\uparrow$.

3. If $p(x, s+1)\downarrow$ and $q(x,s)\downarrow> p(x, s+1)$, then set $q(x, s+1)$ to be the greatest $\alpha \leq p(x, s+1)$ satisfying all four of the following conditions:

    (a) $S_\alpha^{s+1} \cap \hat{S}_\alpha^{s+1} = \emptyset$.

    (b) $x \notin \hat{S}_\alpha^{s+1}$.

    (c) $x \in T_\alpha^{s+1}$.

    (d) $\forall \beta < \alpha$, either β fails one of the three conditions (a)-(c), or $\beta = \langle i', j' \rangle$ and $\alpha = \langle i, j \rangle$ with $i \neq i'$.

Also, enumerate x in $D^{s+1}_{q(x,s+1)}$. (For future reference, notice that if α satisfies (a)-(c), then some $\beta \leq \alpha$ with the same first coordinate as α must satisfy (a)-(d).)

If there is no such α, set $q(x, s+1) = -1$.

4. If $p(x, s+1)\downarrow$ and $q(x, s)\downarrow$ with $0 \leq q(x, s) \leq p(x, s+1)$, we ask whether $x \in B^{s+1}_{q(x,s)}$. If so, or if $q(x, s)$ no longer satisfies the conditions (a)-(d), set $q(x, s+1)$ to be the greatest $\alpha < q(x, s)$ satisfying the conditions (a)-(d) above, and let $x \in D^{s+1}_{q(x,s+1)}$. (If there is no such α, let $q(x, s+1) = -1$.) Otherwise, let $q(x, s+1) = q(x, s)$.

5. If $p(x, s+1)\downarrow$ and $q(x, s)\downarrow= -1$, enumerate $x \in A^{s+1}_0$, and let $q(x, s+1)\uparrow$.

This completes our enumeration of A^{s+1}_0. Next we determine which elements to add to A^{s+1}_1:

1. Find the least $e \leq s$ (if any) such that $\mathcal{Q}_e$ is not yet satisfied and there is an element $x \in W_{e,t} - W_{e,t-1}$ for some $t \leq s$ such that $w(t) > s$, and there exists $y < x$ such that $y \in C^{s+1} - A^{s+1}_0$ and $y \notin A^t_1 \cup \{d^{s+1}_0, \ldots d^{s+1}_e\}$ and no $\mathcal{F}_i$ with $i < e$ wants to put y into A_0. Put the greatest such y into A^{s+1}_1. This forces $A^{s+1}_1 \upharpoonright x \neq A^t_1 \upharpoonright x$, satisfying $\mathcal{Q}_e$ permanently. (If there is no such e, do nothing.)

2. Find the least $e \leq s$ (if any) such that $\mathcal{P}_e$ is not yet satisfied and there is an element $x \in C^{s+1} \cap (W_{e,t} - W_{e,t-1})$ for some $t \leq s$ with $v(t) > s$, such that $x \notin \{d^{s+1}_0, \ldots d^{s+1}_e\}$ and no $\mathcal{F}_i$ with $i < e$ wants to put x into A_0. If no such x lies in $A^s \cup A^{s+1}_0$, then put the least such x into A^{s+1}_1. This forces $x \in A^{s+1}$, satisfying $\mathcal{P}_e$ permanently.

3. Find the least $e \leq s$ (if any) such that $\mathcal{G}_e$ is not yet satisfied and there is an element $x \in (W_{e,s+1} \cap C^{s+1}) - A^{s+1}_0$ with $x \notin \{d^{s+1}_0, \ldots d^{s+1}_e\}$, such that no $\mathcal{F}_i$ with $i < e$ wants to put x into A_0. Put this x into A^{s+1}_1. This satisfies $\mathcal{G}_e$ forever.

Let $A^{s+1} = A_0^{s+1} \cup A_1^{s+1}$. This completes the construction.

Lemma 2.3.4 *$C - A$ is infinite.*

Proof. We prove by induction on e that $d_e = \lim_s d_e^s$ exists. Assume that this holds for all markers d_i with $i < e$, and let $s_0 \geq e$ be a stage such that $d_i^{s_0} = d_i$ for all $i < e$. Now each $\mathcal{F}_j$, $\mathcal{G}_j$, $\mathcal{P}_j$, and $\mathcal{Q}_j$ with $j > e$ cannot put any of the elements $d_0^s, \ldots d_e^s$ into A_1 at stage $s+1$, so none of these requirements ever moves the marker d_e^s. Also, each $\mathcal{G}_i$, $\mathcal{P}_i$, and $\mathcal{Q}_i$ with $i \leq e$ puts at most one element into A, hence moves the markers at most once. Let $s_1 \geq s_0$ be a stage so large that no $\mathcal{G}_i$, $\mathcal{P}_i$, or $\mathcal{Q}_i$ with $i \leq e$ moves any markers at any stage $s \geq s_1$.

By the construction, d_e^s can only be moved at stage $s \geq s_1$ by a requirement $\mathcal{M}_i$ or $\mathcal{F}_i$ with $i \leq e$. Furthermore, when $\mathcal{F}_i$ $(i \leq e)$ moves a marker, it puts an element into A_0, so it is satisfied at that point. Before then it may have tried to put finitely many other elements into A_0 as well, and any of them may go into A_0 or A_1 at a later stage, moving markers in the process. However, since there are only finitely many such elements, d_e is moved only finitely many times on behalf of $\mathcal{F}_i$.

Now $\mathcal{M}_0$ moves d_e at most 2^{e+1} times after stage s_1: once to put d_0 into V_0, possibly twice to put d_1 into V_0, and so on. Once $\mathcal{M}_0$ has finished moving d_e, $\mathcal{M}_1$ moves it at most 2^e more times, to put markers into V_1. Similarly, once each $\mathcal{M}_i$ has moved d_e for the last time, $\mathcal{M}_{i+1}$ may move it at most 2^{e-i} more times. Hence we eventually reach a stage s_2 after which d_e never is moved again. Possibly $d_e^{s_2}\uparrow$, but since C is infinite and every d_i with $i < e$ has already converged to its limit, we know that d_e^t will be defined at some stage $t > s_2$. Since it never moves again, this yields $d_e^t = \lim_s d_e^s$. ■

Lemma 2.3.5 *For each e, the requirements $\mathcal{N}_e$, $\mathcal{P}_e$, $\mathcal{Q}_e$, $\mathcal{F}_e$, and $\mathcal{G}_e$ are all satisfied.*

Proof. We proceed by induction on e. Assume the lemma holds for all $i < e$. We write α for the pair coded by e, and prove first that $\mathcal{N}_\alpha$ is satisfied. Suppose

$(B_\alpha \cap (S_\alpha - A_0)) \cup A_1 = (D_\alpha \cap (S_\alpha - A_0)) \cup A_1$ and $B_\alpha \subseteq C$ and $S_\alpha \sqcup \hat{S}_\alpha = C$. Then $F(\alpha)$ holds and Z_α is infinite. The construction of T_α then guarantees that $\overline{C} \subseteq T_\alpha$. Let G_α be the intersection of all those V_i with $i < \alpha$ such that V_i is infinite, and let $\hat{T}_\alpha = T_\alpha \cap G_\alpha$. Thus $\overline{C} \subseteq \hat{T}_\alpha$, since $\overline{C} \subseteq V_i$ whenever V_i is infinite.

Sublemma 2.3.6 *For each α and each $n < \alpha$, there are only finitely many $x \in \hat{T}_\alpha$ such that $\mathcal{M}_n$ ever wants to put x into A_0.*

Proof. First, if V_n is finite, then $\mathcal{M}_n$ will only want to put finitely many elements into A_0. So we may assume that V_n is infinite, and hence that $\hat{T}_\alpha \subseteq V_n$.

If $\mathcal{M}_n$ wants to put x into A_0 at stage s, then $x \in C^s - A^s$, so $x = d_k^s$ for some k. Moreover, there must be an i with $n < i \leq k$ and a $j > k$ such that $\sigma(n, d_i^s, s) = \sigma(n, d_j^s, s)$ and $d_i^s \notin V_n^s$ and $d_j^s \in V_n^s$. Furthermore, d_i is the leftmost marker which any $\mathcal{M}$-requirement wants to put into A_0 at stage s, and n and j satisfy the minimality requirements of the construction.

Now if $d_k^s \notin V_n^s$, then $d_k^s \notin V_n$, since $C \searrow V_n = \emptyset$, and hence $d_k^s \notin \hat{T}_\alpha$. Therefore we may assume $d_k^s \in V_n^s$. (This guarantees $k \neq i$). Then minimality of n forces $\sigma(n, d_i^s, s) \geq \sigma(n, d_k^s, s)$, and minimality of j forces $\sigma(n, d_i^s, s) > \sigma(n, d_k^s, s)$ (since $d_k^s \in V_n^s$). Hence there is some $m < n$ such that $\sigma(m, d_i^s, s) = \sigma(m, d_k^s, s)$ and $d_i^s \in V_m^s$ and $d_k^s \notin V_m^s$. This forces $d_i^s \in V_m$ and $d_k^s \notin V_m$ (since $d_k^s \in C^s - V_m^s$). If V_m is infinite, then $d_k^s \notin \hat{T}_\alpha$. But if V_m is finite, then d_i^s lies in the finite set

$$V = \bigcup \{V_m : m < n \ \& \ V_m \text{ finite}\}.$$

Hence we need only find a stage t so large that for every $d \in V$, either $d \in A_0^t$ or $\mathcal{M}_n$ wants to put d into A_0 at stage t or $\mathcal{M}_n$ never wants to put d into A_0. Then $\mathcal{M}_n$ will never want to put into A_0 any $x > \max(C^t)$ with $x \in \hat{T}_\alpha$. ∎

We will show that the conclusion of $\mathcal{N}_\alpha$ holds for $\hat{T}_\alpha$:

$$(A_0 \cap S_\alpha \cap \hat{T}_\alpha) \cup A_1 =^* (B_\alpha \cap S_\alpha \cap \hat{T}_\alpha) \cup A_1.$$

Once we have established this for all α, clearly $R(A_0, A_1)$ itself must hold, since for each α we can choose another $\hat{T}_\alpha$ which excludes the (finite) difference set of the two sides and still contains $\overline{C}$.

Suppose first that $x \in A_0 \cap S_\alpha \cap \hat{T}_\alpha$ and $x \notin A_1$, and assume that x is sufficiently large that:

- $x > |Z_\beta|$ for every $\beta < \alpha$ such that Z_β is finite, and
- No $\mathcal{F}_i$ with $i < \alpha$ ever tries to put x into A_0, and
- No $\mathcal{M}_i$ with $i < \alpha$ ever tries to put x into A_0.

The last condition is possible by Sublemma 2.3.6. Notice also that the first condition forces $x \notin T_\beta$ for all $\beta < \alpha$ with $|Z_\beta|$ finite.

Then for all s, either $p(x, s) \geq \alpha$ or $p(x, s)\uparrow$. But since $x \in A_0$, we know that some $p(x, s)\downarrow$. For the least such s we have $x \in C^s$, and hence $x \in T^s_\alpha$, since $C \cap T_\alpha \subseteq T_\alpha \searrow C$.

Now α satisfies conditions (a)-(c) in the construction at stage s, since $F(\alpha)$ holds and $x \in S_\alpha$. So there must exist $\beta = \langle i, j' \rangle \leq \alpha = \langle i, j \rangle$ which satisfies (a)-(d) at stage s.

We claim that this β satisfies conditions (a)-(d) at every stage after s as well. Since $x \in T^s_\beta$, we know that Z_β is infinite and $F(\beta)$ holds, by choice of x. Hence (a) and (c) hold at all subsequent stages. Let t be the first stage at which $q(x, t)$ converged. Then $x \in C^t$, and $x \in T^t_\beta$ since $C \searrow T_\beta = \emptyset$. By the definition of q, we must have had $x \in S^t_\beta \cup \hat{S}^t_\beta$. But $x \notin \hat{S}^s_\beta$ since (b) holds at stage s, and because $s > t$, this forces $x \in S^t_\beta$, so (b) always holds of β.

To show that (d) always holds of β, we choose an arbitrary $\gamma < \beta$ witht he same first coordinate as β. Since β satisfies (d) at stage s, γ must fail one of (a)-(c) at stage s. If γ fails (a) or (b) at stage s, then clearly it fails that same consition at every subsequent stage. Moreover, if γ fails (c) at stage s, then $x \notin T^s_\gamma$, and since $x \in C^s$, this forces $x \notin T_\gamma$. Thus β will always satisfy condition (d).

But since $x \in A_0$, there must also be a stage s' with $q(x, s') = -1$. Since (a)-(d) continue to hold of β, the only way for $q(x, s') < \beta$ to occur is for x to enter B_β.

(Recall that for all s, either $p(x,s) \geq \alpha$ or $p(x,s)\uparrow$.) But $B_\beta = W_i = B_\alpha$ since $\beta = \langle i, j' \rangle$ and $\alpha = \langle i, j \rangle$, so this forces $x \in B_\alpha$. Hence

$$(A_0 \cap S_\alpha \cap \hat{T}_\alpha) \cup A_1 \subseteq^* (B_\alpha \cap S_\alpha \cap \hat{T}_\alpha) \cup A_1.$$

Now suppose that $x \in B_\alpha \cap S_\alpha \cap \hat{T}_\alpha$ and $x \notin A_1$, and assume x is greater than $\max(d_0, \ldots d_\alpha)$, and also greater than the greatest finite $|Z_\beta|$ with $\beta < \alpha$. (Thus $x \notin T_\beta$ for all such β.) Now $x \in C$ since $S_\alpha \subseteq C$, so at some stage s_0, x will enter C and be given a marker: say $x = d_k^{s_0}$. So $x \in C^{s_0}$, and since $x \in T_\alpha$, this forces $x \in T_\alpha^{s_0}$.

If $x \notin A_0$, then we must have $x \in D_\alpha$, since $(B_\alpha \cap (S_\alpha - A_0)) \cup A_1 = (D_\alpha \cap (S_\alpha - A_0)) \cup A_1$ and $x \notin A_1$. (Notice that then x, being in $C - A$, eventually receives some permanent marker $d_{k'}$, with $k' > \alpha$ by choice of x.) For x to have entered D_α, there must have been a stage $s_1 \geq s_0$ with $q(x, s_1) = \gamma = \langle i, j' \rangle$, where $\alpha = \langle i, j \rangle$. (Also, then $p(x, s_1)\downarrow$, and since $x \notin A_1$, $p(x,s)\downarrow$ for all $s \geq s_1$.) But α satisfies conditions (a)-(c) at all stages $s \geq s_0$, so by condition (d) on γ, we must have $\gamma \leq \alpha$. The assumption $x \notin A_0 \cup A_1$ then means that there is some $s_2 > s_1$ such that $q(x,s)\downarrow = q(x, s_2)$ for all $s \geq s_2$. Let $\beta = q(x, s_2) \leq \gamma$. Then $x \in D_\beta - B_\beta$, and furthermore β satisfies the conditions (a)-(d) at all stages $s \geq s_2$.

Now $x \in T_\beta$, to satisfy condition (c), so $x < |Z_\beta|$ and $\beta \leq \gamma \leq \alpha$. If $\beta = \alpha$, then Z_β is infinite since $F(\alpha)$ holds, and if $\beta < \alpha$, then Z_β must be infinite, by our choice of x. Therefore $F(\beta)$ holds, and in particular $S_\beta \sqcup \hat{S}_\beta = C$. Now $x \notin \hat{S}_\beta$ by condition (b), so $x \in S_\beta$. However, with $x \in D_\beta - B_\beta$, this contradicts $F(\beta)$. Hence $x \in A_0$, and

$$(A_0 \cap S_\alpha \cap \hat{T}_\alpha) \cup A_1 \subseteq^* (B_\alpha \cap S_\alpha \cap \hat{T}_\alpha) \cup A_1.$$

This completes our proof that $\mathcal{N}_\alpha$ is satisfied.

Now we continue with the other requirements. Let s_0 be a stage such that no $\mathcal{P}_i$, $\mathcal{Q}_i$, $\mathcal{F}_i$, or $\mathcal{G}_i$ with $i < e$ tries to put any element into A_0 or A_1 at any stage after s_0. ($\mathcal{F}_i$ is different from the other requirements in that it may try to put more than one element into A_0. It only stops trying when one of those elements succeeds in entering A_0. We choose s_0 so that every element which $\mathcal{F}_i$ wants to put into A_0 either is in A^{s_0} or never enters A.) Assume also that s_0 is sufficiently large that $d_i^{s_0} = d_i$ for every $i \leq e$.

Now if $W_e \searrow A$ is infinite, then there must be an x in some $W_{e,s} - A^s$ with $s > s_0$ and $\{d_0, \ldots d_e\}$. No requirement of higher priority will need to put this x anywhere, except possibly some $\mathcal{M}_i$, and according to our construction, $\mathcal{G}_e$ does not respect the priority of the requirements $\mathcal{M}_i$, so $x \in A_1^{s+1}$, and $\mathcal{G}_e$ is satisfied.

Similarly, if W_e is infinite, then there must be an x and an $s > s_0$ such that $x \in W_{e,s} - W_{e,s-1}$ and $x \in C^{v(s)}$, by prompt simplicity of C. If this x is not already in $A^{v(s)-1}$, then the construction puts it into $A_1^{v(s)}$, so $\mathcal{P}_e$ holds. Also, there must be an x and an $s > s_0$ with $x \in W_{e,s} - W_{e,s-1}$ such that $C^s \upharpoonright x \neq C^{w(s)} \upharpoonright x$, by promptness of C. Thus there is a $y < x$ which entered C at some stage t with $s < t \leq w(s)$. We must have $y \notin A^{t-1}$ since $A^{t-1} \subseteq C^{t-1}$. But now $y \notin \{d_0^t, \ldots d_e^t\}$, since these markers had reached their limits by stage s_0 and y only entered C at stage t. Hence the construction will put this y into A_1^t, and $A_1^{w(s)} \upharpoonright x \neq A_1^s \upharpoonright x$, satisfying $\mathcal{Q}_e$.

Continuing with the induction, we need a sublemma to handle $\mathcal{F}_e$.

Sublemma 2.3.7 *For this e and for all sufficiently large x, if $\mathcal{F}_e$ wants to put x into A_0 at some stage, then $x \in A_0$.*

Proof. Choose x so large that it satisfies all of the following:

1. $x > \max\{|Z_\beta| : \beta \leq e \;\&\; Z_\beta \text{ is finite}\}$.

2. No $\mathcal{F}_i$, $\mathcal{G}_i$, $\mathcal{P}_i$, or $\mathcal{Q}_i$ with $i < e$ ever wants to put x into A_0 or A_1.

3. $x \notin \{d_0, \ldots d_e\}$.

Suppose $\mathcal{F}_e$ wants x to enter A_0 at stage s_0. Then $x = d_k^{s_0}$ for some k and $p(x, s_0) \downarrow \leq e$. Now no $\mathcal{G}_j$, $\mathcal{P}_j$, or $\mathcal{Q}_j$ with $j \geq e$ ever manages to put x into A_1, since $\mathcal{F}_e$ takes priority over these. (Since $x \neq d_e$, the only way to have $k \leq e$ is for x eventually to enter A_0. Hence we may assume $k > e$.) Also, for every $\beta < e$, either $x \notin T_\beta$ (if $|Z_\beta| < x$) or $F(\beta)$ holds (if Z_β is infinite). Hence there is an $s_1 \geq s_0$ such that $q(x, s_1)\downarrow$ and $q(x, s_1+1)\downarrow\leq e$.

Now suppose $q(x, s) = \beta$ for some $s \geq s_1$ (so $\beta \leq e$). If $F(\beta)$ failed, then Z_β would have to be finite, so $x \notin T_\beta$ (since $|Z_\beta| < x$) and $q(x, s)$ would never equal β. Therefore, $F(\beta)$ must hold. Suppose $x \notin A_0$. If $x \notin S_\beta$, then $x \in \hat{S}_\beta$ by $F(\beta)$ and so $q(x, s_\beta) < \beta$ for some $s_\beta \geq s_1$. Otherwise $x \in D_\beta \cap (S_\beta - A_0) \subseteq B_\beta$ by $F(\beta)$, so $x \in B_\beta^{s_\beta}$ for some $s_\beta \geq s_1$, and hence $q(x, s_\beta) < \beta$. Thus, by induction on $\beta < e$, eventually we must have $q(x, s) = -1$, and so $x \in A_0^{s+1}$, proving the sublemma. ■

Now if $W_e \searrow A$ is infinite, then $\mathcal{F}_e$ has infinitely many elements at its disposal to try to put into A_0. Hence once we find a sufficiently large $x \in W_e \searrow A$, we know by the sublemma that this x will eventually enter A_0, thus satisfying $\mathcal{F}_e$. This completes the induction of Lemma 2.3.5. ■

Lemma 2.3.8 *The requirements $\mathcal{M}_e$ are all satisfied by our construction.*

Proof. Suppose that $\overline{C} \subseteq W_e$. To prove that $\mathcal{M}_e$ holds, we must show $\overline{A} \subseteq^* W_e$. By induction we assume that $\mathcal{M}_i$ holds for all $i < e$. Let

$$\sigma = \{i < e : \overline{C} \subseteq W_i\}.$$

Now if $i \in \sigma$, then also $\overline{C} \subseteq V_i$, so by inductive hypothesis $\overline{A} \subseteq^* V_i$, whereas if $i \notin \sigma$ (and $i < e$), then V_i is finite. Hence for all but finitely many k we have $\sigma(e, d_k) = \sigma$.

Now let $V_\sigma = V_e \cap \left(\bigcap\{V_i : i \in \sigma\}\right)$. Then $\overline{C} \subseteq V_\sigma$. But C, being promptly simple, is noncomputable, so $V_\sigma \searrow C$ must be infinite. Choose y so large that no element $\geq y$ can be held out of A_0 forever by any requirement $\mathcal{N}_\alpha$ with $\alpha \leq e$, and let s_0 be a stage such that $C^{s_0}\restriction y = C\restriction y$.

Suppose for a contradiction that $\overline{V_e} \cap (C - A)$ is infinite. Then there exists p such that $d_p \notin V_e$ with p so large that $d_p \notin C^{s_0}$ and with $\sigma(e, d_q) = \sigma$. (Hence

$d_p > y$.) Let s_1 be a stage with $d_p^{s_1} = d_p$ and $\sigma(e, d_p, s_1) = \sigma$. Now since $V_\sigma \searrow C$ is infinite, there will be a stage $s > s_1$ at which some element $x \in V_\sigma^{s-1}$ enters C, and is assigned the marker d_q^s (with $q > p$ since $d_p^{s_0} = d_p$). Moreover, we may assume that q is sufficiently large that not only is d_q^s in V_σ, but that $\sigma(e, d_q^s, s) = \sigma$, since every V_i with $i < e$ and $i \notin \sigma$ is finite. Since $d_q^s \in V_\sigma \subseteq V_e$ and $d_p \notin V_e$, $\mathcal{M}_e$ will want to put d_p into A_0 at stage s, and since $d_p > y$, no negative requirement will keep d_p out of A_0. Possibly d_p will be diverted into A_1 by some requirement $\mathcal{G}_j$, $\mathcal{P}_j$, or $\mathcal{Q}_j$, since these do not respect the priority of $\mathcal{M}_e$. If so, then d_p will enter A_1; if not, then d_p will enter A_0. Either way, d_p enters A, contradicting our assumption that the marker d_p had reached its limit at stage s_0.

Hence $\overline{V_e} \cap (C - A)$ is finite, and $\overline{A} \subseteq (C - A) \cup \overline{C} \subseteq^* V_e \subseteq W_e$. Thus $\mathcal{M}_e$ is satisfied, and the lemma is proven. ∎

Knowing that the requirements are all satisfied, we can easily complete the proof of the theorem. The construction ensured that $A_0 \cap A_1 = \emptyset$, and the conjunction of all the $\mathcal{F}_i$ and $\mathcal{G}_i$ implies that $A_0 \sqcup A_1$ is a Friedberg splitting of A. (See pp. 181-182 of [47].) The requirements $\mathcal{P}_i$ together make A a promptly simple set, by definition, and the $\mathcal{Q}_i$ together allow A_1 to satisfy the Promptly Simple Degree Theorem (Thm. XIII.1.6 of [47]), so that A_1 is of prompt degree. To prove that $R(A_0, A_1)$ holds, we note that the requirements $\mathcal{M}_i$, along with Lemma 2.3.4, show that $A = A_0 \sqcup A_1$ is a major subset of C. Moreover, given a $B = W_i$ and a pair $(S_{j'}, \hat{S}_{j''})$ with $S_{j'} \sqcup S_{j''} = C$, we have the D_i and T_α (with $\alpha = \langle i, \langle j', j'' \rangle \rangle$) constructed above. If

$$(B_i \cap (S_{j'} - A_0)) \cup A_1 = (D_i \cap (S_{j'} - A_0)) \cup A_1,$$

then $F(\alpha)$ holds. Since $\mathcal{N}_\alpha$ is satisfied, we know that there exists a T with $\overline{C} \subseteq T$ such that

$$(A_0 \cap S_{j'} \cap T) \cup A_1 =^* (B_i \cap S_{j'} \cap T) \cup A_1.$$

So we can pick a sufficiently large n_α, and let

$$T' = \{x \in T : x \geq n_\alpha\} \cup \{x \in \overline{C} : x < n_\alpha\}.$$

Then $\overline{C} \subseteq T'$ and also $(A_0 \cap S_{j'} \cap T') \cup A_1 = (B_i \cap S_{j'} \cap T') \cup A_1$, since $S_{j'} \cap \overline{C} = \emptyset$. Thus $R(A_0, A_1)$ holds. Finally, since A is a major subset of the set C, A must be of high degree (see [24], page 214). ■

CHAPTER 3

COMPUTABLE CATEGORICITY OF TREES

3.1 Introduction

In a finite language, a countable structure $\mathcal{A}$ whose universe A is a subset of ω is *computable* if A is a computable set and for all functions f and relations R in the language, $f^{\mathcal{A}}$ is a computable function and $R^{\mathcal{A}}$ is a computable relation.

Any computable structure will be isomorphic to infinitely many other computable structures. It may happen, however, that two computable structures are isomorphic, yet that the only isomorphisms between them are noncomputable (as maps from one domain to the other). If so, then these structures lie in distinct *computable isomorphism classes* of the isomorphism type of the structure. On the other hand, if there exists a computable function taking one structure isomorphically to the other, then the two structures lie in the same computable isomorphism class.

The *computable dimension* of a computable structure is the number of computable isomorphism classes of that structure. The most common computable dimensions are 1 and ω, but for each $n \in \omega$, there do exist structures with computable dimension n, by a result of Goncharov ([19]). If the computable dimension of $\mathcal{A}$ is 1, we say that $\mathcal{A}$ is *computably categorical*. This notion is somewhat analogous to the concept of categoricity in ordinary model theory: a theory is *categorical* in a given power κ if all models of the theory of power κ are isomorphic. Computable categoricity is a property of structures, not of theories: a computable structure $\mathcal{A}$ is computably categorical if every other computable structure which is isomorphic to $\mathcal{A}$ is computably isomorphic to $\mathcal{A}$.

A standard example of a categorical theory is the theory of dense linear orders without end points, which is categorical in power ω. One proves this by taking two arbitrary countable dense linear orders and building an isomorphism between them by a back-and-forth construction. The same contruction allows us to prove that the structure $\mathbb{Q}$ is computably categorical. (More formally, let $(\omega, \prec)$ be a computable linear order isomorphic to $(\mathbb{Q}, <)$. Then $(\omega, \prec)$ is computably categorical.)

Characterizations of computable categoricity have been found for certain types of structures. Goncharov and Dzgoev ([20]) and Remmel ([39]) proved that a linear order is computably categorical precisely if it contains finitely many successivities (that is, if only finitely many elements have an immediate successor in the linear order). Remmel also proved that a Boolean algebra is computably categorical if and only if it contains only finitely many atoms ([38]).

In this chapter we consider computable categoricity of trees, and prove that no tree of height ω is computably categorical. The question of computable categoricity of trees of finite height is the subject of joint work by Lempp, Solomon, and the author, presently in progress.

To prove that a tree T is not computably categorical, we will construct a new tree T' isomorphic to T, satisfying the following requirements $\mathcal{R}_e$:

$$\mathcal{R}_e : \varphi_e \text{ total } \implies \text{ there exists } x \in T' \text{ such that } \mathrm{level}_{T'}(x) \neq \mathrm{level}_T(\varphi_e(x)).$$

Clearly $\mathcal{R}_e$ implies that φ_e is not an isomorphism from T' to T. If we can establish $\mathcal{R}_e$ for every e, then, we will have proven that T is not computably categorical.

Our notation is standard, but our definitions demand attention. A *tree* consists of a universe T with a partial order $\prec$ on T such that for every $x \in T$, the set of predecessors of x in T is well-ordered by $\prec$, and T contains a least element under $\prec$. (Hence the tree is computable if T is a computable set and $\prec$ a computable relation.) Throughout this chapter, T will represent the computable tree which we wish to prove not to be computably categorical.

If two nodes x and y in T are incomparable under $\prec$, then we write $x \perp y$. For every node $x \in T$, the size of the set $\{y \in T : y \prec x\}$ will be the *level* of x in T, written $\mathrm{level}_T(x)$. (A more formal definition sets the level of the root to be 0 and inductively defines $\mathrm{level}_T(x) = \sup\{\mathrm{level}_T(y) + 1 : y \prec x\}$, thereby also covering the case of an element with infinitely many predecessors. In this chapter, however, every element of every tree has only finitely many predecessors.)

We view our trees as growing upwards, with a single element r (the *root*, or least element under $\prec$) at the base. Thus the level of the root is 0, its immediate successors under $\prec$ are at level 1, and so on. The level of a node of T is not generally computable, but it is Σ_1, since there exists a computable function $f(x, s) = |\{y <$

$s : y \prec x\}|$ such that for all $x \in T$,

$$\text{level}_T(x) = \lim_s f(x, s).$$

The height of T is defined as follows:

$$\text{ht}(T) = \sup_{x \in T} (\text{level}_T(x) + 1).$$

Thus, the height of T will be the least ordinal α such that no node of T has level α. In this chapter we only consider trees of height ω.

The reader should note that different definitions of subtree and tree homomorphism have been used for different purposes in the literature. By the definition which we use in this chapter, a *homomorphism* from one tree $(T, \prec)$ to another tree $(T', \prec')$ is a map $f : T \to T'$ which respects the partial orders:

$$x \prec y \iff f(x) \prec' f(y).$$

In other papers, a tree is sometimes defined using the infimum function $\wedge$, where the infimum $x \wedge y$ of x and y is the greatest z such that $z \prec x$ and $z \prec y$. Any tree under one definition is also a tree under the other definition, but when the infimum function is used, all homomorphisms are required to respect the infimum function. This is a strictly stronger requirement: all maps respecting $\wedge$ respect $\prec$, because

$$x \preceq y \iff x \wedge y = x,$$

but not conversely. Kruskal's Lemma, which we use in section 3.2, proves the existence of the stronger type of homomorphism.

If the infimum fuction is computable, then the relation $\prec$ is computable, since it is definable in terms of $\wedge$ without quantifiers. Therefore, if the computable trees $(T, \prec)$ and $(T', \prec')$ are isomorphic but not computably isomorphic, then the corresponding structures $(T, \wedge)$ and $(T', \wedge')$ are also isomorphic, but not computably isomorphic. (Notice, however, that $(T, \wedge)$ and $(T', \wedge')$ need not be computable, since computability of $\prec$ does not guarantee that we can compute the infimum function.) Thus, our theorem suffices to prove that even when tree is defined using the infimum, no tree of height ω is computably categorical. The definitions of tree and tree homomorphism using the infimum are probably more common in the literature. We adopt the definitions using $\prec$ because for purposes of our proof, they will be far more useful.

Our definition of subtree arises from our definition of homomorphism. Once again, therefore, it diverges from much of the literature: for our purposes, a tree $(T', \wedge')$ is a *subtree* of $(T, \wedge)$ if $T' \subseteq T$ and the inclusion map is a homomorphism. Thus the infimum of two elements in T may not be the same as their infimum in T'. Also, the root of T may be distinct from the root of T', as in the case of the subtrees $T[x]$, which we will be considering frequently. If x is a node in T, then the subtree $T[x]$ is just the tree

$$T[x] = \{y \in T : x \preceq y\}.$$

The partial order on $T[x]$ is the restriction to $T[x]$ of the partial order $\prec$ on T. Therefore $T[x]$ is a subtree of T with root x. We define the *height of* T *above* x by:

$$\mathrm{ht}_x(T) = \mathrm{ht}(T[x]).$$

The reason for our use of $\prec$ rather than $\wedge$ to define homomorphism and subtree is twofold. First, $\prec$ is the basic relation we used to define the notion of a tree; $\wedge$ was derived from $\prec$. If $\wedge$ were the basic function, then computability questions would be very different. Second, during our proofs about a tree T we will be considering many subsets of T which we will want to regard as subtrees. Under our definition, they will be subtrees (as will any subset of T with a $\prec$-least element), but under the $\wedge$-definition they would not be subtrees.

A *path* γ through T is a maximal linearly ordered subset of T. It may be finite or infinite. Any tree containing an infinite path must have height ω (since we are not considering trees of height $> \omega$). A node is *extendible* if it lies on an infinite path through T, and *non-extendible* otherwise. The extendible nodes of a tree T (if any exist) form a subtree of T, which we denote by T_{ext}. Notice, however, that since we allow T to be infinite-branching, the height of T above a node may be ω even if the node is nonextendible.

3.2 Kruskal's Lemma

Although our results concern infinite trees, we will need the ability to manipulate finite subtrees. For this purpose Kruskal's Lemma is essential.

Theorem 3.2.1 (Kruskal's Lemma) *(See [30], [44].) Let $\{T_i : i \in \omega\}$ be an infinite collection of finite trees. Then there exist $i < j$ in ω such that T_i can be embedded in T_j.*

Every version of Kruskal's Lemma which we will encounter has an analogue of the following corollary:

Corollary 3.2.2 *Let $\{T_i : i \in \omega\}$ be an infinite collection of finite trees. Then there exists $n \in \omega$ such that for every $i > n$, T_i can be embedded in some T_j with $j > i$, and some T_k with $k < i$ can be embedded in T_i.*

Proof. If the set

$$\{i \in \omega : (\forall j > i)\ T_i \text{ does not embed in } T_j\}$$

were infinite, it would itself contradict Kruskal's Lemma. The same is true of

$$\{i \in \omega : (\forall k < i)\ T_k \text{ does not embed in } T_i\}.$$

∎

We can extend Kruskal's Lemma to a version dealing with infinite trees.

Corollary 3.2.3 *Let $\{T_i : i \in \omega\}$ be an infinite collection of trees. (These trees need not be finite, nor even finitely branching.) Then there exists an $i \in \omega$ such that for every finite subtree $T \subseteq T_i$, there exists $j > i$ for which T embeds in T_j.*

Proof. Suppose $\{T_i : i \in \omega\}$ were a collection of trees contradicting the lemma. Then for each i, we would have some finite subtree $S_i \subseteq T_i$ which did not embed into any T_j with $j > i$. In particular, for each $i < j$, S_i would not embed in S_j. Thus the collection $\{S_i : i \in \omega\}$ would contradict Kruskal's Lemma. ∎

Corollary 3.2.4 *Let $\{T_i : i \in \omega\}$ be as in Corollary 3.2.3. Then there is an $n \in \omega$ such that for every $i > n$ and every finite subtree $T \subseteq T_i$, there exists $j > i$ such that T embeds into T_j.*

Proof. If not, then we could find an increasing sequence $i_0 < i_1 < i_2 < \cdots$ such that $\{T_{i_k} : k \in \omega\}$ contradicted Corollary 3.2.3. ■

In this paper we will want to embed trees in such a way that nodes with p predecessors are mapped to nodes with more than p predecessors. That is, the level in the tree T of the node x should be less than the level in T' of its image under the embedding of T into T'. To map nodes to other nodes at greater levels, we need the following stronger version of Kruskal's Lemma, in which one is allowed to "label" nodes of each tree. For our purposes, a *labelling* of a tree T is simply a map from T to ω. Proofs of this result appear in [30] and [44].

Theorem 3.2.5 (Kruskal) *Let $\{T_i : i \in \omega\}$ be an infinite collection of finite trees, each with a labelling l_i. Then there exist $i < j$ in ω and an embedding $f : T_i \to T_j$ such that for every $x \in T_i$, $l_i(x) \leq l_j(f(x))$.*

From Theorem 3.2.5 we derive the following result:

Corollary 3.2.6 *Let $\{T_i : i \in \omega\}$ be an infinite collection of finite trees such that $\sup_i \mathrm{ht}(T_i) = \omega$. Then there is a number $m \in \omega$ such that for every index i and every node $x \in T_i$ with $\mathrm{level}_{T_i}(x) = m$, there exists an embedding f of T_i into some T_j with $j > i$, such that*

$$\mathrm{level}_{T_j}(f(x)) > \mathrm{level}_{T_i}(x).$$

Proof. Suppose no $m \in \omega$ satisfied the theorem. Then for every m, we would have an index i_m and a node $x_m \in T_{(i_m)}$ with $\mathrm{level}_{T_{(i_m)}}(x_m) = m$ such that:

$$\forall \text{ embeddings } f : T_{(i_m)} \to T_j \text{ with } j > i_m, \mathrm{level}_{T_j}(f(x_m)) = \mathrm{level}_{T_{(i_m)}}(x_m). \tag{3.1}$$

Now the set $\{i_0, i_1, i_2 \ldots\}$ will be infinite, since each T_i has finite height. Moreover, the index i_m satisfies Equation 3.1 not only for x_m but also for all predecessors of x_m. Therefore we can choose $i_{m+1} > i_m$ for all m.

For each m, define the labelling l_m on the tree $T_{(i_m)}$ by

$$l_m(x) = \begin{cases} 0, & \text{if } \mathrm{level}_{T_{i_m}}(x) < m \\ 1, & \text{otherwise} \end{cases}$$

Thus $l_m(x_m) = 1$ for all m. However, for any embedding $f : T_{(i_m)} \to T_{(i_k)}$ with $k > m$, we have

$$\mathrm{level}_{T_{i_k}}(f(x_m)) = \mathrm{level}_{T_{(i_m)}}(x_m) = m < k.$$

This forces $l_k(f(x_m)) = 0$. Thus the sequence $\{T_{i_0}, T_{i_1}, T_{i_2}, \ldots\}$ contradicts Theorem 3.2.5. ■

The same result holds for all y above the level m:

Corollary 3.2.7 *Let $\{T_i : i \in \omega\}$ be as in Corollary 3.2.6. Then there is a number $m \in \omega$ such that for every index i and every node $y \in T_i$ with $\mathrm{level}_{T_i}(y) \geq m$, there exists an embedding f of T_i into some T_j with $j > i$, such that*

$$\mathrm{level}_{T_j}(f(y)) > \mathrm{level}_{T_i}(y).$$

Proof. The conclusion follows for every $y \in T_i$ with $\mathrm{level}_{T_i}(y) \geq m$, simply by finding that $x \preceq y$ in T_i with $\mathrm{level}_{T_i}(x) = m$ and applying the embedding given by Corollary 3.2.6 for that x. ■

Finally, we combine the version for infinite trees with the version for embedding nodes at greater levels.

Corollary 3.2.8 *Let $\{T_i : i \in \omega\}$ be any collection of trees. Then there exist an n and an m with the property that for all indices $i > n$, for every finite subtree*

$S \subseteq T_i$, and for any node $x \in S$ with $\text{level}_S(x) \geq m$*, there is an embedding $g : S \to T_j$ of S into some T_j with $j > i$, such that*

$$\text{level}_{T_j}(g(x)) > \text{level}_S(x).$$

Proof. Suppose the statement were false. Now if g is an embedding of S into T_j, it is impossible to have $\text{level}_{T_j}(g(x)) < \text{level}_S(x)$. Therefore, the negation of the statement is as follows:

$$(\forall n)(\forall m)(\exists i > n)(\exists \text{ finite } S \subseteq T_i)(\exists x \in S)$$
$$\left[\begin{array}{c} \text{level}_S(x) \geq m \ \& \\ (\forall j > i)(\forall \text{ embeddings } g : S \to T_j)[\text{level}_{T_j}(g(x)) = \text{level}_S(x)] \end{array}\right]$$

We apply this negation first with $n = 0$ and $m = 0$, yielding an index $i_0 > 0$ and a node x_0 at level ≥ 0 in some finite subtree S_0 of T_{i_0}. Inductively, we apply the negation with $n = i_k$ and $m = k+1$ to get an index $i_{k+1} > i_k$ and a corresponding node x_{k+1} at level $\geq k+1$ of a finite subtree S_{k+1} of $T_{i_{(k+1)}}$. From the negation, we see that every embedding of any S_k into any T_j with $j > i_k$ fixes the level of x_k. In particular, the same holds for any embedding of S_k into any S_j with $j > k$. However, we know that

$$\text{ht}(S_k) > \text{level}_{S_k}(x_k) \geq k,$$

so $\sup_k \text{ht}(S_k) = \omega$. Thus the set $\{S_k : k \in \omega\}$ contradicts Corollary 3.2.7. ∎

Finally, for computability-theoretic purposes, we note that if S and T are finite trees (and we have strong indices for each, i.e. we know the number of nodes of each), then the statement

$$\exists \text{ an embedding } g : S \to T$$

is decidable, uniformly in S and T. From the decidability of this statement, we conclude further that if S is finite with known strong index and T is any computable tree, then the question of embeddability of S into T is a Σ_1 question: it asks whether there exists a finite subtree of T into which S embeds. Therefore, if we know that there exists an embedding of S into T, then we can effectively find such an embedding, via an algorithm uniform in S and T.

3.3 ω-Branching Nodes with $\mathbf{ht}_x(T) = \omega$

We consider computable trees of infinite height. The general theorem, that no such tree is computably categorical, will be proven in the next section. In this section, to prepare for that proof, we prove that a significant subclass of such trees cannot be computably categorical.

We define the limit-supremum of a sequence $\langle n_i \rangle_{i \in \omega}$ to be

$$\limsup_i (n_i) = \inf_j \sup_{i>j} (n_i)$$

T will be a given computable tree under the partial order $\prec$, with height ω, which is ω-branching at a node x_0. (That is, x_0 has infinitely many immediate successors $x_1, x_2, \ldots$.) We assume further that $\limsup_i \mathrm{ht}(T[x_i]) = \omega$. This can occur two ways: either infinitely many x_i's are extendible, or there exist $T[x_i]$'s of arbitrarily large finite heights.

Since the universe of T is computable, we may take it to be ω, pulling back via a 1-1 computable function if necessary to make this so. We will construct a computable tree T' isomorphic to T, such that there is no computable isomorphism between them.

The isomorphism f from T to T' will be a Δ^0_2 function, the limit of a computable sequence of finite partial 1-1 functions f_s, such that the domains $D_s = \mathrm{dom}(f_s) \subset T$ form a strong array of finite sets. We will ensure that $D_s \subseteq D_{s+1}$ for each s, although f_{s+1} need not agree with f_s on D_s. (If it did so for all s, then f would be a computable isomorphism, which is precisely what we wish to avoid!) Also, we will force $\mathrm{range}(f) = \omega$, so that the universe of T' will be ω. The ordering $\prec'$ on T' will be given by lifting the ordering $\prec$ from T via f, thereby guaranteeing that f is an isomorphism. To make $\prec'$ computable, we force the approximations f_s to satisfy the following condition:

Condition 3.3.1 *For all $a, b \in \mathrm{range}(f_s)$, we have $a, b \in \mathrm{range}(f_{s+1})$ and*

$$f^{-1}_{s+1}(a) \prec f^{-1}_{s+1}(b) \iff f^{-1}_s(a) \prec f^{-1}_s(b).$$

To ensure that T and T' are not computably isomorphic, we impose the requirements $\mathcal{R}_e$.

$$\mathcal{R}_e : \quad \varphi_e \text{ total} \implies (\exists x \in T')\ [\mathrm{level}_{T'}(x) \neq \mathrm{level}_T(\varphi_e(x))].$$

This will suffice to prove the proposition.

Proposition 3.3.2 *Let T be a computable tree containing an ω-branching node x_0 with immediate successors $x_1, x_2, \ldots$, such that*

$$\limsup_i \mathrm{ht}(T[x_i]) = \omega.$$

Then T is not computably categorical.

Proof. As previously remarked, we may assume the universe of T to be ω. A *successor tree* of x_0 is a tree of the form $T[x_i]$ with $i \geq 1$. ($\{x_1, x_2, \ldots\}$ are all the immediate successors of x_0, as stated above. This set may not be not computable.) Corollary 3.2.8, applied to the successor trees, provides m and n in ω such that for every finite subtree $S \subseteq T[x_i]$ with $i > n$ and every node $x \in S$ with $\mathrm{level}_S(x) \geq m$, there is an embedding of S into some $T[x_j]$ with $j > i$ which maps x to a node of greater level. We fix these values of m and n for the rest of the proof. (Notice that therefore the proof is not uniform in T.)

Let T_s be the subtree of T with nodes $\{r, x_0, x_1, \ldots x_n\} \cup \{0, 1, 2, \ldots s\}$, under $\prec$, where r is the root of T.

For our purposes, the finite subtrees S will generally be of the form $D_s[y]$, where $D_s \supseteq T_s$ is the domain of f_s and y is an immediate successor of x_0 in D_s (although not necessarily in T). We will call $D_s[y]$ a *successor tree* at stage s. Notice that it may happen that two successor trees which are distinct at stage s acquire a common root at stage $s + 1$, e.g. if $s + 1 = x_i$ for some i, and thus merge into a single successor tree at stage $s + 1$. A given successor tree at stage s, however, can only be merged this way finitely often, since each of its nodes has finite level in T.

The following construction yields a computable tree T' which is isomorphic to T but satisfies every requirement $\mathcal{R}_e$, proving that T is not computably categorical. The *witness nodes* w_e will be nodes in $T[x_0]$, and will be approximated at stage s by a node $w_{e,s}$. The successor tree at stage s containing $w_{e,s}$ will be denoted $S_{e,s}$. This is the successor tree which we use in order to satisfy requirement $\mathcal{R}_e$. The sequence $\langle w_{e,s}\rangle_{s\in\omega}$ will converge to some w_e, and each successor tree in T will contain at most one w_e. The isomorphism f from T to T' will be approximated at stage s by a finite map f_s with domain D_s. If $\varphi_{e,s}(f_s(w_{e,s}))$ converges to a node at the same level of T_s as the level of $f_s(w_{e,s})$ in T'_s, then we redefine f_{s+1} and $w_{e,s+1}$ with $f_{s+1}(w_{e,s+1}) = f_s(w_{e,s})$ at a higher level in T'_{s+1}. (The level of a node in T'_s is just the level of its preimage under f_s in T_s.) Doing this requires us to redefine f_{s+1} on the entire successor tree containing $w_{e,s}$, in order to satisfy Condition 3.3.1, and we will appeal to Corollary 3.2.8 to ensure that the necessary embedding exists. Thus $f(w_e)$ will be the witness required by $\mathcal{R}_e$.

Figure 3.3 gives an example of our basic strategy. $S_{e,s}$ is the successor tree which we use to satisfy $\mathcal{R}_e$. We suppose that we have found at stage s that $\varphi_e(f_s(w_{e,s})) = 6$, which lies at level 2 in D_s. This is bad, because $f_s(w_{e,s})$ lies at level 2 in D'_s, so it appears that φ_e might be an isomorphism from T' to T. $S_{e,s}$ is the successor tree above the node 4 in D_s, and we use Corollary 3.2.8 to find an embedding of $S_{e,s}$ upwards into the successor tree above the node 10 in D_{s+1}. (The embedding is indicated by the arrow to D'_{s+1}.) We use this embedding to make $\text{level}_{D'_{s+1}}(f_{s+1}(w_{e,s+1})) > \text{level}_{D'_s}(f_s(w_{e,s}))$, by defining f_{s+1} so that $f_{s+1}(9) = f_s(4)$, $f_{s+1}(12) = f_s(6)$, and $f_{s+1}(w_{e,s+1}) = f_s(w_{e,s})$. We add new values to range(f_{s+1}) for $f_{s+1}(4)$, $f_{s+1}(6)$, $f_{s+1}(8)$, and $f_{s+1}(10)$. Thus $\text{level}_{D_s}(\varphi_e(f_{s+1}(w_{e,s+1}))) \neq \text{level}_{D'_{s+1}}(f_{s+1}(w_{e,s+1}))$.

Construction: f_0 is the identity map with $\text{dom}(f_0) = T_0$. The witness nodes $w_{e,0}$ and the successor trees $S_{e,0}$ are undefined for all e. We let $D_0 = \text{dom}(f_0)$. (At each stage s, D_s and T_s will both be subtrees of T, with $T_s \subseteq D_s$.) We immediately define the successor trees $T_0[x_i]$ with $1 \leq i \leq n$ to be frozen.

At stage $s+1$, we consider the successor trees of x_0 in D_s. For each successor tree S (if any) of height $\geq m$ which is not frozen and does not contain $S_{e,s}$ for

Figure 3.1: Example of an Upwards Embedding

any $e \leq s$, we choose the least $e \leq s$ such that $S_{e,s}$ is undefined, let $S_{e,s+1} = S$ and choose $w_{e,s+1}$ to be the $<$-least node at the highest level of S. Thus $\text{level}_{S_{e,s+1}}(w_{e,s+1}) \geq m$.

We then consider in turn each e for which $S_{e,s}$ was defined.

Step 1: If there is an $i < e$ and a $z \in T_{s+1}$ such that $x_0 \prec z \prec w_{i,s}$ and $z \prec w_{e,s}$, then we immediately make $S_{j,s+1}$ and $w_{j,s+1}$ undefined for all $j \geq e$, and declare all $S_{j,s}$ with $j > e$ frozen.

(This step ensures that if two successor trees $S_{i,s}$ and $S_{e,s}$ have acquired a common root above x_0, thus becoming the same successor tree, then we use the single new successor tree to play against requirement $\mathcal{R}_i$ only.)

Step 2: Otherwise, we consider $f_s(w_{e,s})$, the potential witness for requirement $\mathcal{R}_e$. If $\varphi_{e,s}(f_s(w_{e,s}))$ diverges, or converges to an element not in D_s, or if $\text{level}_{D_s}(\varphi_{e,s}(f_s(w_{e,s}))) \neq \text{level}_{D_s}(w_{e,s})$, then we define:

$$w_{e,s+1} = w_{e,s}$$

$$f_{s+1} = f_s \text{ on } S_{e,s}$$

$$S_{e,s+1} = \{y \in D_s \cup T_{s+1} : (y \wedge w_{e,s+1}) \succ x_0\}.$$

(Here $y \wedge w_{e,s+1}$ represents the infimum in $D_s \cup T_{s+1}$, which is a finite tree. Taking the infimum over all of T would not be computable.)

(This $S_{e,s+1}$ is just the same successor tree as $S_{e,s}$, along with any new elements that may have appeared in this successor tree at stage s.)

Step 3: If $\text{level}_{D_s}(\varphi_{e,s}(f_s(w_{e,s}))) = \text{level}_{D_s}(w_{e,s})$, then find the least stage $t > s$ with $D_s \subseteq T_t$ such that the following holds:

Condition 3.3.3 *There exists a $z \in T_t$ such that:*

1. *z is an immediate successor of x_0 in T_t, and*

2. *$T_t[z] \cap D_s = \emptyset$, and*

3. There is an embedding g of $S_{e,s}$ into $T_t[z]$ with

$$\text{level}_{T_t}(g(w_{e,s})) > \text{level}_{D_s}(w_{e,s}).$$

Let $S_{e,s+1} = S$, with $w_{e,s+1} = g(w_{e,s})$. (By our choice of g, this forces $\text{level}_{T_t}(w_{e,s+1}) > \text{level}_{D_s}(w_{e,s})$. Also, $\text{level}_{S_{e,s+1}}(w_{e,s+1}) > \text{level}_{S_{e,s}}(w_{e,s}) \geq m$.) For every $x \in S_{e,s}$, define $f_{s+1}(g(x)) = f_s(x)$, and define $f_{s+1}(x)$ to be the least element which is not yet in $\text{range}(f_{s+1}) \cup \text{range}(f_s)$. Declare $S_{e,s}$ to be frozen, so that at no subsequent stage s' will any $w_{i,s'}$ be defined in the successor tree containing $S_{e,s}$. Having executed Step 3 for e, we let $w_{j,s+1}$ and $S_{j,s+1}$ diverge and freeze $S_{j,s}$ for all $j > e$, and do not execute Steps 1, 2, or 3 for any $j > e$.

(We execute Step 3 if $\mathcal{R}_e$ is not satisfied by $f_s(w_{e,s})$. By Corollary 3.2.8, there must exist a successor tree $T[x_j]$ into which the required embedding g exists, because $\text{level}_{S_{e,s}}(w_{e,s}) \geq m$ and $S_{e,s} \subseteq T[x_i]$ for some $i > n$. The successor trees $T[x_1], \ldots T[x_n]$ were all frozen right away at stage 0, so none of them contains $S_{e,s}$. Thus we have found a z such that f_s is completely undefined on the successor tree $S \subset T_t$ containing z, and $S_{e,s}$ embeds into S via a map g. We use this embedding to satisfy $\mathcal{R}_e$, as in the example of Figure 3.3. Freezing $S_{e,s}$ ensures that $f_{s'}$ will never again be redefined on $S_{e,s}$, so that $\lim_{s'} f_{s'}$ must exist.)

Having completed these three steps for each $S_{e,s}$, we now define D_{s+1} to be $\left(\bigcup_e S_{e,s+1}\right) \cup D_s \cup T_{s+1}$. For any $y \in D_s$ such that $f_{s+1}(y)$ is not yet defined, take $f_{s+1}(y) = f_s(y)$. (This includes nodes on already-frozen successor trees, nodes on successor trees of height $\leq m$, and nodes not on $T[x_0]$.) For each $y \in D_{s+1}$, if $f_{s+1}(y)$ is not yet defined, take $f_{s+1}(y)$ to be the least integer not already in $\text{range}(f_{s+1})$ Thus $D_{s+1} = \text{dom}(f_{s+1})$. This completes the construction.

We now prove that this construction really does yield a tree T' which is isomorphic to T but not computably isomorphic to it.

Lemma 3.3.4 *For every e, the sequence $w_{e,s}$ converges to a limit w_e.*

Proof. Assume by induction that the Lemma holds for every $i < e$. Notice that in our construction, once $w_{e,s}$ and $S_{e,s}$ are defined, the only way they can become undefined is in Step 1 (if a new node of $T[x_0]$ appears which is a predecessor of $w_{i,s}$ for some $i < e$) or Step 3 (if $w_{i,s} \neq w_{i,s+1}$ for some $i < e$). Once we reach a stage s_0 such that $w_{i,s} = w_i$ for every $i < e$ and $s \geq s_0$ and every predecessor of every w_i ($i < e$) has appeared in T_{s_0}, we know that once $w_{e,s}$ is defined for some $s \geq s_0$, it will stay defined at all subsequent stages, although its value may change. Also, $w_{e,s}$ is only defined at stages s such that $w_{i,s}$ is also defined for all $i < e$.

By induction, for every $i < e$, $\langle w_{i,s} \rangle_{s\in\omega}$ converges to some w_i. Pick a stage s_0 such that $w_{i,s} = w_i$ and $\text{level}_{T_s}(w_{i,s}) = \text{level}_T(w_i)$ for all $i < e$ and $s \geq s_0$. Now if $s \geq s_0$ and $w_{e,s}$ is not defined, then no $w_{j,s}$ with $j > e$ is defined either. But since $\limsup_i \text{ht}(T[x_i]) = \omega$, there are infinitely many successor trees of height $> m$, so a new one, S, with $S \cap D_{s_0} = \emptyset$, must appear at some stage $s > s_0$. It will not be frozen, since $w_{i,s} = w_i$ for all $i < e$, so it will be chosen as $S_{e,s}$, and one of its nodes of maximal height will be $w_{e,s}$. Then $w_{e,t}$ is defined for every $t > s$, since every predecessor of every w_i with $i < e$ is already in T_s. Thus, by induction, for every e, $w_{e,s}$ is defined for all sufficiently large s.

Once it is defined at a stage beyond s_0, $w_{e,s}$ will only be redefined at a subsequent stage $t+1$ if $\text{level}_{T_t}(\varphi_e(f_t(w_{e,t}))) = \text{level}_{T_t}(w_{e,t})$ and Condition 3.3.3 holds. Moreover, even when it is redefined, we will still have $f_{t+1}(w_{e,t+1}) = f_t(w_{e,t})$. Since the tree T has height ω, we know that for all t,

$$\text{level}_{T_t}(\varphi_e(f_t(w_{e,t}))) \leq \text{level}_T(\varphi_e(f_t(w_{e,t}))) < \omega.$$

But $\langle \text{level}_{T_t}(\varphi_e(f_t(w_{e,t}))) \rangle_{t\in\omega}$ is a non-decreasing sequence, so it can only change value finitely often. Thus, once defined, $w_{e,s}$ will only be redefined finitely often, so it must converge. ■

Lemma 3.3.5 *For every x, $\lim_s f_s(x)$ exists.*

Proof. We know $x \in T_s \subseteq D_s = \text{dom}(f_s)$ for all $s > x$. Furthermore, once $f_s(x)$ is defined, the only way we can have $f_s(x) \neq f_{s+1}(x)$ is if x lies on a successor tree $S_{e,s}$ for which $w_{e,s}$ is redefined or undefined at stage $s + 1$. Once this happens,

$S_{e,s}$ is declared frozen, and $f_t \restriction S_{e,s} = f_{s+1} \restriction S_{e,s}$ for all $t \geq s+1$. Thus, not only does $\langle f_s(x) \rangle_{s \in \omega}$ converge, but in fact it changes value at most once. ∎

We define the function $f = \lim_s f_s$.

Lemma 3.3.6 *The functions f_s satisfy Condition 3.3.1. (Hence the relation $\prec'$ defined on $T' = \text{range}(f)$ by*

$$a \prec' b \iff (\forall s)[a, b \in \text{range}(f_s) \implies f_s^{-1}(a) \prec f_s^{-1}(b)]$$

is computable and gives a tree structure on ω).

Proof. The construction makes it clear that $\text{range}(f_s) \subseteq \text{range}(f_{s+1})$ for all s. Now fix $a, b \in \text{range}(f_s)$. If $f_s^{-1}(a) \neq f_{s+1}^{-1}(a)$, then $f_s^{-1}(a)$ must lie on a successor tree $S_{e,s}$ such that $w_{e,s} \neq w_{e,s+1}$. Hence $f_{s+1}(g(f_s^{-1}(a))) = f_s(f_s^{-1}(a)) = a$, and $f_{s+1}^{-1}(a) = g(f_s^{-1}(a))$, where g is the upward embedding of $S_{e,s}$ into $S_{e,s+1}$ used in the construction. We consider four cases:

Case 1. Suppose $f_s^{-1}(b) \in S_{e,s}$ as well. Then also $f_{s+1}^{-1}(b) = g(f_s^{-1}(b))$, and since g is an embedding, we have

$$f_{s+1}^{-1}(a) \prec f_{s+1}^{-1}(b) \iff f_s^{-1}(a) \prec f_s^{-1}(b).$$

Case 2. Suppose $f_s^{-1}(b) \in T[x_0] - S_{e,s} - \{x_0\}$. Then $f_s^{-1}(b) \perp f_s^{-1}(a)$. By Part 2 of Condition 3.3.3, we know $f_{s+1}^{-1}(b) \in T[x_0] - S_{e,s+1} - \{x_0\}$, so also $f_{s+1}^{-1}(b) \perp f_{s+1}^{-1}(a)$.

Case 3. Suppose $f_s^{-1}(b) \preceq x_0$. Then $f_s^{-1}(b) \prec f_s^{-1}(a)$, and $f_{s+1}^{-1}(b) = f_s^{-1}(b) \preceq x_0 \prec f_{s+1}^{-1}(a)$.

Case 4. If $f_s^{-1}(b) \perp x_0$, then $f_s^{-1}(b) \perp f_s^{-1}(a)$, and also $f_{s+1}^{-1}(b) = f_s^{-1}(b) \perp x_0 \prec f_{s+1}^{-1}(a)$, so $f_{s+1}^{-1}(b) \perp f_{s+1}^{-1}(a)$.

A similar analysis applies if $f_s^{-1}(a) = f_{s+1}^{-1}(a)$ and $f_s^{-1}(b) \neq f_{s+1}^{-1}(b)$. ∎

Lemma 3.3.7 *The tree $(T', \prec')$ is a computable tree isomorphic to T.*

Proof. We defined every f_s to be a 1-1 map, with range$(f_s) \subseteq$ range(f_{s+1}). By Lemma 3.3.5, then, f is also 1-1.

The range of f is ω since at each of the (infinitely many) stages at which we needed a new element for the range of f_s, we took the smallest one available. If $f_{s+1}^{-1}(y) \neq f_s^{-1}(y)$ for some s, then $y = f_s(x)$ for some x on some $S_{e,s}$ which was redefined at stage $s+1$, and $f_{s+1}^{-1}(y) \in S_{e,s+1}$. But $S_{e,s}$ can only be redefined finitely often, since $\text{level}_T(\varphi_e(f(w_e))) < \omega$, so eventually $f_s^{-1}(y)$ will stabilize, forcing $y \in$ range(f).

Moreover, $\text{dom}(f) = \bigcup_s D_s = T$, so f is a bijection from T to T'. Since the partial order $\prec'$ on T' is defined by lifting $\prec$ from T via f, we know that f is an isomorphism. Computability of $\prec'$ follows from Lemma 3.3.6: given $a, b \in T'$, find a stage s such that $a, b \in$ range(f_s). Then $a \prec' b \iff f_s^{-1}(a) \prec f_s^{-1}(b)$. ■

Lemma 3.3.8 *For every e, either $\varphi_e(f(w_e))$ diverges or*

$$\text{level}_{T'}(f(w_e)) \neq \text{level}_T(\varphi_e(f(w_e))).$$

Thus requirement $\mathcal{R}_e$ is satisfied by the element $f(w_e)$.

Proof. Let s_0 be a stage such that for all $s \geq s_0$, $w_{e,s} = w_e$ and $f_s(w_{e,s}) = f(w_e)$. Since $w_{e,s}$ is never redefined after stage s_0, we know that either $\varphi_e(f(w_e))$ diverges, or $\text{level}_{D_s}(\varphi_e(f(w_e))) \neq \text{level}_{D_s}(w_e)$ for all $s \geq s_0$. But since $\bigcup_s D_s = T$, the latter of these implies that $\text{level}_T(\varphi_e(f(w_e))) \neq \text{level}_T(w_e)$. Now $\text{level}_T(w_e) = \text{level}_{T'}(f(w_e))$ since f is an isomorphism, so φ_e maps the element $f(w_e)$ of T' to an element at a different level in T. Thus $\mathcal{R}_e$ is satisfied, and φ_e is not an isomorphism from T' to T. ■

This completes the proof of Proposition 3.3.2.

3.4 Trees of Height ω

3.4.1 Main Theorem

We now arrive at the main result of this chapter.

Theorem 3.4.1 *No tree of height ω is computably categorical.*

The theorem will be proved in subsection 3.4.5, after the necessary propositions have been established.

Corollary 3.4.2 *The computable dimension of a computable tree of height ω is always ω.*

Proof of Corollary. If there are n isomorphic copies $T_0, T_1, \dots T_{n-1}$ of T such that each is computable but no two are computably isomorphic, we construct another one T_n satisfying for all $i < n$ and all $e \in \omega$ the requirements

$$\mathcal{R}_{\langle e,i \rangle} : \quad \varphi_e \text{ total} \implies (\exists x \in T_n)\ [\text{level}_{T_n}(x) \neq \text{level}_{T_i}(\varphi_e(x))].$$

The constructions are the same as in the proofs of Propositions 3.3.2, 3.4.6, and 3.4.14.

Alternatively, one can apply a theorem of Goncharov from [18] which states that if $\mathcal{A}$ is a computable structure which has two computable copies that are Δ^0_2-isomorphic but not computably isomorphic, then $\mathcal{A}$ has computable dimension ω. (The isomorphisms which we construct in our proofs are all Δ^0_2.) ∎

We prove the theorem by proving five propositions, covering five different types of tree. We use the notions of an extendible node and a side tree to define these cases. Recall (from page 69) that a node $x \in T$ is *extendible* if there exists an infinite path through T containing x. The set of all extendible nodes of T, if nonempty, forms a subtree of T, denoted by T_{ext}. T_{ext} need not be computable, even though T is.

The *side tree* above a node x is denoted $S[x]$, and is a subtree of $T[x]$.

$$S[x] = \{y \in T[x] : (\forall z \in T)[x \prec z \preceq y \implies z \notin T_{\text{ext}}]\}$$

(x itself may or may not be extendible.) Equivalently, consider the extendible immediate successors $x_1, x_2, \ldots$ of x. The side tree $S[x]$ is precisely $T[x] - \bigcup_i T[x_i]$. Thus x itself is the only node of $S[x]$ which can be extendible in T, and $S[x]$ contains no infinite paths, although it can have height ω if it is infinite-branching. $S[x]$ is not necessarily computable, but it will be so if the sets

$$\{\text{extendible immediate successors of } x\} \text{ and}$$

$$\{\text{nonextendible immediate successors of } x\}$$

are computably separable.

3.4.2 Three Cases Using Proposition 3.3.2

Proposition 3.4.3 *Suppose the computable tree T has height ω above a nonextendible node y_0. Then T is not computably categorical.*

Proof. Let T and y_0 be as in the proposition. We claim there exists an $x_0 \in T$ with ω-many immediate successors, such that $\text{ht}_{x_0}(T) = \omega$ and T has finite height above every $x \succ x_0$. Indeed, consider the subtree

$$S = \{x \in T : \text{ht}_x(T) = \omega \ \&\ x \text{ is nonextendible } \&\ x \not\preceq y_0\}.$$

S contains a $\prec$-least element (either y_0 or some predecessor of y_0), so S is indeed a subtree. However, S contains no infinite paths, so it must contain terminal nodes, all of which will lie above y_0. We take x_0 to be one of these. (x_0 is terminal in S, that is; T will have height ω above x_0.) Therefore, T has finite height above every $x \succ x_0$, and moreover, this x_0 must be an ω-branch point, since otherwise one of its immediate successors in T would also be in S. Let $x_1, x_2, \ldots$ be the immediate successors of x_0 in T. Then $\sup_i \mathrm{ht}(T[x_i]) = \omega$, because $\mathrm{ht}_{x_0}(T) = \omega$. But $\mathrm{ht}(T[x_i]) < \omega$ for all $i \geq 1$, since otherwise x_i would lie in S. Therefore we must have $\limsup_i \mathrm{ht}(T[x_i]) = \omega$, and so Proposition 3.3.2 applies to T and T is not computably categorical. ∎

Proposition 3.4.4 *Suppose that the computable tree T contains an extendible node x_0 such that the side tree $S[x_0]$ has height ω. Then T is not computably categorical.*

Proof. If x_0 has an immediate successor in $S[x_0]$ above which T has height ω, then we apply Proposition 3.4.3 to this node. If all immediate successors of x_0 in $S[x_0]$ have finite height, then there must be infinitely many of them, say $x_1, x_2, \ldots$. Then $\limsup_{i \geq 1} \mathrm{ht}(T[x_i]) = \omega$, because $\sup_{i \geq 1} \mathrm{ht}(T[x_i]) = \omega$. Moreover, any immediate successor of x_0 in T either lies in $S[x_0]$ or is extendible. Hence Proposition 3.3.2 applies to x_0 itself. ∎

Proposition 3.4.5 *Suppose that in the computable tree T, there is a node $x_0 \in T_{ext}$ with infinitely many immediate successors in T_{ext}. Then T is not computably categorical.*

Proof. $\mathrm{ht}(T[y]) = \omega$ for every immediate successor y of x_0 in T_{ext}, so Proposition 3.3.2 applies to x_0. ∎

3.4.3 An Isolated Path

Proposition 3.4.6 *Suppose there is a node $x_0 \in T$ which lies on exactly one infinite path γ through T. If all side trees at nodes on γ above x_0 have finite height, then T is not computably categorical.*

Proof. Let x_0 be a node on T which lies on exactly one infinite path γ through T, such that all side trees at nodes on γ above x_0 have finite height.

Let $x_0 \prec x_1 \prec x_2 \prec \ldots$ be all the nodes of γ above x_0. We apply Corollary 3.2.4 to the set of side trees $S[x_i]$ above nodes of γ, yielding an n such that for every $i \geq n$ and every finite subtree $S \subseteq S[x_i]$, there is some $j > i$ for which S embeds into $S[x_j]$. Our diagonalization argument will take place entirely above x_n. (Notice that the sequence $\langle x_i \rangle_{i \in \omega}$ cannot necessarily be computed, and that the choice of n from Corollary 3.2.4 is nonuniform.)

We define $T_s = \{r, x_0, x_1, \ldots x_n\} \cup \{0, 1, \ldots s\}$, a tree under $\prec$. (As before, r represents the root of T.) We computably approximate the sequence $\langle x_i \rangle_{i \in \omega}$. For each s, let

$$\{x_n = x_{n,s} \prec x_{n+1,s} \prec \cdots \prec x_{l_s,s}\}$$

be the chain of maximal length in $T_s[x_n]$. (If there is more than one such chain, take the first such in the dictionary order derived from $<$.) Since all side trees have finite height, clearly $x_{i,s} \to x_i$ for each i. Indeed, $x_{i,s} = x_i$ for all s such that $\{x_n, \ldots x_m\} \subseteq T_s$, where $m = \max_{j<i}(j + \mathrm{ht}(S[x_j]))$. (However, $\mathrm{ht}(S[x_j])$ need not be computable in j.)

The requirements $\mathcal{R}_e$ are the same as in Proposition 3.3.2:

$$\mathcal{R}_e : \quad \varphi_e \text{ total} \implies (\exists x \in T') \, [\mathrm{level}_{T'}(x) \neq \mathrm{level}_T(\varphi_e(x))].$$

This time, however, we will say that $\mathcal{R}_e$ is *satisfied at stage* s only if the witness node $w_{e,s}$ is defined and $\varphi_{e,s}(f_s(w_{e,s}))$ converges and lies at a level of T_s different from $\mathrm{level}_{T_s}(w_{e,s})$.

Instead of simply freezing nodes, as in the proof of Proposition 3.3.2, we must freeze them with priority e. Thus, at each stage s, we define *envelopes* $E_{e,s}$ for each e, to provide negative restraints on redefining the isomorphism f on elements of $E_{e,s}$. If x lies in the envelope $E_{e,s}$, then $f_{s+1}(x) \neq f_s(x)$ only if necessary for the sake of a requirement $\mathcal{R}_i$ with $i \leq e$. Thus the envelopes will ensure that the functions f_s converge to a limit f with range ω.

Construction: f_0 is the identity map on T_0, and the witness nodes $w_{e,0}$ are undefined for all e. We define $E_{e,0} = \emptyset$ for all e.

At stage $s + 1$, we search for the least $e \leq s + 1$ such that one of the following holds:

1. $w_{e,s}$ is undefined.

2. For each i with $n \leq i \leq l_{s+1}$, the following holds:

$$x_{i,s+1} \preceq w_{e,s} \implies x_{i,s+1} \preceq w_{e-1,s}.$$

3. $w_{e,s}$ is defined and $\varphi_{e,s}(f_s(w_{e,s}))\downarrow$ and

$$\text{level}_{D_s}(w_{e,s}) = \text{level}_{D_s}(\varphi_e(f_s(w_{e,s}))).$$

(Such an e must exist, because $w_{s+1,s}$ is undefined.) Let $w_{i,s+1} = w_{i,s}$ and

$$E_{i,s+1} = \{i \in D_{s+1} : (\exists z \in E_{i,s}) y \preceq z\}$$

for all $i < e$, and let $w_{j,s+1}$ be undefined and $E_{j,s+1} = \emptyset$ for all $j > e$,

If case (1) holds for e, we let $w_{e,s+1}$ to be the $<$-least node in $D_s[x_n]$ with $\text{level}_{D_s[x_n]}(w_{e,s+1}) \geq e$ which does not lie in any $E_{i,s}$ with $i < e$ and such that

$$(\exists j)[x_{j,s} \preceq w_{e,s+1} \ \& \ x_{j,s} \not\preceq w_{e-1,s+1}].$$

We define $E_{e,s+1} = D_{s+1} = D_s \cup T_{s+1}$. (If no such node exists, then $w_{e,s+1}$ remains undefined, with $E_{e,s+1} = \emptyset$ and $D_{s+1} = D_s \cup T_{s+1}$.)

If case (2) holds, we let $w_{e,s+1}$ diverge with $E_{e,s+1} = \emptyset$ and $D_{s+1} = D_s \cup T_{s+1}$. (This is the case where $w_{e-1,s}$ and $w_{e,s}$ appear to lie in the same side tree along γ, in which case we cannot embed one upwards without disturbing the other.)

Otherwise, case (3) holds. We search for the least $t \geq \max(D_s)$ satisfying either of the following two conditions. Let $m_t = \max\{k : x_{k,t} \preceq w_{e,s}\}$ for each t.

Condition 3.4.7 *There exists $i < e$ such that $x_{m_t,t} \preceq w_{i,t}$.*

Condition 3.4.8 *There exists an embedding g of $D_s[x_{m_t,t}]$ into $T_t[x_{m_t,t}]$ with*

$$\text{level}_{T_t}(g(w_{e,s})) > \text{level}_{D_s}(w_{e,s}).$$

If Condition 3.4.7 holds for t, then we make $w_{e,s+1}$ undefined, and set $E_{e,s+1} = \emptyset$ and $D_{s+1} = D_s \cup T_{s+1}$.

Otherwise, we use the embedding g given by Condition 3.4.8 to satisfy requirement $\mathcal{R}_e$. Let $w_{e,s+1} = g(w_{e,s})$, and for all $y \in D_s[x_{m_t,t}]$, define $f_{s+1}(g(y)) = f_s(y)$. For those $y \in D_s[x_{m_t,t}] - \mathrm{range}(g)$, take $f_{s+1}(y)$ to be the least element of ω that is not yet in range(f_{s+1}) nor in range(f_s). Let $D_{s+1} = D_s \cup T_t$, and let the envelope $E_{e,s+1} = D_{s+1}$.

(For the sake of clarity, we note that if $x_{m_t,t}$ does not lie in D_s, then

$$D_s[x_{m_t,t}] = \{y \in D_s : x_{m_t,t} \preceq y\}.$$

We do have $w_{e,s} \in D_s[x_{m_t,t}]$ by definition of m_t. If $D_s[x_{m_t,t}]$ does not have a single root, then we consider each minimal element in it to have level 0.)

In all three cases, we then define $f_{s+1}(y) = f_s(y)$ for those $y \in D_s$ on which f_{s+1} is not yet defined. Also, for each $y \in D_{s+1} - D_s$ on which f_{s+1} is not yet defined, choose the least element of ω which is not yet in range(f_{s+1}) to be $f_{s+1}(y)$. This completes the construction.

(The idea of the construction is that each witness element $w_{e,s}$ lies in the side tree above some x_i. When we need to satisfy $\mathcal{R}_e$, we do so by embedding the side tree containing $w_{e,s}$ into another side tree at a higher level. We define f_{s+1} so that $f_{s+1}(w_{e,s+1}) = f_s(w_{e,s})$. Since $\mathrm{level}_T(\varphi_e(f_s(w_{e,s})))$ is finite, we will only have to repeat this process finitely often before reaching a stage s such that $f_s(w_{e,s})$ will satisfy $\mathcal{R}_e$ permanently.)

We first must prove that at each stage s at which we search for a t, we eventually find one. This requires a lemma guaranteeing our ability to embed trees upwards in $T[x_n]$.

Lemma 3.4.9 *For every $x_i \succeq x_n$ and every t, there is an embedding g of the tree $T_t[x_i]$ into $T[x_{i+1}]$.*

Proof. By the choice of n and Corollary 3.2.4, we know that every finite subtree of every $S[x_j]$ with $j \geq n$ embeds into some $S[x_k]$ with $k > j$. By induction, then, every finite subtree of every such $S[x_j]$ embeds into infinitely many $S[x_k]$ with $k > j$. Since there are only finitely many side trees $S[x_{j_0}], \ldots S[x_{j_n}]$ which intersect the finite tree T_t, we can embed $S[x_{j_0}] \cap T_t$ into some $S[x_{k_0}]$, then embed $S[x_{j_1}] \cap T_t$ into some $S[x_{k_1}]$ with $k_1 > k_0$, and so on. The union of these embeddings is the desired embedding g. ■

Lemma 3.4.10 *Fix any stage s, and take the corresponding e chosen in the construction. Then there exists a t for which Condition 3.4.8 holds.*

Proof. Since each sequence $\langle x_{i,t} \rangle_{t \in \omega}$ converges to x_i, we know that m_t converges to a limit m as $t \to \infty$. Thus $w_{e,s} \in S[x_m]$, and $m \geq n$. Moreover, there exists t such that $D_s \subseteq T_t$. By Lemma 3.4.9, there is an embedding $g : T_t[x_m] \to T[x_{m+1}]$, and then

$$\mathrm{level}_{D_s}(w_{e,s}) \leq \mathrm{level}_{T_t}(w_{e,s}) < \mathrm{level}_T(g(w_{e,s}))$$

since $\mathrm{level}_{T_t}(x) < \mathrm{level}_T(g(x))$ for every $x \in T_t[x_m]$. ■

Lemma 3.4.11 *For every e in ω, the sequence $\langle w_{e,s} \rangle_{s \in \omega}$ converges to a limit w_e, the sequence $\langle f_s(w_e) \rangle_{s \in \omega}$ converges to a limit $f(w_e)$, and either $\varphi_e(f(w_e))\uparrow$ or $\mathrm{level}_T(\varphi_e(f(w_e))) \neq \mathrm{level}_T(w_e)$. (Since $\mathrm{level}_T(w_e) = \mathrm{level}_{T'}(f(w_e))$, this satisfies $\mathcal{R}_e$.)*

Proof. Assume by induction that there exists a stage s_0 such that for all $s \geq s_0$ and all $i < e$, the hypotheses of the theorem hold: $w_{i,s} = w_i$, $f_s(w_i) = f(w_i)$, and either $\varphi_i(f(w_i))\uparrow$ or $\mathcal{R}_i$ is satisfied by $f(w_i)$ at stage s. Moreover, assume $x_{k,s} = x_k$ for every $k \leq j+1$ and every $s \geq s_0$, where j is maximal with $x_j \preceq w_{e-1}$, Then $m_s \geq j+1$ for every $s \geq s_0$, so $x_{m_s,s} \not\preceq w_i$ for all $i < e$ and $s \geq s_0$. If w_{e,s_0} is undefined, then at the first stage s after s_0 at which $\mathrm{ht}(D_s) > \mathrm{ht}(E_{e-1,s}) + \mathrm{level}_T(x_n)$, $w_{e,s}$ will be defined. Moreover, it will never again become undefined, since Condition 3.4.7 will never again be satisfied and case (2) will never apply.

Now if there is no stage $s \geq s_0$ such that $\varphi_{e,s}(w_{e,s})\downarrow$ and

$$\mathrm{level}_{D_s}(w_{e,s}) = \mathrm{level}_{D_s}(\varphi_{e,s}(f_s(w_{e,s}))),$$

then neither $w_{e,s}$ nor $f_s(w_{e,s})$ will ever be redefined after s_0. Then $\mathcal{R}_e$ will be satisfied by $w_e = \lim_s w_{e,s}$, because $\mathrm{level}_T(\varphi_e(f_s(w_{e,s})))$ is finite. Thus the lemma will be satisfied for e.

If there are stages $s \geq s_0$ where $\varphi_{e,s}(f_s(w_{e,s}))\downarrow$ and

$$\mathrm{level}_{D_s}(w_{e,s}) = \mathrm{level}_{D_s}(\varphi_{e,s}(f_s(w_{e,s}))),$$

then Condition 3.4.7 will not hold at those stages, so at each such s we will find a t satisfying Condition 3.4.8 and follow the corresponding instructions for that t. Thus, $w_{e,s+1}$ will be redefined, but with $f_{s+1}(w_{e,s+1}) = f_s(w_{e,s})$. Moreover, by our choice of g, we will have

$$\mathrm{level}_{D_{s+1}}(w_{e,s+1}) > \mathrm{level}_{D_s}(w_{e,s}).$$

Now $\mathrm{level}_{D_s}(\varphi_{e,s}(f_s(w_{e,s})))$ may increase as s increases, but only finitely often, since $f_s(w_{e,s})$ is constant after s_0 and $\mathrm{level}_T(\varphi_e(f_s(w_{e,s}))) < \omega$. Therefore, we eventually reach a stage s_1 with

$$\mathrm{level}_{D_{s_1}}(\varphi_e(f_{s_1}(w_{e,s_1}))) = \mathrm{level}_T(\varphi_e(f_{s_1}(w_{e,s_1}))),$$

and for all $s > s_1 + 1$, $\mathcal{R}_e$ will be satisfied by $w_{e,s}$. Therefore $w_{e,s}$ will never again be redefined, and $\mathcal{R}_e$ will be satisfied by $w_e = \lim_s w_{e,s}$. ∎

Lemma 3.4.12 *For every $x \in T$, the sequence $\langle f_s(x)\rangle_{s\in\omega}$ converges to a limit. The limit function $f = \lim_s f_s$ has range ω.*

Proof. Fix x. The construction ensures that $x \in T_x \subseteq D_x \subseteq D_s = \mathrm{dom}(f_s)$ for all $s \geq x$. If $x_n \not\prec x$, then $f_s(x) = f_{s+1}(x)$ for all s for which $f_s(x)$ is defined.

Assume, therefore, that $x_n \prec x$. Let $k = \max\{i : x_i \preceq x\}$, so $x \in S[x_k]$. Let s_0 be a stage such that for all $s \geq s_0$ and for all $i \leq k+1$, we have $x_{i,s} = x_i$. Also, let $h = \max\{i + \mathrm{ht}(S[x_i]) : i \leq k+1\}$. Then by Lemma 3.4.11, there exists a stage $s_1 \geq s_0$ such that for all $s \geq s_1$ and for all $i \leq h$, we have $w_{i,s} = w_i$.

Suppose $s \geq s_1$ is a stage such that $f_s \not\subseteq f_{s+1}$, and take the corresponding index e. Then Condition 3.4.8 is satisfied for some $t > s$, yielding an embedding $g : D_s[x_{m_t,t}] \to T_t[x_{m_t,t}]$. By the construction, $w_{e,s} \neq w_{e,s+1}$, so we must have $e > h$. This forces $\text{level}_{D_s}(w_{e,s}) > h$, since each $w_{i+1,s}$ is at a level $> i$, so $w_{e,s} \notin \bigcup_{i \leq k} S[x_i]$ by choice of h. Hence $x_{k+1} \preceq w_{e,s}$, and $m_t \geq k+1$ by definition of m_t (and since $t \geq s_0$). But then $x_{k+1} = x_{k+1,t} \preceq x_{m_t,t}$. Since $x \in S[x_k]$, we have $x_{k+1} \not\preceq x$, so $x_{m_t,t} \not\preceq x$. Hence $x \notin D_s[x_{m_t,t}]$, and so $x \notin \text{dom}(g)$. Therefore $f_{s+1}(x) = f_s(x)$ for all $s \geq s_1$. We define $f = \lim_s f_s$.

To see that $\text{range}(f) = \omega$, let $y \in \omega$. We assume inductively that $\{0, 1, \ldots y - 1\} \subseteq \text{range}(f)$. Therefore, if $y \notin \text{range}(f)$, there would exist a stage at which y would be the least available fresh element, and so there must be a stage s_0 and an $x \in T$ for which $f_{s_0}(x) = y$. Moreover, then $y \in \text{range}(f_s)$ for all $s \geq s_0$.

If there exists some stage $s_1 > s_0$ at which $f_{s_1-1}(x) \neq f_{s_1}(x)$, say for the sake of a requirement $\mathcal{R}_e$, then there must be an x' such that $f_{s_1}(x') = y$. At each such s_1, we will have $x' \in E_{e,s_1}$. Indeed, by taking s_1 so large that all $\mathcal{R}_i$ with $i \leq e$ are satisfied at all stages $s \geq s_1$, we may assume that $x' \in E_{e,s}$ for all $s \geq s_1$. But then $f_s(x') = f_{s+1}(x')$, so $y = f(x') \in \text{range}(f)$. ■

Thus f is a 1-1 Δ^0_2 map from T to ω, hence an isomorphism from T to the tree $(T', \prec')$, where $T' = \omega$ and $\prec'$ is just the ordering $\prec$, induced on T' from T' by f.

Lemma 3.4.13 *The maps f_s satisfy Condition 3.3.1. Thus $\prec'$ is computable.*

Proof. The construction ensures that $D_s \subseteq D_{s+1}$ for all s. For every $x \in D_s - D_s[x_n]$, we have $f_s(x) = f_{s+1}(x)$. Therefore, Condition 3.3.1 clearly holds if either $f_s^{-1}(a)$ or $f_s^{-1}(b)$ is not in $T[x_n]$. So take $x, y \in D_s[x_n]$, with $a = f_s(x)$, $b = f_s(y)$, and let $x' = f_{s+1}^{-1}(a)$ and $y' = f_{s+1}^{-1}(b)$. We have four cases, depending on whether or not $x = x'$ and $y = y'$.

The first case, where $x = x'$ and $y = y'$, is trivial. Also, if $x \neq x'$ and $y \neq y'$, then x and y must both lie in $D_s[x_{m_t,t}]$, for which we find an embedding g into some $T_t[x_{m_t,t}]$. In this case,

$$x \preceq y \iff g(x) \preceq g(y) \iff x' \preceq y'$$

since $g(x) = x'$ and $g(y) = y'$. Thus Condition 3.3.1 is satisfied in these two cases.

Suppose $x \neq x'$ and $y = y'$. Then $x \in D_s[x_{m_t,t}]$. If $y \in D_s[x_{m_t,t}]$ also, then $x' = g(x) \prec g(y) = y'$. If not, then either $y \prec x_{m_t,t}$ (in which case $y \prec x$ and $y \prec g(x) = x'$, since range$(g) \subset T[x_{m_t,t}]$) or $y \perp x_{m_t,t}$ (in which case $y \perp x$ and $y \perp g(x) = x'$, again because range$(g) \subset T[x_{m_t,t}]$).

The preceding paragraph shows that in the third case, not only

$$x \prec y \iff x' \prec y'$$

but also

$$x \perp y \iff x' \perp y'.$$

Hence by symmetry, the fourth case, with $x = x'$ and $y \neq y'$, is also satisfied. ■

Thus $(T', \prec')$ is a computable tree, isomorphic to T, which satisfies every requirement $\mathcal{R}_e$. Hence T is not computably categorical, proving Proposition 3.4.6.

■

3.4.4 No Isolated Paths

Proposition 3.4.14 *Let T be a computable tree such that T_{ext} is non-empty and finite-branching and every $x \in T_{ext}$ lies on infinitely many infinite paths through T. If all side trees in T have finite height, then T is not computably categorical.*

Proof. We use the same requirements $\mathcal{R}_e$ as in Propositions 3.3.2 and 3.4.6. The idea of this construction is that for each e, we devote an entire level l_e of T to satisfying $\mathcal{R}_e$. By the assumptions of the Proposition, we know that there exists at least one extendible node at level l_e, and at most finitely many of them. Also, there may exist any number of nonextendible nodes at level l_e. Since we cannot tell the extendible nodes from the nonextendible ones at any stage s, we consider all the nodes at level $l_{e,s}$ at that stage, and denote them by $v^0_{e,s}, v^1_{e,s}, \ldots v^{n_{e,s}}_{e,s}$.

Now since the Proposition assumes that the side tree above each extendible node has finite height, and since there exist only finitely many extendible nodes at levels $\leq l_e$, there must exist a number d_e such that every node x at level l_e with $\mathrm{ht}_x(T_s) \geq d_e$ must be extendible. We do not know d_e, but at each stage we focus on those nodes at level $l_{e,s}$ in $D_s \supseteq T_s$ above which D_s has maximal height. Thus, we will eventually be considering only extendible nodes and their successors. Above these nodes we look for upward embeddings to use to satisfy $\mathcal{R}_e$. Since every extendible node x lies on infinitely many infinite paths, and since T_{ext} is finite-branching, $T[x]$ must contain a subtree of type $2^{<\omega}$, and any finite tree can be embedded into $2^{<\omega}$ at arbitrarily high levels. Thus we can find upward embeddings of $D_s[x]$ above x whenever needed, as long as x is extendible.

The notation is as in the previous proofs, except that there may be more than one potential witness for a given requirement $\mathcal{R}_e$ at a given stage s. We denote these witnesses by $w^0_{e,s}, w^1_{e,s}, \ldots w^{n_{e,s}}_{e,s}$. Also, we will keep track of the original position of each of these witnesses. When $w^k_{e,s}$ is defined, we will set $v^k_{e,s} = w^k_{e,s}$, but as $w^k_{e,s}$ is embedded further up in the tree, $v^k_{e,s}$ stays fixed. The only stages at which $v^k_{e,s}$ will be redefined are those at which a requirement of higher priority receives attention and those at which $v^k_{e,s}$ acquires a new predecessor. For a given e and s, the elements $v^k_{e,s}$ will be at the same level for all k, and we will denote this level by $l_{e,s}$.

Let r be the root of T. We define $T_s = \{r\} \cup \{0, 1, \ldots s\}$, a tree under $\prec$. Again, we will define envelopes $E_{e,s}$, in order to ensure that $\mathrm{range}(f) = \omega$.

The requirements $\mathcal{R}_e$ are as follows:

$$\mathcal{R}_e : \quad \varphi_e \text{ total} \implies (\exists x \in T')\ [\mathrm{level}_{T'}(x) \neq \mathrm{level}_T(\varphi_e(x))].$$

$\mathcal{R}_e$ *receives attention at stage* s if some witness node $w^k_{e,s}$ is embedded upwards at stage s, if $w^0_{e,s}$ is newly defined at stage s, or if the height of the envelope $E_{e,s}$ increases at stage s. When this happens, all actions previously taken for the sake of requirements $\mathcal{R}_j$ with $j > e$ are injured. However, this will only occur finitely often for each e.

Construction: f_0 is the identity map on T_0, and the witness nodes $w^k_{e,0}$ and their original positions $v^k_{e,0}$ are undefined for all e and k. Also undefined are $n_{e,0}$

and $l_{e,0}$ for all e, and all $E_{e,0}$ are empty.

At stage $s+1$, we execute the following steps for each $e \leq s$, starting with $e = 0$. If a requirement $\mathcal{R}_e$ receives attention, then we do not execute the steps for any $j > e$.

1. If $w^0_{e,s}$ is undefined, and there exists an element x of D_s with

$$\text{level}_{D_s}(x) > \max \bigcup_{i<e} \{\text{level}_{D_s}(y) : y \in E_{i,s}\},$$

then let $l_{e,s+1}$ be its level, and let $w^0_{e,s+1}, \dots w^{n_{e,s+1}}_{e,s+1}$ be all the elements of D_s at level $l_{e,s+1}$. Let $v^k_{e,s+1} = w^k_{e,s+1}$ for each k. Requirement $\mathcal{R}_e$ has now received attention. Let $D_{s+1} = D_s \cup T_{s+1}$, and set $E_{e,s+1} = D_{s+1}$. For each $j > e$ we set

$$E_{j,s+1} = \{y \in D_s : (\exists z \in E_{j,s})\ y \preceq z\}.$$

2. If $w^0_{e,s}$ is undefined, and there does not exist any element x at a sufficiently high level to satisfy condition (1), then let $w_{e,s+1}\uparrow$ also, and set

$$E_{e,s+1} = \{y \in D_s : (\exists z \in E_{e,s})\ y \preceq z\}.$$

Then $\mathcal{R}_e$ has not received attention at this stage.

3. Otherwise, $w^0_{e,s}, \dots w^{n_{e,s}}_{e,s}$ are defined, as are the corresponding $v^k_{e,s}$. Find the least stage $t \geq \max(D_s)$ such that one of the following holds:

   (a) There exists $m \leq n_{e,s}$ and an embedding $g : D_s[v^m_{e,s}] \to T_t[v^m_{e,s}]$ such that
   $$\text{level}_{T_t}(g(w^m_{e,s})) \geq \text{level}_{D_s}(w^m_{e,s}) + s.$$

   (b) There exists $x \in T_t$ with $\text{level}_{T_t}(x) = l_{e,s}$ and $\text{ht}_x(T_t) \geq s$, such that either $x \notin D_s$ or $\text{level}_{D_s}(x) < l_{e,s}$.

   If (b) holds and (a) fails at stage t, let $w^k_{e,s+1} = w^k_{e,s}$ for all $k \leq n_{e,s}$, and let $l_{e,s+1} = l_{e,s}$. For each k, if $\text{level}_{D_s}(v^k_{e,s}) = l_{e,s}$, let $v^k_{e,s+1} = v^k_{e,s}$; otherwise let $v^k_{e,s+1}$ be the predecessor of $v^k_{e,s}$ at level $l_{e,s}$ in D_s. If there exist elements $x \in D_s$ with $\text{level}_{D_s}(x) = l_{e,s}$ such that $x \notin \{v^0_{e,s+1}, \ldots v^{n_{e,s}}_{e,s+1}\}$, then define those x's to be $w^{1+n_{e,s}}_{e,s+1}, w^{2+n_{e,s}}_{e,s+1}, \ldots$, with $v^k_{e,s+1} = w^k_{e,s+1}$ for each, and define $n_{e,s+1}$ to be the greatest superscript required. (If there are no such x, then $n_{e,s+1} = n_{e,s}$.) Define

$$E_{e,s+1} = \{y \in D_s : (\exists z \in E_{e,s})\ [y \preceq z]\}.$$

If $l_{e+1,s}\downarrow$ and $\text{ht}(E_{e,s+1}) \geq l_{e+1,s}$, then we say that $\mathcal{R}_e$ has received attention at stage $s+1$, and for each $j > e$ we set

$$E_{j,s+1} = \{y \in D_s : (\exists z \in E_{j,s})\ [y \preceq z]\}.$$

Otherwise $\mathcal{R}_e$ has not received attention.

If (a) holds at stage t, let m be the least index for which it holds, and let g be the corresponding embedding. If $\varphi_{e,s}(f_s(w^m_{e,s}))\uparrow$, or if

$$\text{level}_{D_s}(\varphi_{e,s}(f_s(w^m_{e,s}))) \neq \text{level}_{D_s}(w^m_{e,s}),$$

then we proceed exactly as in the preceding paragraph. Otherwise, $\mathcal{R}_e$ receives attention as follows. For every node $y \in D_s[v^m_{e,s}]$, define $f_{s+1}(g(y)) = f_s(y)$ and define $f_{s+1}(y)$ vto be the least element of ω which is not already in $\text{range}(f_{s+1}) \cup \text{range}(f_s)$. Let $w^m_{e,s+1} = g(w^m_{e,s})$. We define $l_{e,s+1} = l_{e,s}$. For each k, let $v^k_{e,s+1}$ be that predecessor of $v^k_{e,s}$ at level $l_{e,s}$ in D_s. (Quite possibly, this will be $v^k_{e,s}$ itself.) Also, if there are any $x \in D_s$ at level $l_{e,s}$ which are not in $\{v^k_{e,s+1} : k \leq n_{e,s}\}$, then define those x's to be $w^{1+n_{e,s}}_{e,s+1}, w^{2+n_{e,s}}_{e,s+1}, \ldots$, with $v^k_{e,s+1} = w^k_{e,s+1}$ for each, and define $n_{e,s+1}$ to be the greatest superscript required. (If there are no such x, then $n_{e,s+1} = n_{e,s}$.) Finally, let $D_{s+1} = D_s \cup \text{range}(g) \cup T_{s+1}$, and let $E_{e,s+1} = D_{s+1}$, with $E_{j,s+1} = \emptyset$ for all $j > e$.

4. If $\mathcal{R}_e$ has received attention at stage $s+1$, we make all $n_{j,s+1}$, $l_{j,s+1}$, $v^k_{j,s+1}$

and $w^k_{j,s+1}$ undefined for all $j > e$, and skip all steps for all those j. Otherwise we increment e by 1 and return to Step 1.

Once we have either given attention to a requirement or completed the steps with $e = s$, we define $f_{s+1}(y) = f_s(y)$ for those $y \in D_s$ on which f_{s+1} is not yet defined. Also, for each $y \in D_{s+1} - D_s$ on which f_{s+1} is not yet defined, choose the least element of ω which is not yet in range(f_{s+1}) to be $f_{s+1}(y)$. This completes the construction.

Lemma 3.4.15 *For each s and each $e \leq s$, either 3(a) or 3(b) must hold for some t.*

Proof. Suppose there exists an extendible node y among $\{v^0_{e,s}, \ldots v^{n_{e,s}}_{e,s}\}$. Then by the assumption of the proposition, there is a copy of $2^{<\omega}$ embedded into $T[y]$, and any finite tree can be embedded into $2^{<\omega}$ with the root mapping to a node at an arbitrarily large level of $2^{<\omega}$. Thus 3(a) will eventually hold.

Otherwise, none of $v^0_{e,s}, \ldots v^{n_{e,s}}_{e,s}$ is extendible. Now some node x on level $l_{e,s}$ of T must be extendible. If $x \in D_s$, then we must have $\text{level}_{D_s}(x) < l_{e,s}$, since no node at level $l_{e,s}$ in D_s is extendible. Otherwise $x \notin D_s$, and either way we will eventually reach a stage t at which 3(b) holds of x. ■

Lemma 3.4.16 *For every e the following hold:*

- $\lim_s \text{ht}(E_{e,s})$ *exists and is finite.*

- *The sequence $\langle l_{e,s} \rangle_{s\in\omega}$ converges to some $l_e \in \omega$.*

- *For every $k \in \omega$, either $\langle w^k_{e,s} \rangle_{s\in\omega}$ and $\langle v^k_{e,s} \rangle_{s\in\omega}$ converge to elements w^k_e and v^k_e in ω, or there exists a stage t such that $w^k_{e,s}\uparrow$ and $v^k_{e,s}\uparrow$ for all $s > t$.*

- *The requirement $\mathcal{R}_e$ receives attention at only finitely many stages, and is satisfied.*

Proof. We proceed by induction on e. Fix e, and assume s_0 is a stage satisfying all of the following conditions for every $s \geq s_0$ and every $i < e$:

1. $\mathcal{R}_i$ does not receive attention at stage s;
2. $l_{i,s} = l_i$;
3. Every $v \in T_{\text{ext}}$ with $\text{level}_T(v) = l_e$ satisfies $\text{level}_{T_s}(v) = l_e$, and hence is of the form $v_{e,s}^k$ for some k;
4. $v_{i,s}^k = v_i^k$ and $w_{i,s}^k = w_i^k$ for all k such that $v_{i,s}^k \in T_{\text{ext}}$ (Notice that each level of T_{ext} is finite, since the proposition assumes that T_{ext} is finitely branching. Hence only finitely many $v_{i,s}^k$ lie in T_{ext}.);
5. $\text{ht}(T_s) > l_{e-1}$.

Condition 3 simply says that we have waited until all predecessors of each $v \in T_{\text{ext}}$ at level l_e have appeared in T_{s_0}. This is possible because T_{ext} is finite-branching. Notice that this condition implies the same condition for all $i \leq e$.

Now $l_{e,s}$ is never redefined in the construction, and it can only become undefined at stages at which some $\mathcal{R}_i$ with $i < e$ receives attention. Hence $l_{e,s} = l_{e,s_0+1}$ for all $s > s_0$, so $l_{e,s}$ converges to a limit $l_e = l_{e,s_0+1}$. Also, after stage s_0 in the construction, $v_{e,s}^k$ can only be redefined to be a predecessor of itself, and that only when it has acquired a new predecessor. But by Condition 3, each $v_{e,s}^k$ acquires no new predecessors in T after stage s_0, so each sequence $\langle v_{e,s}^k \rangle_{s \in \omega}$ converges to a limit $v_e^k = v_{e,s_0}^k$.

Similarly, $w_{e,s}^k$ is never undefined after stage s_0, although it may be redefined at certain stages at which $\mathcal{R}_e$ receives attention. If $v_{e,s}^k \notin T_{\text{ext}}$, then $\text{ht}_{v_{e,s}^k}(T)$ is finite, and the corresponding $w_{e,s}^k$ can only be embedded finitely often by step 3(a), since each embedding (at a stage $s+1$) moves it up by at least s levels in D_s. Hence all those sequences $\langle w_{e,s}^k \rangle_{s \in \omega}$ converge.

For each of the finitely many k with $v_{e,s}^k \in T_{\text{ext}}$, it is possible for 3(a) to hold for k at infinitely many stages. However, we only actually apply the embedding g to redefine $w_{e,s}^k$ at stages $s+1$ such that $\varphi_{e,s}(f_s(w_{e,s}^k))\!\downarrow$ and $\text{level}_{D_s}(\varphi_{e,s}(f_s(w_{e,s}^k))) =$

$\mathrm{level}_{D_s}(w^k_{e,s})$. By the construction, we always have $f_{s+1}(w^k_{e,s+1}) = f_s(w^k_{e,s})$, even if $w^k_{e,s+1} \neq w^k_{e,s}$. At each stage $s+1$ at which $w^k_{e,s}$ is redefined, we have

$$\mathrm{level}_{D_{s+1}}(w^k_{e,s+1}) \geq \mathrm{level}_{D_s}(w^k_{e,s}) + s.$$

If this happens sufficiently often, then we will eventually reach a stage s_1 at which $\mathrm{level}_{D_{s_1}}(w^k_{e,s_1}) > \mathrm{level}_T(\varphi_{e,s_1}(f_{s_1}(w_{e,s_1})))$, since T has height ω, and after stage s_1, we will never redefine $w^k_{e,s}$ again, even if 3(a) does apply. Hence each of these sequences $\langle w^k_{e,s}\rangle_{s\in\omega}$ does converge to a limit w^k_e.

Now there must be an element of T_{ext} on level l_e, and this element will be designated at some stage s as $v^k_{e,s}$ for some k. We note first that since all side trees are finite and T_{ext} is finitely-branching, there is a d such that every nonextendible node x at any level $\leq l_e$ satisfies $\mathrm{ht}_x(T) < d$. (Also, assume d is sufficiently large that $l_{e,d} = l_e$.) Once we reach stages $s \geq d$, therefore, 3(a) will never again hold for any m with $v^m_{e,s}$ nonextendible, and 3(b) will not hold for any nonextendible x. Thus only the finitely many extendible nodes $v^k_{e,s}$ will satisfy either 3(a) or 3(b) at any subsequent stage. But every extendible node v at level l_e in T already satifies $\mathrm{level}_{T_{s_0}}(v) = l_e$, by inductive hypothesis, so 3(b) will never hold again. By Lemma 3.4.15, there must exist an m, with v^m_e extendible, which satisfies 3(a) at infinitely many stages. (If there is more than one such, choose the least of them, just as we did at each stage of the construction.)

If $\varphi_e(f_s(w^m_{e,s}))\uparrow$ for the corresponding w^m_e, then $w^m_{e,s}$ is never redefined, and $f_{s+1}(w^m_e) = f_s(w^m_e)$ for all s, so $\varphi_e(f(w^m_e))\uparrow$, where $f = \lim_s f_s$ as defined below. Hence $\mathcal{R}_e$ is satisfied, since φ_e is not total. On the other hand, if $\varphi_e(f_s(w^m_{e,s}))\downarrow$, then for every stage s at which

$$\mathrm{level}_{D_s}(\varphi_{e,s}(f_s(w^m_{e,s}))) = \mathrm{level}_{D_s}(w^m_{e,s}),$$

either there will be a subsequent stage s' at which 3(a) applies to $v^m_{e,s'}$ and $\mathcal{R}_e$

receives attention and $w^m_{e,s'}$ is embedded at a greater level, or else

$$(\forall s' > s)[\text{level}_{D_{s'}}(w^m_{e,s'}) < \text{level}_{D_{s'}}(\varphi_{e,s'}(f_{s'}(w^m_{e,s'})))].$$

In the latter case, $w^m_{e,s'}$ will never again be redefined, leaving $\mathcal{R}_e$ satisfied by the witness $f(w^m_e)$. In the former case, we again have

$$\text{level}_{D_{s'+1}}(\varphi_e(f_{s'+1}(w^m_{e,s'+1}))) < \text{level}_{D_{s'}}(w^m_{e,s'}).$$

But

$$\text{level}_{D_s}(\varphi_e(f(w^m_e))) \leq \text{level}_T(\varphi_e(f(w^m_e))) < \omega,$$

so eventually we reach a stage s with $\text{level}_T(\varphi_e(f(w^m_e))) < \text{level}_{D_s}(w^m_{e,s})$. After this stage, $w^m_{e,s}$ is never redefined, leaving

$$\text{level}_T(\varphi_e(f(w^m_e))) < \text{level}_T(w^m_e) = \text{level}_{T'}(f(w^m_e)).$$

Thus requirement $\mathcal{R}_e$ is satisfied.

We note that since each sequence $\langle w^k_{e,s}\rangle_{s\in\omega}$ converges to w^k_e, none of them changes value more than finitely often. Moreover, the stage d designated above has the property that only finitely many elements $w^k_{e,s}$ are ever redefined after stage d, namely those corresponding to extendible v^k_e.

Moreover, since there are only finitely many stages s at which any of the elements $w^k_{e,s}$ is redefined, we eventually reach a stage s_1 after which none of them is ever redefined. Now E_{e,s_1} is finite. Let s_2 be a stage such that

$$(\forall y \in T)[(\exists z \in E_{e,s_1})[y \preceq z] \implies y \in T_{s_2}].$$

That is, every predecessor of each of the (finitely many) elements $x \in E_{e,s_1}$ appears in T_{s_2}. Then for all $s \geq s_2$, we have $E_{e,s} = E_{e,s_2}$. Hence $\lim_s \text{ht}(E_{e,s}) = \text{ht}(E_{e,s_2})$. Thus $\mathcal{R}_e$ only receives attention finitely often.

This completes the induction. ■

Lemma 3.4.17 *For each x, the sequence $\langle f_s(x)\rangle_{s\in\omega}$ converges. The limit function $f = \lim_s f_s$ has range ω.*

Proof. We need to show that both $\lim_s f_s(x)$ and $\lim_s f_s^{-1}(y)$ exist for all x and y in ω.

First of all, we have $x \in T_s \subseteq D_s$ for every $s \geq x$, so $f_s(x)\downarrow$ for all sufficiently large s. Also, by the construction, we have range$(f_s) \subseteq$ range(f_{s+1}) for every s. Moreover, each time we need a new element for the range of f_{s+1}, we take the least available one, so clearly every $y \in \omega$ lies in range(f_s) for all sufficiently large s.

So suppose $f_s(x) \neq f_{s+1}(x)$ for some s. The only way this can occur in our construction is if 3(a) holds for some e and m, and we execute an upwards embedding g of $D_s[v^m_{e,s}]$ into $T[v^m_{e,s}]$ at stage $s+1$ in order to satisfy $\mathcal{R}_e$. If this happens, then $E_{e,s+1} = D_{s+1} \supseteq$ range(g), so $x \in E_{e,s+1}$. Similarly, if $f_s^{-1}(y) \neq f_{s+1}^{-1}(y)$ for some s, then $f_{s+1}^{-1}(y) \in E_{e,s+1}$.

The only way we could then have $f_t(x) \neq f_{t+1}(x)$ or $f_t^{-1}(y) \neq f_{t+1}^{-1}(y)$ for any $t > s$ is if some $\mathcal{R}_i$ with $i \leq e$ receives attention at stage $t+1$. This could happen for the following reasons:

Case 1: Step 3(a) applies to $\mathcal{R}_i$ for some $i \leq e$, and we execute the corresponding upward embedding g. In this case, $E_{i,t+1} = D_{t+1}$, so $x \in E_{i,t+1}$ and $f_{s+1}^{-1}(y) = g(x) \in E_{i,t+1}$.

Case 2: $w^0_{i,t}\uparrow$ and $w^0_{i,t+1}\downarrow$, for some $i \leq e$. However, although $\mathcal{R}_i$ does receive attention in this case, the construction leaves $E_{e,t} \subseteq B_{e,t+1}$. Hence $x \in E_{e,t+1}$, and $f_{t+1}(x) = f_t(x)$. Similarly, $f_{s+1}^{-1}(y) = f_s^{-1}(y) \in E_{e,t+1}$.

Case 3: $\text{ht}(E_{i,t+1}) > l_{i+1,t}$ for some $i < e$. Again, the construction leaves $E_{e,t} \subseteq E_{e,t+1}$, so $x \in E_{e,t+1}$ and $f_{t+1}(x) = f_t(x)$ and $f_{s+1}^{-1}(y) = f_s^{-1}(y) \in E_{e,t+1}$.

Thus, for every $t > s$, we have both x and $f_t^{-1}(y)$ in $Ev_{i,t}$ for some $i \leq e$. Therefore, $f_{t+1}(x) \neq f_t(x)$ and $f_{t+1}^{-1}(y) \neq f_t^{-1}(y)$ each can occur only for the sake of an upwards embedding on behalf of some $\mathcal{R}_i$ with $i \leq e$. By Lemma 3.4.16, this can only occur finitely often. Hence the sequences $\langle f_s(x)\rangle_{s\in\omega}$ and $\langle f_s^{-1}(y)\rangle_{s\in\omega}$ both converge, making $f = \lim_s f_s$ a Δ^0_2-bijection from ω to ω. ∎

As usual, we lift the partial order $\prec$ from T to an order $\prec'$ on T', making f an isomorphism from T to T'.

Lemma 3.4.18 *The functions f_s satisfy Condition 3.3.1. Hence $\prec'$ is computable.*

Proof. We have already seen that $\mathrm{range}(f_s) \subseteq \mathrm{range}(f_{s+1})$. Take $a, b \in \mathrm{range}(f_s)$. The only way for $f_{s+1}^{-1}(b) \neq f_s^{-1}(b)$ is if $f_s^{-1}(b)$ lies in some subtree $D_s[v_{e,s}^m]$ which is embedded upward via some g as part of Step 3(a) for some e at stage $s+1$. If $f_s^{-1}(a)$ is also embedded upward at stage $s+1$, then since g is a homomorphism of trees, we have:

$$f_s^{-1}(a) \prec f_s^{-1}(b) \iff g(f_s^{-1}(a)) \prec g(f_s^{-1}(b)) \iff f_{s+1}^{-1}(a) \prec f_{s+1}^{-1}(b).$$

Otherwise, $f_s^{-1}(a) \notin D_s[v_{e,s}^m]$. In this case:

$$\begin{aligned} f_s^{-1}(a) \prec f_s^{-1}(b) &\iff f_s^{-1}(a) \prec v_{e,s}^m \\ &\iff f_{s+1}^{-1}(a) \prec v_{e,s}^m \\ &\iff f_{s+1}^{-1}(a) \prec f_{s+1}^{-1}(b). \end{aligned}$$

The case $f_{s+1}^{-1}(b) = f_s^{-1}(b)$ is simpler, since this implies $f_s^{-1}(b) \notin D_s[v_{e,s}^m]$. Thus, if $f_s^{-1}(a) \prec f_s^{-1}(b)$, we know that $f_s^{-1}(a) = f_{s+1}^{-1}(a)$, so $f_{s+1}^{-1}(a) \prec f_{s+1}^{-1}(b)$ and conversely as well. ■

Thus $(T', \prec')$ is a computable tree, isomorphic to T via f, yet not computably isomorphic to T, since every requirement $\mathcal{R}_e$ is satisfied. Therefore, T is not computably categorical. This completes the proof of Proposition 3.4.14. ■

3.4.5 Proof of the Theorem

Proof of Theorem 3.4.1. We need only confirm that the preceding propositions cover all possible cases. First, if T contains no extendible nodes, then Proposition

3.4.3 applies to the root of T, since $\mathrm{ht}(T) = \omega$. If T_{ext} is nonempty and infinite-branching, then Proposition 3.4.5 covers this case. If T_{ext} is nonempty and finite-branching, then we ask whether there exist side trees of height ω. If so, then Proposition 3.4.4 gives the result. Otherwise, every side tree has finite height. If every extendible node lies on infinitely many infinite paths, we apply Proposition 3.4.14. If there exists a node $x \in T_{\mathrm{ext}}$ which lies on only finitely many infinite paths through T, then by following those finitely many infinite paths upwards until they all diverge, we find a node $x_0 \in T_{\mathrm{ext}}$ which fits Proposition 3.4.6.) ■

CHAPTER 4

ORBITS OF COMPUTABLY ENUMERABLE SETS: AVOIDING AN UPPER CONE

4.1 Introduction

We now return to the subject of the lattice $\mathcal{E}$ of computably enumerable sets under inclusion. The relation between this lattice and the upper semi-lattice of the same sets under Turing reducibility is such that often properties of a set in the former allow us to infer properties of the same set in the latter, or vice versa. The $\mathcal{E}$-definable property of maximality, for instance, enabled Martin to characterize the high c.e. degrees as those which contained a maximal set ([33]), and other $\mathcal{E}$-definable properties discovered by Harrington and Soare imply Turing completeness, Turing incompleteness, and non-lowness (see [21] and [23]).

The study of $\mathcal{E}$ often focusses on automorphisms of the lattice and the orbits of c.e. sets under those automorphisms. We say that two c.e. sets are *automorphic* if they lie in the same orbit. Again, the Turing-degree properties of a set often yield insight into the orbit of the set. Harrington and Soare have shown (in [22]) that the orbit of a noncomputable c.e. set must contain a set of high degree, and the same paper proves Harrington's theorem that the orbit of a noncomputable c.e. set cannot be contained in the lower cone $\{B \in \mathcal{E} : B \leq_T A\}$ below any c.e. set A (unless A is Turing-complete, of course). On the other hand, Wald showed in [51] that the orbit of a low c.e. set must intersect the lower cone below any given promptly simple set C. (This result fails to hold for certain non-prompt sets C, however, by a result of Downey and Harrington.)

In this chapter we use the Turing-definable property of lowness to avoid an upper cone. Specifically, our main theorem is:

Theorem 4.1.1 *For every low c.e. set A and every noncomputable c.e. set C, there exists an automorphism of $\mathcal{E}$ mapping A to a set B such that $C \not\leq_T B$.*

Thus, the orbit of A cannot be contained in the upper cone above C.

The main tool for proving this result is the New Extension Theorem of Soare, as stated in [48]. The lowness of A allows us to predict with fair certainty (i.e. with only finitely many incorrect guesses) which elements of any given c.e. set will eventually enter A and which will stay in its complement $\overline{A}$.

Much of the machinery in this chapter is identical to that used in [22], [48], and [51]. We have deliberately tried to keep our notation and intuitions the same as in those papers whenever possible, in order that readers familiar with the constructions in those papers will find it easier to follow this one. One noticeable distinction is the use of $\mathcal{K}_\alpha$, which was defined in [22] (equation (14), p. 625) to mean precisely the opposite of its meaning in [48], [51], and the present chapter. *Caveat lector!*

All sets mentioned in this chapter will be c.e. unless specifically stated otherwise. (Complements, of course, need not be c.e.)

4.2 Construction

4.2.1 Defining e-States on a Tree

To prove Theorem 4.1.1, we must construct an automorphism of $\mathcal{E}$. By a result of Soare (in [47], XV.2.6), it suffices to construct an automorphism of $\mathcal{E}^*$, the quotient of $\mathcal{E}$ by the ideal of finite sets. Thus we must map every c.e. set U_e to some other c.e. set $\hat{U}_e$ in such a way that unions and intersections are preserved up to finitely many elements. Ordinarily we would employ an e-state construction for this purpose, where by the e-state of an element x at stage s we simply mean

$$\{i < e : x \in U_{i,s}\}$$

and the general e-state of x is

$$\{i < e : x \in U_i\}.$$

(Thus, for instance, the 4-state 0101 indicates that an element lies in U_1 and U_3, but not in U_0 or U_2.) The corresponding e-state for sets $\hat{U}_e$ in the range of the automorphism would be defined in exactly the same way. We have a copy of ω, denoted by $\widehat{\omega}$, containing the elements of sets in the range, and we write $\hat{x}$ to stand for such an element.

To ensure that the map be onto, we would use a second enumeration $V_0, V_1, \ldots$ of all c.e. sets and make sure that for each e there is a c.e. set $\hat{V}_e$ which maps to V_e. This gives rise to an additional e-state, that with respect to the sets $\hat{V}_e$, and the full e-state of x would be $\langle \sigma, \tau \rangle$ where σ is the e-state relative to the sets U_e and τ is the e-state relative to the sets $\hat{V}_e$. We would then need to create our automorphism in such a way that for every full e-state (relative to sets in the domain) which contained infinitely many elements x, the corresponding full e-state (relative to the sets $\hat{U}_e$ and V_e in the range) contained infinitely many elements $\hat{x}$, and conversely. (For details, see [47], XV.4.3.)

In the present theorem, however, we have additional negative requirements $\mathcal{Q}_e$

to ensure that the image of A under the automorphism does not lie in the upper cone above C. These requirements follow the *Sacks preservation strategy* for B, the image of A, and are stated below, after we define the necessary machinery. (A description of the Sacks preservation strategy in a simpler situation is given in [47], VII.3.1.)

In order to construct the automorphism while respecting the negative requirements, we must make guesses about which e-states really do contain infinitely many elements. Since elements can move from one e-state to another between stages, the number of elements in a given e-state fluctuates. Some e-states accumulate more and more elements, and wind up in the end with infinitely many; we say that such e-states are *well-resided*. For other e-states, there are infinitely many elements which enter that state at some stage but only finitely many which remain there for good. These e-states are *well-visited*, but not well-resided. (The well-resided states are also considered to be well-visited.) Finally, an e-state which is not well-visited has only finitely many elements that ever enter that state. We write $\mathcal{K}$ to represent the set of well-resided e-states, $\mathcal{M}$ to represent the set of well-visited e-states, and $\mathcal{N}$ to represent the set of e-states which are well-visited but not well-resided. Thus $\mathcal{K} = \mathcal{M} - \mathcal{N}$.

Our guesses about these possibilities for each e-state lead us to employ a tree construction. Each node α of the tree T at level e will represent a guess about which e-states are well-visited and which of those are well-resided. Indeed, the c.e. sets we build will depend on our guesses: for each $\alpha \in T$ with $|\alpha| \equiv 1 \pmod 5$, we will have a set U_α. Therefore, we will not speak of e-states, but rather of α-states, which are just e-states relative to the sets $U_{\alpha\restriction 1}, U_{\alpha\restriction 6}, U_{\alpha\restriction 11}, \ldots U_\alpha$. The *true path* f through T will correspond to the correct guesses, and the collection $\{U_\alpha : \alpha \subset f \ \& \ |\alpha| \equiv 1 \pmod 5\}$ will include every c.e. set W_e (up to finite difference).

We will use $\mathcal{M}_\alpha$ to denote the set of α-states which α believes to be well-visited. The set containing those states which α believes to be well-visited but not well-resided will be partitioned into two subsets $\mathcal{B}_\alpha \sqcup \mathcal{R}_\alpha$, according to the method which α believes is used to remove elements from those states. Also, for each α we write

$$e_\alpha = \max\{k \in \omega : 5k < |\alpha|\}.$$

Therefore, if $\beta = \alpha^-$ is the immediate predecessor of α in T, then the set U_α is defined if and only if $e_\alpha > e_\beta$. We also have the sets V_α on the $\widehat{\omega}$-side which ensure that the automorphism is onto. Then $\hat{e}_\alpha$ is defined by

$$\hat{e}_\alpha = \max\{k \in \omega : 5k + 1 < |\alpha|\},$$

and the set V_α is defined if and only if $\hat{e}_\alpha > \hat{e}_\beta$. (For the purposes of this paper, we could use a modulus smaller than 5, but we will adhere to the usage in previous papers.)

T will contain a unique node ρ of length 1, and we will ensure that $U_\rho = A$. The set $\hat{U}_\rho$ which we build will be the image of A under the automorphism, so this is the set B for which we must worry about the negative restraints. We will often speak of $\overline{A}$-states and $\overline{B}$-states. These terms refer to full α-states which exclude U_ρ and $\hat{U}_\rho$, respectively. If x is in an $\overline{A}$-state at stage s, then $x \notin A_s$, and if $\hat{x}$ is in a $\overline{B}$-state at stage s, then $\hat{x} \notin B_s$.

We think of the sets U_α as being "red" sets, containing elements $x \in \omega$, by which we mean that the elements x are enumerated in these sets by a player called "RED." The other player in the game, "BLUE," tries to match the moves of RED by moving his own elements $\hat{x}$ (from the other copy $\widehat{\omega}$ of ω) among the sets $\hat{U}_\alpha$, so that the map taking U_α to $\hat{U}_\alpha$ will be an automorphism. Again, to ensure surjectivity of this map, RED will also play sets V_α containing the elements $\hat{x} \in \widehat{\omega}$, so that every computably enumerable set is represented (up to finite difference) by at least one V_α along the true path, and it will be up to BLUE to build corresponding sets $\hat{V}_\alpha$ of the elements $x \in \omega$. Ultimately BLUE's goal is that each full α-state on the ω-side should contain infinitely many elements x if and only if the corresponding full α-state on the $\widehat{\omega}$-side contains infinitely many elements $\hat{x}$.

In light of this RED/BLUE dichotomy, the class $\mathcal{N}_\alpha$ of α-states which are well-visited but not well-resided will be partitioned into disjoint subclasses $\mathcal{R}_\alpha$ and $\mathcal{B}_\alpha$. The latter contains every state which is emptied out by BLUE, i.e. such that cofinitely many of the elements which enter that state eventually leave the state because they are enumerated into some other blue set. (Here we include B as a blue set.) $\mathcal{R}_\alpha$ contains every state which is emptied out by RED. Of course, an α-state ν can be emptied out by both players, since there could be infinitely many elements enumerated into a red set and infinitely many others enumerated into a blue set. Such states are assigned to either $\mathcal{R}_\alpha$ or $\mathcal{B}_\alpha$ (but not both!) according to which player empties out the corresponding γ-state, where $\gamma \subseteq \alpha$ is the least predecessor of α such that the γ-state corresponding to ν is not well-resided.

4.2.2 Definitions

To the extent possible, we take our definitions straight from [22] and [51]. One change is the use of the superscript 0, so that (for instance) $\mathcal{M}_\alpha^0$ and $\widehat{\mathcal{M}}_\alpha^0$ will replace $\mathcal{M}_\alpha^{\overline{A}}$ and $\widehat{\mathcal{M}}_\alpha^{\overline{B}}$.

To define the tree T, we need the formal definition of an α-state.

Definition 4.2.1 An *α-state* is a triple $\langle \alpha, \sigma, \tau \rangle$ where $\sigma \subseteq \{0, \ldots, e_\alpha\}$ and $\tau \subseteq \{0, \ldots, \hat{e}_\alpha\}$. The only λ-state is $\nu_{-1} = \langle \lambda, \emptyset, \emptyset \rangle$. If $0 \notin \sigma$, then we call the state an $\overline{A}$-state or a $\overline{B}$-state.

As in [51], we define our tree T with a specific node ρ at level 1, since the corresponding c.e. sets U_0 and $\hat{U}_0$ are A and B. Also, here we specify the sets U_i and V_i. Pick some e such that $W_i = A$, and define:

$$\begin{aligned} U_{0,s} &= W_{i,s} \\ U_{e,s} &= W_{e,s} \quad \text{for all } e > 0 \\ V_{e,s} &= W_{e,s} \quad \text{for all } e. \end{aligned}$$

Definition 4.2.2 We define the tree T as follows:

Let the empty node λ be the root of T and ρ the unique node at level 1, defined as follows:

$$\begin{array}{cc} \mathcal{M}^0_\lambda = \widehat{\mathcal{M}}^0_\lambda = \emptyset & \mathcal{M}^0_\rho = \widehat{\mathcal{M}}^0_\rho = \{\langle \rho, \emptyset, \emptyset \rangle, \langle \rho, \{0\}, \emptyset \rangle\} \\ \mathcal{R}^0_\lambda = \mathcal{B}^0_\lambda = \emptyset & \mathcal{R}^0_\rho = \mathcal{B}^0_\rho = \emptyset \\ k_\lambda = -1 & k_\rho = -1 \\ e_\lambda = -1 & e_\rho = 0 \\ \hat{e}_\lambda = -1 & \hat{e}_\rho = -1 \end{array}$$

For every $\beta \in T$ with $\beta \neq \lambda$, we put $\alpha = \beta^\frown \langle \mathcal{M}^0_\alpha, \mathcal{R}^0_\alpha, \mathcal{B}^0_\alpha, k_\alpha \rangle$ in T (and write $\beta = \alpha^-$) providing the following conditions hold:

(i) β is consistent (as defined in Definition 4.2.5 below),

(ii) $\mathcal{M}^0_\alpha$ is a set of $\overline{A}$-α-states, $\mathcal{R}^0_\alpha \cup \mathcal{B}^0_\alpha \subseteq \mathcal{M}^0_\alpha$, and $\mathcal{R}^0_\alpha \cap \mathcal{B}^0_\alpha = \emptyset$,

(iii) $\mathcal{M}^0_\alpha \restriction \beta \subseteq \mathcal{M}^0_\beta$,

(iv) $[e_\alpha = e_\beta \ \& \ \hat{e}_\alpha = \hat{e}_\beta] \implies \mathcal{M}^0_\alpha = \mathcal{M}^0_\beta$,

(v) $\mathcal{R}^{<\alpha}_\alpha =_{\text{dfn}} \{\nu \in \mathcal{M}^0_\alpha : \nu \restriction \beta \in \mathcal{R}^0_\beta\} \subseteq \mathcal{R}^0_\alpha$,

(vi) $\mathcal{B}^{<\alpha}_\alpha =_{\text{dfn}} \{\nu \in \mathcal{M}^0_\alpha : \nu \restriction \beta \in \mathcal{B}^0_\beta\} \subseteq \mathcal{B}^0_\alpha$,

(vii) $\mathcal{R}^{\alpha}_\alpha =_{\text{dfn}} \mathcal{R}^0_\alpha - \mathcal{R}^{<\alpha}_\alpha \neq \emptyset \implies |\alpha| \equiv 3 \bmod 5$,

(viii) $\mathcal{B}^{\alpha}_\alpha =_{\text{dfn}} \mathcal{B}^0_\alpha - \mathcal{B}^{<\alpha}_\alpha \neq \emptyset \implies |\alpha| \equiv 4 \bmod 5$.

In addition, each $\alpha \in T$ has associated dual sets $\widehat{\mathcal{M}}^0_\alpha$, $\widehat{\mathcal{R}}^0_\alpha$, and $\widehat{\mathcal{B}}^0_\alpha$ which are determined from $\mathcal{M}^0_\alpha$, $\mathcal{B}^0_\alpha$ and $\mathcal{R}^0_\alpha$ by

$$\widehat{\mathcal{M}}^0_\alpha = \{\hat{\nu} : \nu \in \mathcal{M}^0_\alpha\} \tag{4.1}$$

$$\mathcal{B}^\alpha_\alpha =_{\text{dfn}} \{\nu : \hat{\nu} \in \widehat{\mathcal{R}}^\alpha_\alpha\} \tag{4.2}$$

$$\widehat{\mathcal{B}}^\alpha_\alpha =_{\text{dfn}} \{\hat{\nu} : \nu \in \mathcal{R}^\alpha_\alpha\} \tag{4.3}$$

Also, α has associated integers e_α and $\hat{e}_\alpha$ (depending only on $|\alpha|$) defined by

$$e_\alpha = \max\{k \in \omega : 5k < |\alpha|\} \qquad \hat{e}_\alpha = \max\{k \in \omega : 5k + 1 < |\alpha|\}.$$

We identify the finite object $\langle \mathcal{M}^0_\alpha, \mathcal{R}^0_\alpha, \mathcal{B}^0_\alpha, k_\alpha \rangle$ with an integer under some effective coding, so that we may regard T as a subtree of $\omega^{<\omega}$. Therefore the partial order on T will be denoted by $\subseteq$. We write $\alpha <_L \gamma$ to denote that α is to the left of γ on the tree, i.e. that there exists $\delta \in T$ and $m < n$ in ω with $\delta\frown m \subseteq \alpha$ and $\delta\frown n \subseteq \gamma$.

The consistency required by part (i) above is defined as follows.

Definition 4.2.3 A node $\alpha \in T$ is $\mathcal{R}^0$-*consistent* if

$$(\forall \nu_0 \in \mathcal{R}^0_\alpha)(\exists \nu_1)[\nu_0 <_R \nu_1 \ \ \& \ \ \nu_1 \in \mathcal{M}^0_\alpha], \tag{4.4}$$

The node α is $\widehat{\mathcal{R}}^0$-*consistent* if

$$(\forall \hat{\nu}_0 \in \widehat{\mathcal{R}}^0_\alpha)(\exists \hat{\nu}_1)[\hat{\nu}_0 <_R \hat{\nu}_1 \ \ \& \ \ \nu_1 \in \widehat{\mathcal{M}}^0_\alpha], \tag{4.5}$$

If α is both $\mathcal{R}^0$-consistent and $\widehat{\mathcal{R}}^0$-consistent, then we say that α is $\mathcal{R}$-*consistent*; otherwise α is $\mathcal{R}$-*inconsistent*

Definition 4.2.4 A node $\alpha \in T$, with $\beta = \alpha^-$, is $\mathcal{M}$-*consistent* if

$$e_\alpha > e_\beta \implies (\forall \nu_0 \in \mathcal{M}^0_\alpha)(\forall\ \alpha\text{-states } \nu_1)[\nu_1 \restriction \beta \in \mathcal{M}^0_\beta \implies \nu_1 \in \mathcal{M}^0_\alpha].$$

Definition 4.2.5 The node α is *consistent* if it is both $\mathcal{R}$-consistent and $\mathcal{M}$-consistent.

Notice that we can compute uniformly for any α whether it is consistent or not, since there are only finitely many α-states.

The superscript "0" in $\mathcal{M}^0_\alpha$, etc. is intended to make clear that we are only concerned with $\overline{A}$-states (and $\overline{B}$-states, in the dual). After all, $U_0 = A$, so any $\overline{A}$-state $\nu = \langle \alpha, \sigma, \tau \rangle$ will have $\sigma(0) = 0$ (as defined below). Similarly, $\hat{\sigma}(0) = 0$ for $\overline{B}$-states $\hat{\nu}$.

In Subsection 4.2.4 we will approximate the true path f through T by a uniformly computable sequence of nodes $\{f_s\}_{s\in\omega}$. A node α will lie on f if and only if α is the leftmost node at level $|\alpha|$ in T such that $\alpha \subseteq f_s$ for infinitely many s.

The nodes of the true path are the only nodes for which we ultimately need the construction to work, but since all we have is an approximation to the true path, we must follow the dictates of that approximation at each stage. Each element x ($\hat{x}$) will be assigned to a given node $\alpha(x, s)$ ($\alpha(\hat{x}, s)$) at each stage. $\alpha(x, s)$ may be redefined at stage $s + 1$ to equal an immediate successor of $\alpha(x, s)$. Moreover, if the true path moves to the left of $\alpha(x, s)$, then $\alpha(x, s)$ may be redefined so that $\alpha(x, s + 1) <_L \alpha(x, s)$ or so that $\alpha(x, s + 1)$ is a predecessor of $\alpha(x, s)$. However, $\alpha(x, s + 1)$ will never move back to the right of $\alpha(x, s)$. The construction will ensure that $\alpha(x) = \lim_s \alpha(x, s)$ exists and that cofinitely many x wind up being assigned to nodes on f, with the finitely many remaining ones all being assigned to nodes to the left of f.

We use the elements assigned to node α and its successors vat stage s to help build U_α, writing:

$$S_{\alpha,s} = \{x \in \omega : \alpha(x, s) = \alpha\}$$

$$\hat{S}_{\alpha,s} = \{\hat{x} \in \widehat{\omega} : \alpha(\hat{x}, s) = \alpha\}$$

$$R_{\alpha,s} = \{x \in \omega : \alpha \subseteq \alpha(x, s)\}$$

$$Y_{\alpha,s} = \cup_{t \leq s} R_{\alpha,t}.$$

The duals $\hat{R}_{\alpha,s}$ and $\hat{Y}_{\alpha,s}$ are defined similarly. Each of these sets is computable. However, in the limits, only Y_α is even c.e.:

$$S_\alpha = \{x \in \omega : \alpha(x) = \alpha\}$$

$$R_\alpha = \{x \in \omega : \alpha \subseteq \alpha(x)\}$$

$$Y_\alpha = \cup_t R_{\alpha,t}.$$

We now give the formal definition of the α-state of an element $x \in \omega$ or $\hat{x} \in \widehat{\omega}$. In general we will only be interested in $\nu(\alpha, x, s)$ when $\alpha \subseteq \alpha(x, s)$, but the definition applies for any $\alpha \in T$.

Definition 4.2.6 (i) The *α-state of x at stage s*, $\nu(\alpha, x, s)$, is the triple $\langle \alpha, \sigma(\alpha, x, s), \tau(\alpha, x, s)\rangle$ where

$$\sigma(\alpha, x, s) = \{e_\beta : \beta \subseteq \alpha \ \ \& \ \ e_\beta > e_{\beta-} \ \ \& \ \ x \in U_{\beta,s}\},$$

$$\tau(\alpha, x, s) = \{\hat{e}_\beta : \beta \subseteq \alpha \ \ \& \ \ \hat{e}_\beta > \hat{e}_{\beta-} \ \ \& \ \ x \in \widehat{V}_{\beta,s}\}.$$

(ii) The *final α-state of x* is $\nu(\alpha, x) = \langle \alpha, \sigma(\alpha, x), \tau(\alpha, x)\rangle$ where $\sigma(\alpha, x) = \lim_s \sigma(\alpha, x, s)$ and $\tau(\alpha, x) = \lim_s \tau(\alpha, x, s)$.

The α-state of an element $\hat{x} \in \widehat{\omega}$ is defined similarly, with $\widehat{U}_{\beta,s}$ in place of $U_{\beta,s}$ and $V_{\beta,s}$ in place of $\widehat{V}_{\beta,s}$.

For each $\alpha \in T$ we define the following classes of $\overline{A}$-α-states

$$\mathcal{E}^0_\alpha = \{\nu : (\exists^\infty x)(\exists s)[x \in \overline{A}_s \cap (S_{\alpha,s} - \bigcup \{S_{\alpha,t}) : t < s\} \ \ \& \ \ \nu(\alpha, x, s) = \nu]\}$$

$$\text{and } \mathcal{F}^0_\alpha = \{\nu : (\exists^\infty x)(\exists s)[x \in R_{\alpha,s} \ \ \& \ \ \nu(\alpha, x, s) = \nu \ \ \& \ \ x \notin A_s]\}.$$

Thus $\mathcal{E}^0_\alpha$ consists of states well visited by elements x when they first *enter* R_α and $\mathcal{F}^0_\alpha$ of those states well-visited by elements at some stage while they *remain* in R_α, so $\mathcal{E}^0_\alpha \subseteq \mathcal{F}^0_\alpha$. For each $\alpha \in T$, $\mathcal{M}^0_\alpha$ represents α's "guess" at the true $\mathcal{F}^0_\alpha$ such that if $\alpha \subset f$ then $\mathcal{M}^0_\alpha = \mathcal{F}^0_\alpha$. For $\alpha \subset f$ we shall achieve $\mathcal{M}^0_\alpha = \mathcal{F}^0_\alpha$ by ensuring the following properties of $\mathcal{M}^0_\alpha$,

$$\mathcal{E}^0_\alpha \subseteq \mathcal{M}^0_\alpha, \tag{4.6}$$

$$\begin{gathered}\text{(a.e. } x)[\text{if } x \in Y_{\alpha,s},\ \nu_0 = \nu(\alpha, x, s) \in \mathcal{M}^0_\alpha, \\ \text{and RED causes enumeration of } x \text{ so that} \\ \nu_1 = \nu(\alpha, x, s+1) \text{ then } \nu_1 \in \mathcal{M}^0_\alpha],\end{gathered} \tag{4.7}$$

$$\begin{gathered}\text{(a.e. } x)[\text{if } x \in Y_{\alpha,s},\ \nu_0 = \nu(\alpha, x, s) \in \mathcal{M}^0_\alpha \\ \text{and BLUE causes enumeration of } x \text{ so that} \\ \nu_1 = \nu(\alpha, x, s+1) \text{ then } \nu_1 \in \mathcal{M}^0_\alpha].\end{gathered} \tag{4.8}$$

(Here (a.e. x) denotes “for almost every x”.) Two main constraints on BLUE's moves will be (4.6) and (4.8). Clearly, (4.6), (4.7), and (4.8) guarantee

$$\mathcal{F}^0_\alpha \subseteq \mathcal{M}^0_\alpha. \tag{4.9}$$

During Step 1 of the construction in Subsection 4.2.4 we shall move elements $x \in R_{\alpha^-,s}$ into $S_{\alpha,s+1}$ whenever possible in order to ensure

$$\mathcal{M}^0_\alpha \subseteq \mathcal{E}^0_\alpha. \tag{4.10}$$

Hence, by (4.9), (4.10), and $\mathcal{E}^0_\alpha \subseteq \mathcal{F}^0_\alpha$ we will have, for $\alpha \subset f$,

$$\mathcal{M}^0_\alpha = \mathcal{F}^0_\alpha = \mathcal{E}^0_\alpha. \tag{4.11}$$

On the $\widehat{\omega}$-side we have dual definitions for the above items by replacing $\omega, x, U_\alpha, \widehat{V}_\alpha$ by $\widehat{\omega}, \hat{x}, \widehat{U}_\alpha, V_\alpha$ respectively. These dual items will be denoted by $\hat{\nu}(\alpha, \hat{x}, s)$, $\widehat{S}_\alpha, \widehat{R}_\alpha, \widehat{Y}_\alpha, \widehat{\mathcal{E}}^0_\alpha, \widehat{\mathcal{F}}^0_\alpha$, and $\widehat{\mathcal{M}}^0_\alpha$. We write hats over the α-states, e.g. $\hat{\nu}_1 = \hat{\nu}(\alpha, \hat{x}, s)$, to indicate α-states for elements $\hat{x} \in \widehat{\omega}$. (In fact, though, an α-state on either side consists only of the node α, a subset of $\{e_0, \dots e_\alpha\}$, and a subset of $\{\hat{e}_0, \dots \hat{e}_\alpha\}$, so it is acceptable to write $\nu_1 = \hat{\nu}(\alpha, \hat{x}, s)$, or $\hat{\nu} \in \mathcal{M}_\alpha$, as we shall need to do in certain situations.) We shall ensure

$$\widehat{\mathcal{M}}^0_\alpha = \{\hat{\nu} : \nu \in \mathcal{M}^0_\alpha\}, \tag{4.12}$$

which implies by (4.11) that the well visited α-states on both sides coincide.

Having said that every $\alpha \in T$ should have an associated set $\mathcal{M}^0_\alpha$ such that $\mathcal{M}^0_\alpha = \mathcal{F}^0_\alpha$ if $\alpha \subset f$, we note that although this is the *property* we want $\mathcal{M}^0_\alpha$ to have, we cannot simply *define* $\mathcal{M}^0_\alpha$ to be α's guess at $\mathcal{F}^0_\alpha$ because that definition would be circular. (The definition of $\mathcal{F}^0_\alpha$ depends on U_α, and the construction of U_α in Section 4.2.4 will depend on $\mathcal{M}^0_\alpha$.) Rather we must define here a certain set $\mathcal{F}^{0+}_\beta$ which depends only on β, and then let $\mathcal{M}^0_\alpha$ be α's guess at $\mathcal{F}^{0+}_\beta$ so that $\mathcal{M}^0_\alpha = \mathcal{F}^{0+}_\beta$ $(= \mathcal{F}^0_\alpha)$ for $\alpha \subset f$.

Fix $\alpha \in T$ such that $e_\alpha > e_\beta$ for $\beta = \alpha^-$. Define the r.e. set $Z_{e_\alpha} = \bigcup_s Z_{e_\alpha,s}$ where

$$Z_{e_\alpha,s+1} =_{\text{dfn}} \{x : x \in U_{e_\alpha,s+1} \ \& \ x \in Y_{\beta,s}\}. \tag{4.13}$$

Define the α-state function $\nu^+(\alpha,x,s)$ exactly as for $\nu(\alpha,x,s)$ in Definition 4.2.6 but with $Z_{e_\alpha,s}$ in place of $U_{\alpha,s}$.

Define

$$\mathcal{F}_\beta^{0+} = \{\nu : (\exists^\infty x)(\exists s)[x \in Y_{\beta,s} \ \& \ \nu^+(\alpha,x,s) = \nu \ \& \ x \notin A_s]\}, \tag{4.14}$$

$$\begin{aligned} k_\beta^+ = \min\{y : (\forall x > y)(\forall s) \\ [[x \in Y_{\beta,s} \ \& \ \nu^+(\alpha,x,s) = \nu_1] \implies \nu_1 \in \mathcal{F}_\beta^{0+}]\}. \end{aligned} \tag{4.15}$$

If $e_\alpha > e_\beta$ we also define $\widehat{\mathcal{F}}_\beta^{0+} = \{\hat{\nu} : \nu \in \mathcal{F}_\beta^{0+}\}$. (Note that Z_{e_α} and hence $\mathcal{F}_\beta^{0+}$ and k_β^+ depend only upon β not α and thus α can make guesses $\mathcal{M}_\alpha^0$ and k_α for $\mathcal{F}_\beta^{0+}$ and k_β^+.)

If $\hat{e}_\alpha > \hat{e}_\beta$ we first define $\widehat{\mathcal{F}}_\beta^{0+}$ and k_β^+ using the duals of (4.14) and (4.15) (with $\hat{Y}_{\beta,s}$, $V_{\hat{e}_\alpha}$, $\hat{Z}_{\hat{e}_\alpha}$, and $\nu^+(\alpha,\hat{x},s)$ in place of $Y_{\beta,s}$, U_{e_α}, Z_{e_α}, and $\nu^+(\alpha,x,s)$, respectively), and then we define $\mathcal{F}_\beta^{0+} = \{\nu : \hat{\nu} \in \mathcal{F}_\beta^{0+}\}$. (Note that there is no $\hat{k}_\beta^+$ only k_β^+.)

Every $\alpha \in T$ will have associated items $\mathcal{M}_\alpha^0$ and k_α such that $\mathcal{M}_\alpha^0 = \mathcal{F}_\beta^{0+}$ and $k_\alpha = k_\beta^+$ for $\alpha \subset f$. We allow x to enter Y_α only if $x > k_\alpha$. If $e_\alpha = e_\beta$ and $\hat{e}_\alpha = \hat{e}_\beta$ we define $\mathcal{F}_\beta^{0+} = \mathcal{F}_\beta^0$, $\widehat{\mathcal{F}}_\beta^{0+} = \widehat{\mathcal{F}}_\beta^0$, and $k_\beta^+ = k_\beta$. If

$$(\exists x)(\exists s)[x \in Y_{\alpha,s} \ \& \ \nu(\alpha,x,s) \notin \mathcal{M}_\alpha^0] \tag{4.16}$$

then we say that α is *provably incorrect* at all stages $t \geq s$ and we ensure that $\alpha \not\subset f$.

Definition 4.2.7 Given α-states $\nu_0 = \langle \alpha, \sigma_0, \tau_0 \rangle$ and $\nu_1 = \langle \alpha, \sigma_1, \tau_1 \rangle$.

(i) $\nu_0 \leq_R \nu_1$ if $\sigma_0 \subseteq \sigma_1$ and $\tau_0 = \tau_1$.

(ii) $\nu_0 \leq_B \nu_1$ if $\tau_0 \subseteq \tau_1$ and $\sigma_0 = \sigma_1$.

(iii) $\hat{\nu}_0 \leq_R \hat{\nu}_1$ if $\hat{\sigma}_0 = \hat{\sigma}_1$ and $\hat{\tau}_0 \subseteq \hat{\tau}_1$.

(iv) $\hat{\nu}_0 \leq_B \hat{\nu}_1$ if $\hat{\sigma}_0 \subseteq \hat{\sigma}_1$ and $\hat{\tau}_0 = \hat{\tau}_1$.

(v) $\nu_0 <_R \nu_1$ ($\nu_0 <_B \nu_1$) if $\nu_0 \leq_R \nu_1$ ($\nu_0 \leq_B \nu_1$) and $\nu_0 \neq \nu_1$, and similarly for $\hat{\nu}_0 <_R \hat{\nu}_1$ and $\hat{\nu}_0 <_B \hat{\nu}_1$.

The intuition is that if x is in α-state $\nu_0 = \nu(\alpha, x, s)$ and $\nu_0 <_R \nu_1$ ($\nu_0 <_B \nu_1$) then RED (BLUE) can enumerate x in the necessary U sets ($\widehat{V}$ sets) causing $\nu_1 = \nu(\alpha, x, s+1)$. For $\hat{\nu}_0$ and $\hat{\nu}_1$ the role of σ and τ is reversed because on the $\widehat{\omega}$-side BLUE (RED) plays the $\widehat{U}$ sets (V sets), and hence

$$[\nu_0 <_R \nu_1 \iff \hat{\nu}_0 <_B \hat{\nu}_1] \ \& \ [\nu_0 <_B \nu_1 \iff \hat{\nu}_0 <_R \hat{\nu}_1]. \tag{4.17}$$

To construct an automorphism we must show for $\alpha \subset f$ that

$$\widehat{\mathcal{K}}^0_\alpha = \{\hat{\nu} : \nu \in \mathcal{K}^0_\alpha\}. \tag{4.18}$$

To achieve (4.18) note that unlike $\mathcal{E}^0_\alpha$ and $\mathcal{F}^0_\alpha$, $\mathcal{K}^0_\alpha$ is Π^0_3 not Π^0_2 so α cannot guess at $\mathcal{K}^0_\alpha$ directly but only at a certain Σ^0_2 approximation $\mathcal{N}^0_\alpha$ to $\mathcal{M}^0_\alpha - \mathcal{K}^0_\alpha$. We divide $\mathcal{N}^0_\alpha$ into the disjoint union of sets $\mathcal{R}^0_\alpha$ and $\mathcal{B}^0_\alpha$ which correspond to those $\nu \in \mathcal{N}^0_\alpha$ which α believes are being emptied by RED and BLUE respectively.

To define $\mathcal{R}^0_\alpha$ and $\mathcal{B}^0_\alpha$ fix $\alpha \in T$, let $\beta = \alpha^-$, and assume that $\mathcal{R}^0_\gamma$, $\mathcal{B}^0_\gamma$ and their duals $\widehat{\mathcal{R}}^0_\gamma$, $\widehat{\mathcal{B}}^0_\gamma$ have been defined for all $\gamma \subset \alpha$. We decompose $\mathcal{R}^0_\alpha$ into the disjoint union,

$$\mathcal{R}^0_\alpha = \mathcal{R}^\alpha_\alpha \sqcup \mathcal{R}^{<\alpha}_\alpha, \text{ where} \tag{4.19}$$

$$\mathcal{R}^{<\alpha}_\alpha =_{\text{dfn}} \{\nu \in \mathcal{M}^0_\alpha : \nu \restriction \beta \in \mathcal{R}^0_\beta\}, \text{ and} \tag{4.20}$$

$$\mathcal{R}^\alpha_\alpha =_{\text{dfn}} \mathcal{R}^0_\alpha - \mathcal{R}^{<\alpha}_\alpha. \tag{4.21}$$

Note that $\mathcal{R}^{<\alpha}_\alpha$ is determined by $\mathcal{R}^0_\beta$, $\beta \subset \alpha$, but $\mathcal{R}^\alpha_\alpha$ may contain new elements and for $\alpha \subset f$ it has the meaning described below in (4.23). Likewise, let $\mathcal{B}^0_\alpha = \mathcal{B}^\alpha_\alpha \sqcup \mathcal{B}^{<\alpha}_\alpha$, where $\mathcal{B}^{<\alpha}_\alpha$ is defined as in (4.20) but with $\mathcal{B}^0_\beta$ in place of $\mathcal{R}^0_\beta$.

If $|\alpha| \not\equiv 3 \bmod 5$ define $\mathcal{R}^{\alpha}_{\alpha} = \widehat{\mathcal{B}}^{\alpha}_{\alpha} = \emptyset$. If $|\alpha| \equiv 3 \bmod 5$ we let $\mathcal{M}^0_{\alpha} = \mathcal{M}^0_{\beta}$ (since α-states are β-states because $e_\alpha = e_\beta$ and $\hat{e}_\alpha = \hat{e}_\beta$), we define the Π^0_2 predicate,

$$F(\beta, \nu) \equiv (\forall x)[[x > |\beta| \ \& \ x \in Y_\beta] \implies \nu(\alpha, x) \neq \nu], \tag{4.22}$$

and we allow $\mathcal{R}^{\alpha}_{\alpha} \neq \emptyset$ with the intention that for $\alpha \subset f$,

$$\mathcal{R}^{\alpha}_{\alpha} = \{\nu : \nu \in \mathcal{M}^0_{\alpha} - (\mathcal{R}^{<\alpha}_{\alpha} \cup \mathcal{B}^{<\alpha}_{\alpha}) \ \& \ F(\beta, \nu)\}. \tag{4.23}$$

In 4.3 we defined

$$\widehat{\mathcal{B}}^{\alpha}_{\alpha} = \{\hat{\nu} : \nu \in \mathcal{R}^{\alpha}_{\alpha}\}. \tag{4.24}$$

Similarly, if $|\alpha| \not\equiv 4 \bmod 5$ define $\widehat{\mathcal{R}}^{\alpha}_{\alpha} = \mathcal{B}^{\alpha}_{\alpha} = \emptyset$. If $\alpha \equiv 4 \bmod 5$ we allow $\widehat{\mathcal{R}}^{\alpha}_{\alpha} \neq \emptyset$ (using the duals of (4.19)–(4.23) where e.g. in the dual of (4.22) we use $\widehat{Y}_\beta$ in place of Y_β), and we recall from (4.2) the definition

$$\mathcal{B}^{\alpha}_{\alpha} = \{\nu : \widehat{\nu} \in \widehat{\mathcal{R}}^{\alpha}_{\alpha}\}. \tag{4.25}$$

At most one of $\mathcal{R}^{\alpha}_{\alpha}$ and $\widehat{\mathcal{R}}^{\alpha}_{\alpha}$ is nonempty so by (4.2), (4.3), and (4.23),

$$\mathcal{R}^{\alpha}_{\alpha} \cap \mathcal{B}^{\alpha}_{\alpha} = \emptyset \ \& \ ((\mathcal{R}^{\alpha}_{\alpha} \cup \mathcal{B}^{\alpha}_{\alpha}) \cap (\mathcal{R}^{<\alpha}_{\alpha} \cup \mathcal{B}^{<\alpha}_{\alpha}) = \emptyset), \tag{4.26}$$

and hence

$$\mathcal{R}^0_{\alpha} \cap \mathcal{B}^0_{\alpha} = \emptyset. \tag{4.27}$$

If $\alpha \subset f$ then $\nu \in \mathcal{R}^0_{\alpha}$ implies $F(\alpha^-, \nu)$ and hence

$$(\forall \nu \in \mathcal{R}^0_{\alpha})(\forall x \in Y_\alpha)(\forall s)[\nu(\alpha, x, s) = \nu \implies (\exists t > s)[\nu(\alpha, x, t) \neq \nu]]. \tag{4.28}$$

It will be BLUE's responsibility to change the α-state of x if $\nu(\alpha, x, s) \in \mathcal{B}^0_{\alpha}$ and $x \in R_\alpha$. However, $\mathcal{B}^0_{\alpha} \cap \mathcal{R}^0_{\alpha} = \emptyset$ so if $\nu(\alpha, x, s) = \nu \in \mathcal{R}^0_{\alpha}$ then BLUE can wait for RED to change the α-state of each x to meet (4.28), by restraining x from entering any blue set until we reach a stage $t > s$ such that $\nu(\alpha, x, s) <_R \nu(\alpha, x, t)$.

Definition 4.2.8 Given $\beta \subseteq \alpha \in T$ and an α-state $\nu_0 = \langle \alpha, \sigma_0, \tau_0 \rangle$ or a set $\mathcal{C}_\alpha$ of α-states,

(i) $\nu_0 \restriction \beta = \langle \beta, \sigma_1, \tau_1 \rangle$ where we define $\sigma_1 = \sigma_0 \cap \{0, \ldots, e_\beta\}$ and we define $\tau_1 = \tau_0 \cap \{0, \ldots, \hat{e}_\beta\}$,

(ii) $\nu_1 \preceq \nu_0$ (read " ν_0 extends ν_1") if $\nu_0 \restriction \beta = \nu_1$,

(iii) $\mathcal{C}_\alpha \restriction \beta = \{\nu \restriction \beta : \nu \in \mathcal{C}_\alpha\}$.

(iv) Given a finite set of α-states $\{\langle \alpha, \sigma_i, \tau_i \rangle : i \in I\}$, we define $\cup\{\langle \alpha, \sigma_i, \tau_i \rangle : i \in I\} =_{\text{dfn}} \langle \alpha, \sigma, \tau \rangle$, where $\sigma = \cup\{\sigma_i : i \in I\}$, and where we define $\tau = \cup\{\tau_i : i \in I\}$.

4.2.3 The New Extension Theorem

Soare developed his New Extension Theorem to simplify the process of constructing automorphisms. Using the NET, one can divide the construction into three distinct parts and concentrate on each separately, rather than having to satisfy all three simultaneously. The idea is that in building an automorphism which maps A to B, at each stage $s+1$ we can consider three classes of elements: those elements x which are still in $\overline{A}_{s+1}$; those x which enter A at stage $s+1$; and those x which were already in A_s. (On the $\widehat{\omega}$ side, we have the same three classes: $\hat{x} \in \overline{B}_{s+1}$, $\hat{x} \in B_{s+1} - B_s$, and $\hat{x} \in B_s$.) Indeed, the NET constructs the automorphism on the third class itself, leaving only two types of element for us to worry about.

In the construction of the tree T in the preceding section, we defined the sets $\mathcal{M}^0_\alpha$, $\widehat{\mathcal{M}}^0_\alpha$, etc. for each $\alpha \in T$. In [22], a similar construction required the inclusion of A-states as well as $\overline{A}$-states in $\mathcal{M}_\alpha$. With the New Extension Theorem, however, we need only consider $\overline{A}$- and $\overline{B}$-states. The NET requires that for each α on the true path, $\mathcal{M}^0_\alpha = \widehat{\mathcal{M}}^0_\alpha$ and $\mathcal{N}^0_\alpha = \widehat{\mathcal{N}}^0_\alpha$. Together, these will guarantee that

$$\mathcal{K}^0_\alpha = \mathcal{M}^0_\alpha - \mathcal{N}^0_\alpha = \widehat{\mathcal{M}}^0_\alpha - \widehat{\mathcal{N}}^0_\alpha = \widehat{\mathcal{K}}^0_\alpha$$

so that the well-resided $\overline{A}$-α-states correspond precisely to the well-resided $\overline{B}$-α-states.

The second class of elements contains those x which enter A at stage $s+1$, and those $\hat{x}$ entering B_{s+1}. The New Extension Theorem requires us to record the α-state of each such x at stage s, as a sort of snapshot of its status at the moment it enters A, and similarly for each $\hat{x}$ that enters B. We define for each α:

$$\mathcal{G}^A_\alpha = \{\nu \in \mathcal{M}^0_\alpha : (\exists^\infty x)(\exists s)[x \in A_{s+1} - A_s \ \&\ \nu(\alpha, x, s) = \nu]\}$$

$$\widehat{\mathcal{G}}^B_\alpha = \{\hat{\nu} \in \widehat{\mathcal{M}}^0_\alpha : (\exists^\infty \hat{x})(\exists s)[\hat{x} \in B_{s+1} - B_s \ \&\ \hat{\nu}(\alpha, \hat{x}, s) = \hat{\nu}]\}.$$

Thus $\mathcal{G}^A_\alpha$ contains those $\overline{A}$-α-states such that infinitely many elements x are in that state at the moment of entering A, and similarly for $\widehat{\mathcal{G}}^B_\alpha$. The NET then requires that for each α on the true path, the α-states in $\mathcal{G}^A_\alpha$ must correspond precisely to those in $\widehat{\mathcal{G}}^B_\alpha$.

If we can accomplish these two conditions, then the New Extension Theorem guarantees that the third part of the automorphism construction can be carried out as well, and therefore that there exists an automorphism mapping each U_α ($\alpha \subset f$) to the corresponding $\hat{U}_\alpha$.

Theorem 4.2.9 (New Extension Theorem, Soare [48]) *Given a computable priority tree T as defined above with infinite true path f, suppose that each of the collections $\{U_\alpha\}_{\alpha\subset f}$ and $\{V_\alpha\}_{\alpha\subset f}$ contains every computably enumerable set, up to finite difference. If for each $\alpha \subset f$ we have:*

(T1) $\mathcal{K}^0_\alpha = \widehat{\mathcal{K}}^0_\alpha$, *and*

(T2) $\mathcal{G}^A_\alpha = \widehat{\mathcal{G}}^B_\alpha$,

then there exists an automorphism of $\mathcal{E}$ mapping U_α to $\hat{U}_\alpha$ for each $\alpha \subset f$.

It is left to us to satisfy our own requirements for U_ρ and $\hat{U}_\rho$, namely that $U_\rho = A$ (which we have already ensured, simply by arranging our enumeration of the c.e. sets to begin with A) and that $\hat{U}_\rho$ does not lie in the upper cone above C (which is the hard part).

Definition 4.2.10 The *true path* $f \in [T]$ is defined by induction on n. Let $\beta = f\restriction n$ be consistent. Then $f\restriction(n+1)$ is the $<_L$-least $\alpha \in T$, $\alpha \supset \beta$, of length $m = n+1$ such that:

(i) $m \equiv 1 \bmod 5 \implies \mathcal{M}^0_\alpha = \mathcal{F}^{0+}_\beta \;\;\&\;\; k_\alpha = k^+_\beta$,

(ii) $m \equiv 2 \bmod 5 \implies \widehat{\mathcal{M}}^0_\alpha = \widehat{\mathcal{F}}^{0+}_\beta \;\;\&\;\; k_\alpha = k^+_\beta$,

(iii)

$$\begin{aligned}&m \equiv 3 \bmod 5 \implies\\ &[\mathcal{R}^\alpha_\alpha = \{\nu : \nu \in \mathcal{M}^0_\alpha - (\mathcal{R}^{<\alpha}_\alpha \cap \mathcal{B}^{<\alpha}_\alpha) \;\;\&\;\; F(\beta,\nu)\}\\ &\;\;\&\;\; \widehat{\mathcal{B}}^\alpha_\alpha = \{\hat\nu : \nu \in \mathcal{R}^\alpha_\alpha\}],\end{aligned}$$

(iv)

$$\begin{aligned}&m \equiv 4 \bmod 5 \implies\\ &[\widehat{\mathcal{R}}^\alpha_\alpha = \{\hat\nu : \hat\nu \in \widehat{\mathcal{M}}^0_\alpha - (\widehat{\mathcal{R}}^{<\alpha}_\alpha \cup \widehat{\mathcal{B}}^{<\alpha}_\alpha) \;\;\&\;\; \widehat{F}(\beta,\nu)\}\\ &\;\;\&\;\; \mathcal{B}^\alpha_\alpha = \{\nu : \hat\nu \in \widehat{\mathcal{R}}^\alpha_\alpha\}],\end{aligned}$$

(v) unless otherwise specified in (i)–(iv), $\mathcal{M}^0_\alpha$, $\mathcal{R}^0_\alpha$, $\mathcal{B}^0_\alpha$, k_α, and their duals take the values $\mathcal{M}^0_\beta$, $\mathcal{R}^0_\beta$, $\mathcal{B}^0_\beta$, k_β, and their duals, respectively.

(If β were inconsistent, it would be a terminal node and the true path would end at β. We will show in Lemmas 4.3.9 and 4.3.11, however, that this cannot be the case.)

For a consistent $\beta = f\restriction n$, $\mathcal{F}^{0+}_\beta$ is just a finite set of states and k^+_β is an integer, so clearly α exists. Note that each of the conditions in Definition 4.2.10 is Π^0_2. Hence, there is a computable collection of c.e. sets $\{D_\alpha\}_{\alpha\in T}$ such that $\alpha \subset f$ iff $|D_\alpha| = \infty$. Fix a simultaneous computable enumeration $\{D_{\alpha,s}\}_{\alpha\in T, s\in\omega}$.

We impose the following positive requirements, for all $\alpha \in T$, all α-states ν, and all $i \in \omega$, to ensure that $\mathcal{G}^A = \widehat{\mathcal{G}}^B$:

$$\mathcal{P}_{\langle\alpha,\nu,i\rangle} : \nu \in \mathcal{G}^A_\alpha \implies |\{\hat{x} : (\exists s)[\hat{x} \in B_{\text{at } s+1} \cap \hat{Y}_{\alpha,s} \;\&\; \hat\nu(\alpha,\hat{x},s) = \nu]\}| \geq i$$

Clearly each $\mathcal{P}_{\langle\alpha,\nu,i\rangle}$ will only put finitely many elements into B. Indeed, since $\mathcal{P}_{\langle\alpha,\nu,i-1\rangle}$ has higher priority than $\mathcal{P}_{\langle\alpha,\nu,i\rangle}$, each $\mathcal{P}_{\langle\alpha,\nu,i\rangle}$ will only require that a single element enter B.

The negative requirements $\mathcal{Q}_e$ are the standard ones for the Sacks strategy for avoiding an upper cone:

$$\mathcal{Q}_e: \quad C \neq \{e\}^B.$$

To satisfy these, we define the length functions $l(e,s)$ and restraint functions $r(e,s)$ (as in [47] VII.3.1):

$$l(e,s) = \max\{x : (\forall y < x)[\{e\}_s^{B_s}(y)\downarrow = C_s(y)]\}$$

$$r(e,s) = \max\{u(B_s; e, x, s) : x \leq l(e,s)\}.$$

In the construction, we will restrain (with priority e) all elements $< r(e,s)$ from entering B at stage s. Thus we will preserve the computation $\{e\}^B(y)$ for every $y \leq l(e,s)$, including $y = l(e,s)$ itself. If $\lim_s l(e,s) = \infty$, then C would be computable, contrary to hypothesis. Moreover, for each e, $l(e,s)$ will be nondecreasing as a function of s, except at the finitely many stages s at which $\mathcal{N}_e$ is injured, i.e. at which $B_{s+1} \restriction (r(e,s)+1) \neq B_s \restriction (r(e,s)+1)$. Therefore, there exists a finite limit $l(e) = \lim_s l(e,s)$. Then the computation $\{e\}^B(l(e))$ must either diverge or converge to a value distinct from $C(l(e))$. Hence $\mathcal{Q}_e$ will be satisfied.

If A were an arbitrary set, then it would be extremely difficult, perhaps impossible, to satisfy the requirements $\mathcal{Q}_e$. The difficulty would be that if all the elements x in some $\overline{A}$-α-state ν enter A, then we have to put all the elements $\hat{x}$ from the corresponding $\overline{B}$-α-state $\hat{\nu}$ into B, probably violating some requirement $\mathcal{Q}_e$ in the process. Each time this happened, we could allow finitely many elements $\hat{x}$ to remain in $\hat{\nu}$ rather than entering B, but if it happened infinitely often, then $\hat{\nu}$ would be a well-resided state and ν would not be.

The assumption that A is low allows us to avoid this difficulty. We use a variation of Robinson's Trick (see [41]), as expressed in Soare's Lowness Lemma in [48], to predict which elements x in the $\overline{A}$-α-state ν will eventually enter A.

Our prediction may be wrong, but if all elements in ν eventually enter A, then the prediction will only be wrong on finitely many of those elements. A corresponding finite number of elements $\hat{x}$ may have to stay in $\hat{\nu}$ rather than entering B, but that is acceptable, since then $\hat{\nu}$ will lie in $\widehat{\mathcal{N}}^0_\alpha$, just as ν lies in $\mathcal{N}^0_\alpha$. Essentially Robinson's Trick gives us believable evidence that certain elements x will never enter A, and we use this knowledge to ensure that the requirements $\mathcal{Q}_e$ will not prevent us from matching up $\overline{A}$-states and $\overline{B}$-states.

Recall Definition 4.2.3, which stated that a node $\alpha \in T$ is $\mathcal{R}$-*consistent* if it satisfies both of the following:

$$(\forall \nu_0 \in \mathcal{R}^0_\alpha)(\exists \nu_1)[\nu_0 <_R \nu_1 \ \ \& \ \ \nu_1 \in \mathcal{M}^0_\alpha]; \tag{4.29}$$

$$(\forall \hat{\nu}_0 \in \widehat{\mathcal{R}}^0_\alpha)(\exists \hat{\nu}_1)[\hat{\nu}_0 <_R \hat{\nu}_1 \ \ \& \ \ \hat{\nu}_1 \in \widehat{\mathcal{M}}^0_\alpha]. \tag{4.30}$$

Lowness of A allows us to ensure that every α on the true path is $\mathcal{R}$-consistent. Without lowness, the equation for $\mathcal{R}^0$ would be impossible, since states could be emptied out into A with no advance warning to us.

BLUE will ensure that α is $\mathcal{R}$-consistent for $\alpha \subset f$ by waiting to enumerate x in any blue sets until RED has enumerated x in some red set. Now (4.3), (4.17), and (4.29) imply for $\alpha \subset f$ that

$$(\forall \hat{\nu}_0 \in \widehat{\mathcal{B}}^0_\alpha)(\exists \hat{\nu}_1)[\hat{\nu}_0 <_B \hat{\nu}_1 \ \ \& \ \ \hat{\nu}_1 \in \widehat{\mathcal{M}}^0_\alpha]. \tag{4.31}$$

By repeatedly applying (4.31) BLUE can achieve $\hat{\nu}_1 \in \widehat{\mathcal{M}}^0_\alpha - \widehat{\mathcal{B}}^0_\alpha$, namely

$$(\exists \text{ function } \hat{h}_\alpha)[\hat{h}_\alpha : \widehat{\mathcal{B}}^0_\alpha \to (\widehat{\mathcal{M}}^0_\alpha - \widehat{\mathcal{B}}^0_\alpha) \ \ \& \ \ (\forall \hat{\nu} \in \widehat{\mathcal{B}}^0_\alpha)[\hat{\nu} <_B \hat{h}_\alpha(\hat{\nu})]]. \tag{4.32}$$

(The function $\hat{h}$ is called the *target function*.)

It will be BLUE's responsibility to move any element $\hat{x} \in \widehat{R}_\alpha$ for which $\hat{\nu}(\alpha, \hat{x}, s) = \hat{\nu}_0 \in \widehat{\mathcal{B}}^0_\alpha$ to the *target state* $\hat{\nu}_1 = \hat{h}_\alpha(\hat{\nu}_0)$ so that BLUE can achieve,

$$(\forall \hat{x} \in \widehat{R}_\alpha)(\forall s)[\hat{\nu}(\alpha, \hat{x}, s) \in \widehat{\mathcal{B}}^0_\alpha \implies (\exists t > s)[\hat{\nu}(\alpha, \hat{x}, t) \in \widehat{\mathcal{M}}^0_\alpha - \widehat{\mathcal{B}}^0_\alpha]], \tag{4.33}$$

and hence BLUE will cause every state $\hat{\nu}_0 \in \widehat{\mathcal{B}}^0_\alpha$ to be emptied. To achieve (4.33) on $\widehat{R}_\alpha$ it suffices to achieve the following on $\widehat{S}_\gamma$ for each $\gamma \supseteq \alpha$,

$$(\forall \hat{x} \in \hat{S}_\gamma)(\forall s)[\nu(\gamma, \hat{x}, s) \in \widehat{\mathcal{B}}^0_\gamma \implies (\exists t > s)[\nu(\gamma, \hat{x}, t) \in \widehat{\mathcal{M}}^0_\gamma - \widehat{\mathcal{B}}^0_\gamma]]. \tag{4.34}$$

(For BLUE to achieve (4.34) from the hypothesis of (4.33) there is a subtle but crucial point. Suppose $\nu_0 \in \mathcal{R}^0_\alpha$, so $\hat{\nu}_0 \in \widehat{\mathcal{B}}^0_\alpha$. Hence $\hat{\nu}'_0 \in \widehat{\mathcal{B}}^0_\gamma$ for all $\gamma \supset \alpha$ such that $\hat{\nu}'_0 \restriction \alpha = \hat{\nu}_0$. Now for every $\hat{x}$ in region $\widehat{R}_\alpha$ such that $\hat{\nu}(\alpha, \hat{x}, s) = \hat{\nu}_0 \in \widehat{\mathcal{B}}^0_\alpha$, BLUE is required by (4.33) to enumerate $\hat{x}$ in blue sets to achieve $\hat{\nu}(\alpha, \hat{x}, t) = \hat{\nu}_1 >_B \hat{\nu}_0$ for some $t > s$. However, if $\hat{x} \in \hat{S}_{\gamma,s}$ for some $\gamma \supset \alpha$ then BLUE can only make γ-*legal* moves, i.e. BLUE must ensure that $\hat{\nu}(\gamma, \hat{x}, s) \in \widehat{\mathcal{M}}^0_\gamma$. Hence, on the γ-level if $\hat{\nu}'_0 = \hat{\nu}(\gamma, \hat{x}, s)$ and $\hat{\nu}'_0 \restriction \alpha = \hat{\nu}_0 \in \hat{\mathcal{B}}^0_\alpha$ then $\nu_0 \in \mathcal{R}^0_\alpha$ so $\nu'_0 \in \mathcal{R}^0_\gamma$ and BLUE needs a γ-*target* $\hat{\nu}'_1 >_B \hat{\nu}'_0$ for $\hat{x}$, not merely an α-target $\hat{\nu}_1 >_B \hat{\nu}_0$. To obtain this γ-target $\hat{\nu}'_1$, BLUE can hold some $y \in S_\gamma$ in γ-state ν'_0 until RED is forced to cause $\nu(\alpha, y, t) = \nu_1 >_R \nu_0$, for some $t > s$, and hence $\nu(\gamma, y, t) = \nu'_1 >_R \nu'_0$, thus ensuring that γ is $\mathcal{R}$-consistent and giving a target γ-state $\hat{\nu}'_1$ for $\hat{x}$. This action may have to be repeated for each of the infinitely many $\gamma \supseteq \alpha$ even for those $\gamma <_L f$. Hence, (4.33) constitutes a very strong BLUE constraint on the entire downward cone $\widehat{R}_\alpha$. This procedure for producing an appropriate target j-state $\hat{\nu}'_1$ for $j > e$ when an e-state $\hat{\nu}_0$ is emptied is taken from the effective automorphism machinery in [47, Chapter XV], and [46], where it also plays a central role.)

We often refer to the dual of (4.32) which asserts

$$(\exists \text{ function } h_\alpha)[h_\alpha : \mathcal{B}^0_\alpha \to (\mathcal{M}^0_\alpha - \mathcal{B}^0_\alpha) \ \& \ (\forall \nu \in \mathcal{B}^0_\alpha)[\nu <_B h_\alpha(\nu)]], \tag{4.35}$$

and which enables us to achieve the dual of (4.34), namely

$$(\forall x \in S_\gamma)(\forall s)[\nu(\gamma, x, s) \in \mathcal{B}^0_\gamma \implies (\exists t > s)[\nu(\gamma, x, t) \in \mathcal{M}^0_\gamma - \mathcal{B}^0_\gamma]]. \tag{4.36}$$

Finally, we have ensured

$$(\forall\gamma \subset f)(\forall\nu_0 \in \mathcal{M}^0_\gamma)[(\exists^{<\infty}x)[x \in Y_\gamma \ \& \ \nu(\gamma, x) = \nu_0] \\ \implies (\exists\alpha)_{\gamma\subset\alpha\subset f}[\{\nu_1 \in \mathcal{M}^0_\alpha : \nu_1{\restriction}\gamma = \nu_0\} \subseteq \mathcal{R}^0_\alpha \cup \mathcal{B}^0_\alpha]]. \tag{4.37}$$

To check (4.37) fix $\gamma \subset f$ and $\nu_0 \in \mathcal{M}^0_\gamma$. Now $Y_\gamma =^* \omega$ since $\gamma \subset f$, so if the hypothesis of (4.37) holds then we can choose b such that

$$(\forall x \in \omega)[x > b \implies \nu(\gamma, x) \neq \nu_0].$$

Choose $\alpha \subset f$ such that $\alpha \supset \gamma$, $|\alpha| > b$ and $|\alpha| \equiv 3 \bmod 5$. Consider any $\nu_1 \in \mathcal{M}^0_\alpha$ such that $\nu_1{\restriction}\gamma = \nu_0$. If $\nu_1 \notin \mathcal{R}^{<\alpha}_\alpha \cup \mathcal{B}^{<\alpha}_\alpha$ then $F(\alpha^-, \nu_1)$ holds so $\nu_1 \in \mathcal{R}^\alpha_\alpha$ by (4.23), and hence $\nu_1 \in \mathcal{R}^0_\alpha$ by (4.19).

4.2.4 Construction

To parallel the construction in [22], the steps presented in this section will be denoted as Steps 0–5 and $\hat{0}$–$\hat{5}$ for the construction, with final Steps 10, $\widehat{10}$, and 11 at which we define f_{s+1} and other necessary items. (In the construction in [22], Steps 10 and $\widehat{10}$ were substeps of Step 11. We have separated the two because the actions in our Step 11 must be performed at every stage, whereas the action in our Steps 10 and $\widehat{10}$ must not be performed unless the preceding steps do not apply.) Steps $\hat{1}$–$\hat{5}$ and $\widehat{10}$ are the obvious duals to Steps 1–5, and will not be stated. There is no dual of Step 11.

Our construction is as follows:

Stage $s = 0$. For all $\alpha \in T$ define $U_{\alpha,0} = V_{\alpha,0} = \widehat{U}_{\alpha,0} = \widehat{V}_{\alpha,0} = \emptyset$, and define $m(\alpha, 0) = 0$. Define $Y_{\lambda,0} = \widehat{Y}_{\lambda,0} = \emptyset$, and $f_0 = \rho$. Define every $Q^\alpha_{\nu,i,0} = \emptyset$ and every marker $\Gamma^\alpha_{\nu,i,0}$ to be unassigned. Define $A_0 = B_0 = \emptyset$. Let $l(e, 0) = r(e, 0) = 0$ for every e.

Stage $s + 1$. Find the least $n < 11$ such that Step n applies to some $x \in Y_{\alpha,s}$ and perform the intended action. If there is no such n, then find the least $n < 11$ such that Step $\hat{n}$ applies to some $\hat{x} \in \widehat{Y}_{\alpha,s}$, and perform the indicated action. Having completed that, apply Step 11, and go to stage $s+2$.

(In Steps 0–5 and $\hat{0}$–$\hat{5}$ we let $\alpha \in T$, $\alpha \neq \lambda$, be arbitrary, let $\beta = \alpha^-$, and let $x \in Y_{\lambda,s}$ ($\hat{x} \in \widehat{Y}_{\lambda,s}$) be arbitrary.)

The sets $\{\tilde{A}_s\}_{s\in\omega}$ represent a given computable enumeration of A, from which we will derive our own enumeration $\{A_s\}_{s\in\omega}$ to satisfy the New Extension Theorem.

Step 0 (Moving elements into A).

Substep 0.1 (Enumerated elements.) If $x \in (Y_{\lambda,s} \cap \tilde{A}_{s+1}) - (Y_{\lambda,s-1} \cap \tilde{A}_s)$,

(0.1.1) Where $\nu(\alpha(x,s),x,s) = \nu$, add to $\mathcal{L}^{\mathcal{G}}$ a new pair $\langle \beta, \hat{\nu} \restriction \beta \rangle$ for every $\beta \subseteq \alpha(x,s)$,

(0.1.2) Enumerate x into A_{s+1}, and

(0.1.3) Designate every Γ-marker attached to x as unassigned.

Substep 0.2 (Assigning a Γ-marker to an x believed not to go into A.) In the following, to *challenge* x with regard to marker type j (= 1, 2, 3) and α-node ν means to do the following:

(i) Where i is the least number such that the marker $\Gamma^{j,\alpha}_{\nu,i}$ is currently unassigned, enumerate x into $Q^{j,\alpha}_{\nu,i}$.

(ii) Find the least t such that either

(a) $h(q^{j,\alpha}_{\nu,i}, t)\downarrow = 1$ or

(b) $x \in \tilde{A}_t$.

In case (a), assign marker $\Gamma^{j,\alpha}_{\nu,i}$ to x. In case (b),

(iii) If $j = 1$ or 2, add to $\mathcal{L}^{\mathcal{G}}$ a pair $\langle \beta, \hat{\nu} \restriction \beta \rangle$ for every $\beta \subseteq \alpha(x,s)$; if $j = 3$, add a pair $\langle \beta, \hat{\nu} \restriction \beta \rangle$ for every $\beta \subsetneq \alpha(x,s)$;

(iv) Enumerate x into A_{s+1} immediately; and

(v) Designate every Γ-marker attached to x as unassigned.

Then Substep (0.2) consists of repeating the following three instructions:

(0.2.1) If some element x is to be moved into some Y_α in $\overline{A}$-state ν by Step 1 or 2, then challenge x with regard to marker type 1 and α-state ν.

(0.2.2) If some element x is to be put into $\overline{A}$-state ν by one of Steps 1–5 or 11C, then challenge x with regard to marker type 2 and α-state ν.

(0.2.3) If there is some element x such that, as a result of x being enumerated into U_{e_α} and/or the action of Steps 1–5 or 11C, $\nu^+(x,\alpha)$ will become equal to $\overline{A}$-state ν, then challenge x with regard to marker type 3 and α-state ν.

We repeat these instructions until none of these three challenges described enters case (b) (that is, none of them causes an element to enter A_{s+1}).

Step $\hat{\mathbf{0}}$. (Moving elements into B.)

Find the first unmarked pair $\langle \alpha, \hat{\nu}_0\rangle$ in $\mathcal{L}^{\mathcal{G}}$ satisfying all of the following:

($\hat{0}$.1) For some k, $\mathcal{P}_{\langle \alpha,\nu_0,k\rangle}$ is not satisfied;

($\hat{0}$.2) α is consistent;

($\hat{0}$.3) there exist elements $\hat{y}_0 < \hat{y}_1 < \hat{y}_2 < \cdots < \hat{y}_{2k}$ such that for each $i \leq 2k$, both of the following hold:

$$(\exists t \leq s)[\hat{y}_i \in R_{\alpha,t} \ \ \& \ \ \hat{\nu}(\alpha,\hat{y}_i,t) = \hat{\nu}_0], \text{ and}$$

$$\hat{y}_i \notin B_s \text{ or } (\exists t < s)[\hat{y}_i \in B_{\text{at } t+1} \ \ \& \ \ \hat{\nu}(\alpha,\hat{y}_i,t) \neq \hat{\nu}_0];$$

($\hat{0}$.4) $\hat{y}_{2k} > 2 \cdot \langle \alpha,\nu_0,k\rangle$;

($\hat{0}$.5) $\hat{y}_{2k} > r(e,s)$ for every $e \leq \langle \alpha,\nu_0,k\rangle$;

($\hat{0}$.6) $\hat{\nu}(\alpha,\hat{y}_{2k},s) = \hat{\nu}_0$.

Action. Enumerate $\hat{y}_{2k}$ into B_{s+1}. (Notice that by $(\hat{0}.6)$, $\hat{y}_{2k} \notin B_s$.) Also, mark the first unmarked copy of $\langle \alpha, \hat{\nu}_0 \rangle$ on $\mathcal{L}^{\mathcal{G}}$.

Step 1. (Prompt pulling of x from R_β to S_α to ensure $\mathcal{M}_\alpha^0 \subseteq \mathcal{E}_\alpha^0$.) Suppose $\langle \alpha, \nu_1 \rangle$ is the first unmarked entry on the list $\mathcal{L}_s$ such that the following conditions hold for some x, where $\nu_1 = \langle \alpha, \sigma_1, \tau_1 \rangle$,

(1.1) $x \in R_{\beta,s} - Y_{\alpha,s}$, and α is $\mathcal{R}$-consistent;

(1.2) $x > k_\alpha$ and $x > |\alpha|$;

(1.3) x is α-eligible (*i.e.*, $\neg(\exists t)[x \leq t \leq s \;\; \& \;\; f_t < \alpha]$);

(1.4) $\neg[\alpha(x,s) <_L \alpha]$;

(1.5) $x > m(\alpha, s)$;

(1.6) $\nu(\beta, x, s) = \nu_1 \restriction \beta$;

(1.7) $e_\alpha > e_\beta \implies \nu^+(\alpha, x, s) = \nu_1$.

Action. Choose the least x corresponding to $\langle \alpha, \nu_1 \rangle$, and do the following.

(1.8) Mark the α-entry $\langle \alpha, \nu_1 \rangle$ on $\mathcal{L}_s$.

(1.9) Move x to S_α.

(1.10) If $e_\alpha > e_\beta$ and $e_\alpha \in \sigma_1$ then enumerate x in $U_{\alpha,s+1}$.

(1.11) If $\hat{e}_\alpha > \hat{e}_\beta$ and $\hat{e}_\alpha \in \tau_1$ then enumerate x in $\widehat{V}_{\alpha,s+1}$. (Hence, $\nu(\alpha, x, s+1) = \nu_1$. Also $\nu_1 \in \mathcal{M}_\alpha^0$ because $\langle \alpha, \nu_1 \rangle \in \mathcal{L}$ implies $\nu_1 \in \mathcal{M}_\alpha^0$.)

Step 2. (Move x from S_β to S_α so $Y_\alpha =^* \omega$.) Suppose there is an x such that

(2.1) $x \in S_{\beta,s}$,

(2.2) $x > |\alpha|$ and $x > k_\alpha$.

(2.3) x is α-eligible,

(2.4) $x < m(\alpha, s)$,

(2.5) α is the $<_L$-least $\gamma \in T$ with $\gamma^- = \beta$ satisfying (2.1)–(2.4).

Action. Choose the least pair $\langle \alpha, x \rangle$ and

(2.6) move x from S_β to S_α.

(In Step 2 we need (2.4) so Y_α will not grow while α is waiting for another prompt pulling under Step 1.)

Step 3. (For α $\mathcal{M}$-inconsistent to ensure $\alpha \not\subset f$.) Suppose for $\alpha \in T$ there exists $x > k_\alpha$ such that,

(3.1) $e_\alpha > e_\beta$,

(3.2) $x \in S_{\alpha,s}$,

(3.3) $\nu(\alpha, x, s) = \nu_0 \in \mathcal{M}^0_\alpha$,

(3.4) $(\exists \nu_1)[\nu_0 <_{\mathrm{B}} \nu_1 \ \ \& \ \ \nu_1 \restriction \beta \in \mathcal{M}^0_\beta \ \ \& \ \ \nu_1 \notin \mathcal{M}^0_\alpha]$.

Action. Choose the least such pair $\langle \alpha, x \rangle$ and,

(3.5) enumerate x in $\widehat{V}_{\delta,s+1}$ for all $\delta \subset \alpha$ such that $e_\delta \in \tau_1$. (This action causes $\nu(\alpha, x, s+1) = \nu_1$. Hence, α is provably incorrect at all stages $t \geq s+1$ so $\alpha \not\subset f$.)

Step 4. (Delayed RED enumeration into U_α.) Suppose $x \in R_{\alpha,s}$ and

(4.1) $e_\alpha > e_\beta$,

(4.2) $x \notin U_{\alpha,s}$,

(4.3) $x \in Z_{e_\alpha,s} =_{\mathrm{dfn}} U_{e_\alpha,s} \cap Y_{\beta,s-1}$.

Action. Choose the least such pair $\langle \alpha, x \rangle$ and,

(4.4) enumerate x in $U_{\alpha,s+1}$.

Step 5. (BLUE emptying of state $\nu_0 \in \mathcal{B}^0_\alpha$.) Suppose for $\alpha \in T$ there exists x such that either Case 1 or Case 2 holds.

Case 1. Suppose

(5.1) $\nu(\alpha, x, s) = \nu_0 \in \mathcal{B}^0_\alpha$, say $\nu_0 = \langle \alpha, \sigma_0, \tau_0 \rangle$,

(5.2) $x \in S_{\alpha,s}$,

(5.3) α is a consistent node.

Action. Choose the least such pair $\langle \alpha, x \rangle$. Let $\nu_1 = h_\alpha(\nu_0) >_B \nu_0$, where h_α is a target function satisfying (4.35). Let $\nu_1 = \langle \alpha, \sigma_1, \tau_1 \rangle$.

(5.4) Enumerate x into $\widehat{V}_\delta$ for all $\delta \subseteq \alpha$ such that $\hat{e}_\delta > \hat{e}_{\delta-}$ and also $e_\delta \in \tau_1 - \tau_0$. (Hence, $\nu(\alpha, x, s+1) = \nu_1$.)

Case 2. Suppose that (5.1) holds and

(5.5) $x \in S_{\gamma,s}$ where $\gamma^- = \alpha$, and

(5.6) γ is not a consistent node.

Action. Perform the same action as in Case 1 to achieve $\nu(\alpha, x, s+1) = \nu_1$.

(In (5.6) note that $\gamma \in T$ implies (5.3) for $\alpha = \gamma^-$ since inconsistent nodes are terminal, so h_α exists in Case 2. Note in Step 5 Case 2 that the enumeration may not be γ-legal, *i.e.*, perhaps $\nu(\gamma, x, s+1) \notin \mathcal{M}^0_\gamma$, but this will not matter because we shall prove that $\gamma \not\subset f$ if γ is inconsistent. Hence, it only matters that the enumeration is α-legal, *i.e.*, $\nu(\alpha, x, s) \in \mathcal{M}^0_\alpha$.)

Step 10. (Filling Y_λ.) Choose the least $x < s$ such that $x \notin Y_{\lambda,s}$ Put x in S_λ.

Step 11. (Defining f_{s+1}, $m(\alpha, s+1)$, $\mathcal{L}_{s+1}$, $Y_{\lambda,s+1}$, and B_{s+1}.)

Substep 11A. (Defining f_{s+1}.) First we define δ_t by induction on t for $t \leq s+1$. Let $\delta_0 = \rho$ (as given in Definition 4.2.2, the definition of T). Given δ_t, let $v \leq s$ be maximal such that $\delta_t \subseteq f_v$ if v exists, or let $v = 0$ otherwise. (Let $\{D_{\gamma,v}\}_{\gamma\in T, v\in\omega}$ be the simultaneous recursive enumeration specified on page 122.) Choose the $\leq_L$-least $\alpha \in T$ such that $\alpha^- = \delta_t$ and $D_{\alpha,s+1} \neq D_{\alpha,v}$ if α exists and define $\delta_{t+1} = \alpha$. If α does not exist define $\delta_{t+1} = \delta_t$. Finally, define $f_{s+1} = \delta_{s+1}$.

Substep 11B. (Defining $m(\alpha, s+1)$, $\mathcal{L}_{s+1}$, and their duals.) For each $\alpha \subseteq f_{s+1}$, if every α-entry $\langle\alpha, \nu\rangle$ on $\mathcal{L}_s$ and every α-entry $\langle\alpha, \hat{\nu}\rangle$ on $\widehat{\mathcal{L}}_s$ is marked we say that the lists are α-*marked* and we

(11.1) define $m(\alpha, s+1) = m(\alpha, s) + 1$, and

(11.2) add to the bottom of list $\mathcal{L}_s$ ($\widehat{\mathcal{L}}_s$) a new (unmarked) α-entry $\langle\alpha, \nu\rangle$ ($\langle\alpha, \hat{\nu}\rangle$) for every such α and every $\nu \in \mathcal{M}^0_\alpha$. Let the resulting list be $\mathcal{L}_{s+1}(\widehat{\mathcal{L}}_{s+1})$.

If the lists are not both α-marked then let $m(\alpha, s+1) = m(\alpha, s)$, $\mathcal{L}_{s+1} = \mathcal{L}_s$, and $\widehat{\mathcal{L}}_{s+1} = \widehat{\mathcal{L}}_s$.

Substep 11C. (Emptying R_α to the right of f_{s+1}.) For every α such that $f_{s+1} <_L \alpha$, *initialize* α, by removing every $x \in S_{\alpha,s}$ ($\hat{x} \in \hat{S}_{\alpha,s}$), and putting x in S_β ($\hat{x}$ in $\hat{S}_\beta$) for $\beta = \alpha \cap f_{s+1}$ (where $\alpha \cap \delta$ denotes the longest γ such that $\gamma \subseteq \alpha$ and $\gamma \subseteq \delta$).

For each $x \in Y_{\lambda,s+1}$ ($\hat{x} \in \hat{Y}_{\lambda,s+1}$) such that $x \notin A_{s+1}$ ($\hat{x} \notin B_{s+1}$), let $\alpha(x, s+1)$ ($\alpha(\hat{x}, s+1)$) denote the unique γ such that $x \in S_{\gamma,s+1}$. If $x \in A_{s+1}$, then $\alpha(x, s+1)$ diverges, and similarly for $\hat{x} \in B_{s+1}$.

Define the length function $l(e, s+1)$ and the restraint function $r(e, s+1)$ for stage $s+1$ as follows:

$$l(e,s) = \max\{x : (\forall y < x)[\{e\}_{s+1}^{B_{s+1}}(y)\downarrow = C_{s+1}(y)]\}$$

$$r(e,s) = \max\{u(B_{s+1}, e, x, s+1) : x \leq l(e, s+1)\}.$$

(Here u represents the standard use function for relative Turing machines.)

This completes stage $s+1$ and the construction.

Remark 4.2.11 *Notice that the only step which can put elements into $B = \hat{U}_\rho$ is Step $\hat{0}$. All of Steps 1-5 and their duals are dedicated toward the $\overline{A}/\overline{B}$ part of the game. Steps $\hat{1}$, $\hat{3}$, and $\hat{5}$ may put elements $\hat{x}$ into certain sets $\hat{U}_\alpha$ in order to change $\hat{\nu}(\alpha, \hat{x}, s+1)$. In Steps $\hat{1}$ and $\hat{5}$, however, this can only happen when the desired $\hat{\nu}(\alpha, \hat{x}, s+1)$ is a $\overline{B}$-state, so we are not required to put $\hat{x}$ into B. Also, Step $\hat{3}$ never applies with $\beta = \lambda$ because ρ, the unique node at level 1 of T, is $\mathcal{M}$-consistent by definition 4.2.4. Thus these steps never require any $\hat{x}$ to enter B.*

4.3 Proof of the Theorem

We now prove that the preceding construction satisfies Theorem 4.1.1:

1. In §4.3.1, we verify the restrictions of certain tree properties to $\overline{A}$ and $\overline{B}$.
2. In §4.3.2, we use these tree properties to verify the correctness of $\mathcal{M}^0$, $\mathcal{M}^0$, $\mathcal{N}^0$, and $\mathcal{N}^0$.
3. In §4.3.3, we use the above verification to check that $\mathcal{G}^A = \widehat{\mathcal{G}}^B$.

4.3.1 Tree Properties

The construction of [22] is designed to ensure that certain properties of the tree T (the *tree properties*) hold automatically for every $\alpha = \beta^+$ on the true path:

1. $\mathcal{M}_\alpha = \mathcal{F}_\beta^+$,
2. $\widehat{\mathcal{M}}_\alpha = \widehat{\mathcal{F}}_\beta^+$ and
3. k_α is a correct guess.

Since our construction employs the New Extension Theorem, we need only verify the correctness of the restrictions of these properties to $\overline{A}$- and $\overline{B}$-states. The New Extension Theorem takes care of the A/B aspect of the game, and we handle the $\mathcal{G}^A/\widehat{\mathcal{G}}^B$ aspect in Steps 0 and $\hat{0}$, which we have added to the original construction of [22].

To help handle the $\mathcal{G}^A/\widehat{\mathcal{G}}^B$ game, however, our construction defined the first level of the tree artificially, so that it contains only the node ρ. Therefore we must give special proofs of the tree properties (restricted to $\overline{A}$ and $\overline{B}$) for ρ.

(We will assume that A is infinite and coinfinite, for otherwise A would be computable and would itself witness that the orbit of A is not contained in the upper cone above the noncomputable set C.)

Property 4.3.1 $\mathcal{M}_\rho = \mathcal{F}_\lambda^+$.

Proof. By Step 10, every element x of ω eventually enters Y_λ. (Lemma 4.3.6 below implies that Step 10 acts infinitely often. The proofs of the lemmas of subsection 4.3.2 do not rely on the properties of this subsection at all.) Every element of the infinite set A eventually enters some A_s by Step 0, and no element of the infinite set $\overline{A}$ ever does. Thus, there are infinitely many x such that for some s, $x \in Y_{\lambda,s}$ and $x \in A_s$, and there are infinitely many x such that for some s, $x \in Y_{\lambda,s}$ and $x \notin A_s$, so $\mathcal{F}_\lambda^+ = \{\langle \rho, \emptyset, \emptyset\rangle, \langle \rho, \{0\}, \emptyset\rangle\} = \mathcal{M}_\rho$. ■

(In particular, then, $\mathcal{M}_\rho^0 = \mathcal{F}_\lambda^{0+}$.)

Property 4.3.2 $\widehat{\mathcal{M}}_\rho^0 = \widehat{\mathcal{F}}_\lambda^{0+}$.

Proof. $\widehat{\mathcal{M}}_\rho^0$ contains $\langle \rho, \emptyset, \emptyset\rangle$, which is the only possible $\overline{B}$-ρ-state. By Step 10, every element $\hat{x}$ of $\widehat{\omega}$ eventually enters Y_λ. As noted in Remark 4.2.11, only Step $\hat{0}$ ever puts any elements into $\hat{U}_0$, and it waits to do so until such elements are already in Y_λ. Thus, $\langle \rho, \emptyset, \emptyset\rangle \in \widehat{\mathcal{F}}_\lambda^{0+}$, so $\widehat{\mathcal{F}}_\lambda^{0+} = \widehat{\mathcal{M}}_\rho^0$. ■

Property 4.3.3 *No element of $\overline{A}$ or $\overline{B}$ remains permanently in a non-well-resided ρ-state. (Thus, the guess $k_\rho = -1$ is correct.)*

Proof. If $x \in \overline{A}$ ($\hat{x} \in \overline{B}$), then x ($\hat{x}$) is permanently in the ρ-state $\nu = \langle \rho, \emptyset, \emptyset\rangle$, which we have just seen is well-visited. To see that this state is well-resided, we must note that $\overline{A}$ and $\overline{B}$ are infinite. We assumed this for $\overline{A}$. For $\overline{B}$, we note that by Remark 4.2.11, Step $\hat{0}$ is the only step to put any elements into B, and for each $\langle \alpha, \nu, i\rangle$, it puts at most one element $\hat{y}$ into B, with $\hat{y} > 2 \cdot \langle \alpha, \nu, i\rangle$. Hence $\overline{B}$ must be infinite. ■

This completes the verification of the restricted versions of the tree properties for ρ. It remains to see that Properties 4.3.2 and 4.3.3 hold for all states, not just $\overline{B}$-states. This will be the very last line in the verification of Theorem 4.1.1, once we have proven that B is infinite. Since all D_α, $|\alpha| > 1$, are defined as in [22], these properties hold automatically for all $\alpha \supseteq \rho$ with α on the true path f:

1. $\mathcal{M}_\alpha = \mathcal{F}_\alpha^{0+}$,

2. $\mathcal{M}_\alpha = \mathcal{F}_\alpha^{0+}$, and

3. k_α is the upper bound for the set of all $x \in \overline{A}$ and $\hat{x} \in \overline{B}$ that remain permanently in a non-well-visited α-state.

4.3.2 Verification that $\mathcal{M}^0 = \widehat{\mathcal{M}}^0$, and $\mathcal{N}^0 = \widehat{\mathcal{N}}^0$

For purposes of parallelism, we arrange our Lemmas 4.3.1–4.3.12 to match Lemmas 5.1 through 5.12 of [22] and [51]. All twelve of these lemmas have duals, which we will not state or prove except when the proof of the dual requires a distinct technique, principally in Lemma 4.3.11, which yields a nice insight into the construction and the reasons why Theorem 4.1.1 actually holds.

Of course, our lemmas hold for the $\overline{A}/\overline{B}$ game, whereas in [22] they held for the entire universe of elements. Also, our first Lemma matches Lemma 5.0 of [51].

Lemma 4.3.0 *(i) If the $\overline{A}$-state ν lies in $\mathcal{E}_\alpha^0$, then there exists an infinite set $\{x_i\}_{i\in\omega} \subseteq \overline{A}$ such that*

$$(\forall i)[\lim_s \Gamma^{1,\alpha}_{\nu,i,s} = x_i \;\; \& \;\; (\exists s)[x_i \in S_{\alpha,s} - Y_{\alpha,s-1} \;\; \& \;\; \nu(\alpha, x_i, s) = \nu]].$$

(ii) If the $\overline{A}$-state ν lies in $\mathcal{F}_\alpha^0$, then there exists an infinite set $\{x_i\}_{i\in\omega} \subseteq \overline{A}$ such that

$$(\forall i)[\lim_s \Gamma^{2,\alpha}_{\nu,i,s} = x_i \;\; \& \;\; (\exists s)[x_i \in R_{\alpha,s} \;\; \& \;\; \nu(\alpha, x_i, s) = \nu]].$$

(iii) If the $\overline{A}$-state ν lies in $\mathcal{F}_\alpha^{0+}$, then there exists an infinite set $\{x_i\}_{i\in\omega} \subseteq \overline{A}$ such that

$$(\forall i)[\lim_s \Gamma^{3,\alpha}_{\nu,i,s} = x_i \;\; \& \;\; (\exists s)[x \in R_{\alpha,s} \;\; \& \;\; \nu^+(\alpha, x_i, s) = \nu]].$$

Proof. All of these proofs are similar; we therefore give just the proof for (i), which serves with appropriate modifications for the other two:

Assume by induction that we have found distinct elements $x_0, x_1, \ldots x_{i-1}$ as required in (i), and let t be a stage by which

(a) $\lim_s \Gamma^{1,\alpha}_{\nu,i',s} = \Gamma^{1,\alpha}_{\nu,i',t'}$ for all $t' \geq t$ and $i' < i$; and

(b) $\lim_s h(q^{1,\alpha}_{\nu,i}, s) = h(q^{1,\alpha}_{\nu,i}, t')$ for all $t' \geq t$.

If $h(q^{1,\alpha}_{\nu,i}, t) = 0$, then at every stage $t' > t$, every x which enters $S_{\alpha,t'+1}$ in α-state ν enters $A_{t'+1}$ immediately, by Substep 0.2. This contradicts $\nu \in \mathcal{E}^0_\alpha$. Hence $h(q^{1,\alpha}_{\nu,i}, t) = 1$, and $Q^{1,\alpha}_{\nu,i} \cap \overline{A} \neq \emptyset$, so some $y \in \overline{A}$ eventually goes into $Q^{1,\alpha}_{\nu,i}$. When this y enters $Q^{1,\alpha}_{\nu,i}$, it has $\Gamma^{1,\alpha}_{\nu,i}$ assigned to it (since y does not go into A), and since y never enters A this marker is permanently assigned to y.

Since only finitely many $\Gamma^{1,\alpha}_{\nu}$ markers may be attached to a given y, the set $\{x_i\}_{i\in\omega}$, where x_i is the y to which marker $\Gamma^{1,\alpha}_{\nu,i}$ is permanently assigned, must be infinite. ∎

The construction makes the following lemma clear. (When, e.g., Step 1 of the construction applies to a node α and an element x, we will say, "Step 1_α applies to x.")

Lemma 4.3.1 *At stage $s+1$,*

(i) if x enters R_α, $\alpha \neq \lambda$, then Step 1 or Step 2 applies to α and x;

(ii) if x moves from S_α to S_δ then one of the following steps must apply to x: Step 1_δ for $\delta <_{\mathrm{L}} \alpha$ or $\delta^- = \alpha$; Step 2_δ for δ such that $\delta^- = \alpha$; or Substep $11C_\alpha$, so $f_{s+1} <_{\mathrm{L}} \alpha$;

(iii) if $x \in S_{\alpha,s}$ is enumerated in a red set U_α at stage $s+1$ then Step 1 or Step 4 must apply to x;

(iv) if $x \in S_{\alpha,s}$ is enumerated in a blue set $\widehat{V}_\alpha$ then Step 1, Step 3, or Step 5 must apply to x. ∎

Lemma 4.3.2 (True Path Lemma) *The true path $f = \liminf_s f_s$.*

Proof. This is clear from the definition of f_s in Step 11A and from the choice of the sets D_α. ∎

Lemma 4.3.3 *For all $\alpha \in T$,*

(i) $f <_L \alpha \implies R_{\alpha,\infty} = \emptyset$,

(ii) $\alpha <_L f \implies Y_\alpha =^* \emptyset$,

(iii) $\alpha \subset f \implies Y_{<\alpha} =_{\text{dfn}} \bigcup \{Y_\delta : \delta <_L \alpha\} =^* \emptyset$.

Proof. Part (i) holds because whenever $f_{s+1} <_L \alpha$, Step 11C sets $S_{\alpha,s+1} = \emptyset$. For part (ii), if $\alpha <_L f$, pick an s such that $\alpha <_L f_t$ for all $t \geq s$. Then $Y_\alpha = Y_{\alpha,s} =^* \emptyset$. Finally, for part (iii), if $\alpha \subset f$, then $Y_{<\alpha} \subseteq \{0, 1, \ldots s\}$, where s is a stage such that $f_t \not<_L \alpha$ for all $t \geq s$. ■

In Lemma 4.3.4, since it is now possible for an element x to disappear from the game by being enumerated into A (or B, in the dual lemma), we must slightly modify the statement of (iv) from [22] by restricting x to elements of $\overline{A}$ (and $\hat{x}$ to $\overline{B}$, in the dual), as shown:

Lemma 4.3.4 *For every $\alpha \in T$ such that $\alpha \neq \lambda$, if $\beta = \alpha^-$, then*

(i) $Y_\alpha \setminus Y_\beta = \emptyset$ and $Y_\alpha \subseteq Y_\beta$,

(ii) $(\forall x)(\exists^{\leq 1} s)[x \in R_{\alpha,s+1} - R_{\alpha,s}]$,

(iii) $U_\alpha \setminus Y_\alpha = \widehat{V}_\alpha \setminus Y_\alpha = \emptyset$, and

(iv) If $\alpha \subset f$, then

$$(\exists v_\alpha)(\forall x \in \overline{A})(\forall s \geq v_\alpha)[x \in R_{\alpha,s} \implies (\forall t \geq s)[x \in R_{\alpha,t}]]$$

(and correspondingly with $\overline{B}$ in the dual).

Proof. Part (i) follows from Lemma 4.3.1(i).

For (ii), we note from Lemma 4.3.1(ii) that if $x \in R_{\alpha,t} - R_{\alpha,t+1}$, then $x \in S_{\delta,t+1}$ for some δ, and either $\delta <_L \alpha$, or α was initialized at stage $t+1$. In the former case, x can never re-enter R_α (by 4.3.1(ii), again). If α was initialized, then $\delta = f_{t+1} \subset \alpha$, and x could only return to R_α by applications of Step 1 or Step 2. However, we know that $x < t$ by Step 10 (since $x \in R_{\alpha,t}$), so the restrictions (1.3) and (2.3) in Steps 1 and 2 rule out the return of x to R_α.

For (iii), any of Steps 1, 3, 4 and 5 can put an x into some $U_{\alpha,s+1}$ or $\hat{V}_{\alpha,s+1}$, but each of them either requires $x \in Y_{\alpha,s}$ or puts $x \in Y_{\alpha,s+1}$.

Finally, (iv) assumes $\alpha \subset f$, so by Lemma 4.3.3(iii), $Y_{<\alpha}$ is finite. Let v_α be a stage so large that $f_s <_L \alpha$ only if $s < v_\alpha$, and also that every $y \in Y_{<\alpha}$ never again either enters or leaves R_α. (By Part (ii) of this lemma, each of the finitely many $y \in Y_{<\alpha}$ enters R_α at most once.) Lemma 4.3.1(ii) makes it clear that the only way for *any* $x \in \overline{A}$ to leave R_α at any stage is for it to enter $Y_{<\alpha}$ or for $f_{s+1} <_L \alpha$. Neither of these can occur at any stage $s > v_\alpha$, by our choice of v_α. ■

Lemma 4.3.5 *For all* $x \in \overline{A}$

(i) $\alpha(x) =_{\text{dfn}} \lim_s \alpha(x,s)$ *exists, and*

(ii) x *is enumerated in at most finitely many r.e. sets* U_γ, $\widehat{V}_\gamma$, *and hence for* $\alpha = \alpha(x)$,

$$\nu(\alpha,x) =_{\text{dfn}} \lim_s \nu(\alpha,x,s) \text{ exists.}$$

(And similarly with $\overline{B}$ *in the dual.)*

Proof. Lemma 4.3.1(ii) gives the conditions under which $\alpha(x,s+1) \neq \alpha(x,s)$ can occur. Let $\gamma = f \restriction x$ be the initial segment of the true path with length x, and choose $s > v_\gamma$ with $f_s \restriction x = \gamma$. Step 11C forces either $\alpha(x,s) <_L \gamma$ or $\alpha(x,s) \subseteq \gamma$. (It is impossible for $\gamma \subsetneq \alpha(x,s)$ since $|\gamma| = x$.) Moreover, Step 11C will never apply to x after stage s.

Now Steps 1 and 2 can only move x into S_α if $x > |\alpha|$. Also, each α has only finitely many predecessors in T, and x cannot be moved back and forth among these predecessors infinitely often because of Lemma 4.3.4(ii). Therefore, if $\alpha(x,s+1) \neq \alpha(x,s)$ occurs infinitely often, then there must be infinitely many stages at which either $\alpha(x,s+1) <_L \alpha(x,s)$. However, there is no infinite sequence $\{\delta_1 <_L \delta_2 <_L \delta_3 <_L \ldots\}$ in T with every $|\delta_i| < x$. This proves part (i).

Part (ii) follows from (i) because $\alpha(x,s)$ eventually converges to some $\alpha(x)$, and there are only finitely many possible $\alpha(x)$-states. Once x leaves some $\alpha(x)$-state, it can never return to that state, because the sets U_γ and $\hat{V}_\gamma$ which we enumerate are c.e. Moreover, x will never be enumerated in any U_γ or $\hat{V}_\gamma$ unless $\gamma \subseteq \alpha(x)$. ■

Lemma 4.3.6 *If the hypotheses of some Step* 0–5*, or* $\hat{0}$–$\hat{5}$ *remain satisfied, then that step eventually applies. Also, Step 10 applies infinitely often.*

Proof. If Steps 10 and $\widehat{10}$ never applied after some stage s_0, then there would only be finitely many elements x and $\hat{x}$ in Y_λ and $\hat{Y}_\lambda$, to which the steps preceding Step 10 would apply at every stage after s_0. Each of these steps performs some action when applied, either moving an x or an $\hat{x}$ into a new S_α or enumerating it into some U_α, V_α, $\hat{U}_\alpha$, or $\hat{V}_\alpha$. However, such actions can only occur finitely often for any given x or $\hat{x}$, by Lemma 4.3.5, so eventually Step 10 or Step $\widehat{10}$ must apply, providing a new element x or $\hat{x}$. In order for Step 10 or $\widehat{10}$ to apply, the hypotheses of all the other steps must be unsatisfied. This proves the lemma. ∎

Lemma 4.3.7 *If* $\alpha \subset f$*,* $\rho \subsetneq \alpha$*, and* $\beta = \alpha^-$ *then*

(i) $(\forall \gamma <_L f)[m(\gamma) =_{\mathrm{dfn}} \lim_s m(\gamma, s) < \infty]$,

(ii) $m(\alpha) =_{\mathrm{dfn}} \lim_s m(\alpha, s) = \infty$,

(iii) $\mathcal{E}^0_\alpha \supseteq \mathcal{M}^0_\alpha = \mathcal{F}^{0+}_\beta$,

(iv) $\widehat{\mathcal{E}}^0_\alpha \supseteq \widehat{\mathcal{M}}^0_\alpha = \widehat{\mathcal{F}}^{0+}_\beta$*, and*

Proof. For part (i), we note that for each $\gamma <_L f$, Substep 11B can only apply finitely often. Hence $\lim_s m(\gamma, s)$ must be finite.

Turning to (ii), we let α and β be as given in the lemma. The definition of the true path (Definition 4.2.10) yields $\mathcal{M}^0_\alpha = \mathcal{F}^{0+}_\beta$ and $\widehat{\mathcal{M}}^0_\alpha = \widehat{\mathcal{F}}^{0+}_\beta$. By Substep 11B, $m(\alpha, s)$ is nondecreasing as a function of s; we claim that it increases infinitely often. Otherwise there would exist a stage s_0 with $m(\alpha, s) = m(\alpha, s_0)$ for all $s \geq s_0$.

Claim: Every α-entry $\langle \alpha, \nu_1 \rangle$ on $\mathcal{L}$ ($\langle \alpha, \widehat{\nu}_1 \rangle$ on $\widehat{\mathcal{L}}$) is eventually marked.

As in [51], we modify the proof of this claim in the non-dual case, since it is now possible for elements to leave the game before they can enter S_α. We will use Lemma 4.3.0(iii) to guarantee a supply of elements ($\{x_i\}_{i \in \omega}$) that remain in $\overline{A}$ because their Γ^3-tags are never removed.

If some entry $\langle \alpha, \nu_1 \rangle$ on $\mathcal{L}$ were never marked, then no more α-entries would ever be added to $\mathcal{L}$ after $\langle \alpha, \nu_1 \rangle$. Choose a stage s_1 large enough that neither any

α-entries on $\mathcal{L}$ nor any entry on $\mathcal{L}$ preceding $\langle\alpha, \nu_1\rangle$ is ever marked after stage s_1, that $Y_{<\alpha,s_1} = Y_{<\alpha}$ (using Lemma 4.3.3), and that $Y_{\alpha,s_1}\lceil m(\alpha, s_0) = Y_\alpha\lceil m(\alpha, s_0)$. Now requirement (2.4) prevents Step 2 from enumerating any $x > m(\alpha, s_0)$ into R_α after stage s_1, and Step 1 will never again put any x into R_α because by (1.8), that would involve marking an unmarked α-entry on $\mathcal{L}$.

Now $\nu_1 \in \mathcal{M}^0_\alpha$ since $\langle\alpha, \nu_1\rangle \in \mathcal{L}$. Also $\mathcal{M}^0_\alpha = \mathcal{F}^{0+}_\beta$, since $\alpha \subset f$. Hence Lemma 4.3.0(iii) applied to β provides an infinite collection of elements $\{x_i\}_{i\in\omega} \subset \overline{A}$. By the choice of s_1 all but finitely many x_i satisfy (1.1)–(1.7). (Satisfying (1.5) uses the assumption that $\lim_s m(\alpha, s)$ is finite.) Thus, some such x_i is moved to S_α under Step 1 at some stage $s + 1 > s_1$, and the entry $\langle\alpha, \nu_1\rangle$ is then marked, contrary to hypothesis. This establishes the claim for $\mathcal{L}$.

With the claim, we see that Substep 11B will apply to α at some stage $s > s_1$, forcing $m(\alpha, s) > m(\alpha, s - 1)$.

(The proof of (ii) in the dual case is simpler, because we never enumerate any element of $\widehat{S}_{\beta,s}$ into B.)

(iii) now follows (and (iv) similarly) because for any $\nu_1 \in \mathcal{M}^0_\alpha$, (ii) forces infinitely many entries $\langle\alpha, \nu_1\rangle$ to be added to $\mathcal{L}$, and for each to enter, all previous such entries must have been marked. The only way for an entry to be marked is for an x in α-state ν_1 to enter S_α, and if this happens infinitely often, then $\nu_1 \in \mathcal{E}^0_\alpha$.

■

Lemma 4.3.8 $\alpha \subset f \implies$

(i) $R_{\alpha,\infty} =^* Y_\alpha \cap \overline{A} =^* Y_\lambda \cap \overline{A} = \overline{A}$*; and*

(ii) Y_α *is infinite. (And similarly for the dual lemma, with* B *for* A*.)*

Proof. By Lemma 4.3.6(i) Step 10 must eventually put every element $x \in \omega$ into Y_λ. By induction we may assume that $R_{\beta,\infty} =^* Y_\beta \cap \overline{A} =^* \overline{A}$ and Y_β is infinite, for $\beta = \alpha^-$. By Lemma 4.3.7 $m(\alpha) = \infty$, and $m(\gamma) < \infty$ for all $\gamma <_L \alpha$ with $\gamma^- = \beta$.

Now by Lemma 4.3.3, $Y_{<\alpha} =^* \emptyset$. Also, cofinitely many of the elements $x \in (Y_\beta - Y_\alpha) \cap \overline{A}$ will eventually enter S_β. Therefore, cofinitely many such x will satisfy (2.1)–(2.5) at some stage, and will be moved to S_α by Step 2. Once there, cofinitely many of them will remain in R_β forever, by Lemma 4.3.4(iv).

Part (ii) follows immediately from part (i), since $\overline{A}$ is infinite (as is $\overline{B}$, in the dual case).

The proof of the dual case is nearly the same, except that we make the indicated changes in the second paragraph, and the last paragraph is replaced by the following:

To see that $\widehat{Y}_\alpha$ is infinite, observe that since $\widehat{Y}_\beta$ is infinite, infinitely many elements must enter $\widehat{S}_\beta$ via Step $\hat{1}$ or Step $\hat{2}$. By the above reasoning, almost all of these must eventually enter $\widehat{S}_\alpha$. ∎

Lemma 4.3.9 *$\alpha \subset f \implies \alpha$ is $\mathcal{M}$-consistent.*

Proof. Let $\alpha \subset f$ and $\beta = \alpha^-$. Assume for a contradiction that α is not $\mathcal{M}$-consistent. Then $e_\alpha > e_\beta$ and there exist $\nu_0 \in \mathcal{M}^0_\alpha$, $\nu_1 \notin \mathcal{M}^0_\alpha$, $\nu_0 <_B \nu_1$ and $\nu_1 \restriction \beta \in \mathcal{M}^0_\beta$. By Definition 4.2.2, α is a terminal node on T, so $S_{\alpha,s} = R_{\alpha,s}$ for all s. Thus, by Lemma 4.3.4(iv), for some v_α, no $x \in S_{\alpha,s} \cap \overline{A}$ later leaves S_α.

By Lemma 4.3.7, $\nu_0 \in \mathcal{E}^0_\alpha$. Thus, by Lemma 4.3.0(i), we have an infinite set $\{x_i\}_{i\in\omega} \subseteq \overline{A}$ such that

$$(\forall i)(\exists s)[x_i \in S_{\alpha,s+1} - S_{\alpha,s} \;\;\&\;\; \nu(\alpha, x_i, s+1) = \nu_0].$$

Let x be any such x_i with $x > k_\alpha$ and the corresponding $s > v_\alpha$.

Now Step 0 can never change the α-state of x, since $x \in \overline{A}$, and Steps 1 and 2 cannot move x at any stage $t > s$, since they could only act to move x to a different region S_γ. Thus, Step 3_α must eventually apply to x at some stage $t+1 > s+1$, moving x from ν_0 either to ν_1, or to some other state ν_1' such that $\nu_0 <_B \nu_1'$ and $\nu_1' \restriction \beta \in \mathcal{M}^0_\beta$ and $\nu_1' \notin \mathcal{M}^0_\alpha$. Then α is provably incorrect at all stages $v \geq t+1$, so $\alpha \not\subset f$.

In the dual, there is no need to appeal to an analogue of Lemma 4.3.0(i), since we do not need $\hat{x} \in \overline{B}$. We simply note that since $\alpha \subset f$, we have $\hat{\nu}_0 \in \widehat{\mathcal{M}}^0_\alpha = \widehat{\mathcal{E}}^0_\alpha$, so there will be infinitely many $\hat{x} > k_\alpha$ and $s > \hat{v}_\alpha$ available to us with $\hat{x} \in$

$\hat{S}_{\alpha,s+1} - \hat{S}_{\alpha,s}$ and $\hat{\nu}(\alpha, \hat{x}, s+1) = \hat{\nu}_0$. As with x above, Step $\hat{3}$ must eventually move each such $\hat{x}$ into some blue set. Since α is inconsistent, $\hat{x}$ cannot enter B at any stage $t > \hat{v}_\alpha$ so it enters a state $\hat{\nu}_1 \notin \widehat{\mathcal{M}}^0_\alpha$. Again, this forces $\alpha \not\subset f$. ■

Lemma 4.3.10 *If* $\alpha \subset f$ *then*

(i) $\widehat{\mathcal{M}}^0_\alpha = \{\hat{\nu} : \nu \in \mathcal{M}^0_\alpha\}$,

(ii) $\mathcal{M}^0_\alpha = \mathcal{F}^0_\alpha = \mathcal{E}^0_\alpha$, *and*

(iii) $\widehat{\mathcal{M}}^0_\alpha = \widehat{\mathcal{F}}^0_\alpha = \widehat{\mathcal{E}}^0_\alpha$.

Proof. Fix $\alpha \subset f$, and let $\beta = \alpha^-$. Now (i) holds by the definition of $\widehat{\mathcal{M}}^0_\alpha$. By induction we may assume (ii) and (iii) for β. We know $\mathcal{E}^0_\alpha \subseteq \mathcal{F}^0_\alpha$ by their definitions, and $\mathcal{M}^0_\alpha \subseteq \mathcal{E}^0_\alpha$ by Lemma 4.3.7. Thus, to prove (ii) (and (iii)) it suffices to prove $\mathcal{F}^0_\alpha \subseteq \mathcal{M}^0_\alpha$, (and $\widehat{\mathcal{F}}^0_\alpha \subseteq \widehat{\mathcal{M}}^0_\alpha$).

Case 1. $e_\alpha = e_\beta$ and $\hat{e}_\alpha = \hat{e}_\beta$.

Then $\mathcal{M}^0_\alpha = \mathcal{M}^0_\beta$. Also $\mathcal{F}^0_\alpha \subseteq \mathcal{F}^0_\beta$ since $Y_\alpha \subseteq Y_\beta$. Finally, $\mathcal{M}^0_\beta = \mathcal{F}^0_\beta$ by the inductive hypothesis (ii) for β. Hence,

$$\mathcal{F}^0_\alpha \subseteq \mathcal{F}^0_\beta = \mathcal{M}^0_\beta = \mathcal{M}^0_\alpha,$$

so (ii) holds for α. Likewise, $\widehat{\mathcal{F}}^0_\alpha \subseteq \widehat{\mathcal{M}}^0_\alpha$, so (iii) holds for α.

Before considering Case 2 we need a technical sublemma.

Sublemma. If $e_\alpha > e_\beta$, $\nu_2 = \langle \alpha, \sigma_2, \tau_2 \rangle \in \mathcal{F}^{0+}_\beta$, and $\nu_1 = \langle \alpha, \sigma_1, \tau_2 \rangle$, where $\sigma_1 = \sigma_2 - \{e_\alpha\}$, then $\nu_1 \in \mathcal{F}^{0+}_\beta$ also.

Proof. Suppose $\nu_2 \in \mathcal{F}^{0+}_\beta$. Then $\nu_3 = \nu_2 \restriction \beta \in \mathcal{F}^0_\beta$, and $\mathcal{F}^0_\beta = \mathcal{E}^0_\beta$ by the inductive hypothesis (ii) for β. Hence, by the definition of $\mathcal{E}^0_\beta$,

$$(\exists^\infty x)(\exists s)[x \in Y_{\beta,s} - Y_{\beta,s-1} \;\&\; \nu(\beta, x, s) = \nu_3].$$

However, for each such x and s, we have $x \notin Z_{e_\alpha,s}$ (by the definition of $Z_{e_\alpha,s}$) so $\nu^+(\alpha, x, s) = \nu_1$. Hence, $\nu_1 \in \mathcal{F}^{0+}_\beta$ by the definition of $\mathcal{F}^{0+}_\beta$ in (4.14). This proves the Sublemma.

Case 2. $e_\alpha > e_\beta$.

We prove $\mathcal{F}^0_\alpha \subseteq \mathcal{M}^0_\alpha$ and its dual $\widehat{\mathcal{F}}^0_\alpha \subseteq \widehat{\mathcal{M}}^0_\alpha$ in the next five claims. (The proof of Case 3, $\hat{e}_\alpha > \hat{e}_\beta$, is entirely dual and will be omitted.)

Claim 1. $\mathcal{F}^0_\alpha \subseteq \mathcal{M}^0_\alpha$.

Proof. Suppose $\nu_1 \in \mathcal{F}^0_\alpha$. Let $\nu_1 = \langle \alpha, \sigma_1, \tau_1 \rangle$. Then

$$(\exists^\infty x)(\exists s)[x \in Y_{\alpha,s} \;\;\&\;\; \nu(\alpha, x, s) = \nu_1]. \tag{4.38}$$

Note that $Y_{\alpha,s} \subseteq Y_{\beta,s}$ and $\nu(\alpha, x, s) \leq_{\mathrm{R}} \nu^+(\alpha, x, s)$ because $U_{\alpha,s} \subseteq Z_{e_\alpha,s}$. First suppose

$$(\exists^\infty x)(\exists s)[x \in Y_{\alpha,s} \;\;\&\;\; \nu^+(\alpha, x, s) = \nu_1]. \tag{4.39}$$

Then $\nu_1 \in \mathcal{F}^{0+}_\beta$ by definition of $\mathcal{F}^{0+}_\beta$ because $Y_{\alpha,s} \subseteq Y_{\beta,s}$, and $\mathcal{F}^{0+}_\beta = \mathcal{M}^0_\alpha$ since $\alpha \subset f$.

If (4.39) fails, then for almost every x in (4.38), $\nu^+(\alpha, x, s) = \nu_2 >_R \nu_1$, so $\nu_2 = \langle \alpha, \sigma_2, \tau_1 \rangle$ where $e_\alpha \notin \sigma_1$ and $\sigma_2 = \sigma_1 \cup \{e_\alpha\}$. Now $\nu_2 \in \mathcal{F}^{0+}_\beta$ since $Y_{\alpha,s} \subseteq Y_{\beta,s}$, so $\nu_1 \in \mathcal{F}^{0+}_\beta = \mathcal{M}^0_\alpha$ by the Sublemma.

Claim 2. $\widehat{\mathcal{F}}^0_\alpha \subseteq \widehat{\mathcal{M}}^0_\alpha$.

Proof. We establish Claim 2 by the next three claims which are the duals of (4.6), (4.7), and (4.8).

Claim 3. $\widehat{\mathcal{E}}^0_\alpha \subseteq \widehat{\mathcal{M}}^0_\alpha$.

Proof. Assume $\hat{\nu}_1 \in \widehat{\mathcal{E}}^0_\alpha$. Hence,

$$(\exists^\infty \hat{x})(\exists s)[\hat{x} \in \widehat{S}_{\alpha,s+1} - \widehat{Y}_{\alpha,s} \;\;\&\;\; \hat{\nu}(\alpha, \hat{x}, s+1) = \hat{\nu}_1].$$

For every such $\hat{x}$ and s, $\hat{x}$ must have entered $\widehat{S}_{\alpha,s+1}$ under Step $\hat{1}$ or Step $\hat{2}$. If Step $\hat{1}$ applied then we marked an entry $\langle \alpha, \hat{\nu}_1 \rangle$ on $\widehat{\mathcal{L}}_s$ so $\hat{\nu}_1 \in \widehat{\mathcal{M}}^0_\alpha$ by the definition of $\widehat{\mathcal{L}}$ in Step 11. If Step $\hat{2}$ applied then $\hat{x} \notin \widehat{U}_{\alpha,s+1}$ because $\hat{x} \notin \widehat{U}_{\alpha,s}$

by Lemma 4.3.4(iii) and no enumeration takes place at stage $s+1$ under Step $\hat{2}$. Hence, $e_\alpha \notin \sigma_1$, where $\nu_1 = \langle \alpha, \sigma_1, \tau_1 \rangle$.

Let $\nu_3 = \nu_1 \restriction \beta$. Now $\hat{\nu}_3 \in \widehat{\mathcal{F}}^0_\beta = \widehat{\mathcal{M}}^0_\beta$ so $\nu_3 \in \mathcal{M}^0_\beta = \mathcal{F}^0_\beta$ and thus either $\nu_1 \in \mathcal{F}^{0+}_\beta$ or $\nu_2 \in \mathcal{F}^{0+}_\beta$ where $\nu_2 = \langle \alpha, \sigma_1 \cup \{e_\alpha\}, \tau_1 \rangle$. But if $\nu_2 \in \mathcal{F}^{0+}_\beta$ then $\nu_1 \in \mathcal{F}^{0+}_\beta$ by the Sublemma. In either case $\nu_1 \in \mathcal{F}^{0+}_\beta = \mathcal{M}^0_\alpha$, so $\hat{\nu}_1 \in \widehat{\mathcal{M}}^0_\alpha$.

Claim 4. If $\hat{x} \in \widehat{Y}_{\alpha,s}$, $\hat{\nu}_1 = \nu(\alpha, \hat{x}, s) \in \widehat{\mathcal{M}}^0_\alpha$, $s > v_\alpha$ of Lemma 4.3.4(iv), and RED causes enumeration of $\hat{x}$ so that $\hat{\nu}_2 = \hat{\nu}(\alpha, \hat{x}, s+1)$ then $\hat{\nu}_2 \in \widehat{\mathcal{M}}^0_\alpha$.

Proof. Suppose this enumeration occurs. Then $\hat{\nu}_1 <_R \hat{\nu}_2$ so $\nu_1 <_B \nu_2$ by (4.17). Now $\nu_1 \in \mathcal{M}^0_\alpha$ since $\hat{\nu}_1 \in \widehat{\mathcal{M}}^0_\alpha$. But α is $\mathcal{M}$-consistent by Lemma 4.3.9, so $\nu_2 \in \mathcal{M}^0_\alpha$, and hence $\hat{\nu}_2 \in \widehat{\mathcal{M}}^0_\alpha$.

Claim 5. If $\hat{x} \in \widehat{Y}_{\alpha,s}$, $\hat{\nu}_1 = \hat{\nu}(\alpha, \hat{x}, s) \in \widehat{\mathcal{M}}^0_\alpha$, $s > v_\alpha$ of Lemma 4.3.4(iv), and BLUE causes enumeration of $\hat{x}$ so that $\hat{\nu}_2 = \hat{\nu}(\alpha, \hat{x}, s+1)$ then either $\hat{\nu}_2 \in \widehat{\mathcal{M}}^0_\alpha$ or $\hat{\nu}_2$ is a B-state.

Proof. Suppose $\hat{x} \in \widehat{Y}_{\alpha,s}$ and BLUE causes this enumeration at stage $s+1$, so $\hat{\nu}_1 <_B \hat{\nu}_2$. Since $s > v_\alpha$, $\hat{x} \in \widehat{R}_{\alpha,s} \cap \widehat{R}_{\alpha,s+1}$. Hence, either Step $\hat{1}$, Step $\hat{3}$, Step $\hat{5}$, or Step $\hat{0}$ applies to $\hat{x}$ at stage $s+1$ for some $\gamma \supseteq \alpha$. Assume that $\hat{\nu}_2$ is not a B-state. (Thus Step $\hat{0}$ cannot have applied.) If Step $\hat{1}_\gamma$ or Step $\hat{5}_\gamma$ applies then $\hat{\nu}_3 = \hat{\nu}(\gamma, \hat{x}, s+1) \in \widehat{\mathcal{M}}^0_\gamma$ so $\hat{\nu}_2 = \hat{\nu}_3 \restriction \alpha \in \widehat{\mathcal{M}}^0_\alpha$. (Here Step $\hat{5}_\gamma$ means Step $\hat{5}$ Case 1 for $\hat{x} \in \widehat{Y}_{\gamma,s}$ or Step $\hat{5}$ Case 2 for $\hat{x} \in \widehat{Y}_{\delta,s}$ where $\gamma = \delta^-$.) If Step $\hat{3}_\gamma$ applies, then $\gamma \supsetneq \alpha$ (since α is $\mathcal{M}$-consistent and γ is not) and $\hat{\nu}_3 = \hat{\nu}(\gamma^-, \hat{x}, s+1) \in \widehat{\mathcal{M}}^0_{\gamma^-}$ by (3.4) so $\hat{\nu}_2 = \hat{\nu}_3 \restriction \alpha \in \widehat{\mathcal{M}}^0_\alpha$. This completes the proof of Claim 5.

Claim 2 now follows, since for any $\hat{\nu} \in \widehat{\mathcal{F}}^0_\alpha - \widehat{\mathcal{E}}^0_\alpha$,

$$(\exists^\infty \hat{x})(\exists s)\hat{x} \in \hat{R}_{\alpha,s} \ \& \ \hat{\nu}(\alpha, \hat{x}, s) = \hat{\nu} \ \& \ \hat{\nu}(\alpha, \hat{x}, s-1) \neq \hat{\nu}.$$

(Notice that $\widehat{\mathcal{F}}^0_\alpha$ contains only $\overline{B}$-states, by definition.)

This completes the proof of Case 2, and that of Lemma 4.3.10. ■

Lemma 4.3.11 $\alpha \subset f \implies \alpha$ *is* $\mathcal{R}$*-consistent.*

Proof. To prove $\mathcal{R}^0$-consistency of α, assume for a contradiction that $\alpha \subset f$ and α is not $\mathcal{R}^0$-consistent. Choose $\nu_1 \in \mathcal{R}^0_\alpha$ such that for all $\nu_2 \in \mathcal{M}^0_\alpha$, $\nu_1 \not<_R \nu_2$. Being inconsistent, α is a terminal node on T, so $S_{\alpha,s} = R_{\alpha,s}$ for all s. Thus, by Lemma 4.3.4(v), there exists a stage v_α such that $S_{\alpha,s} \cap \overline{A} \subseteq S_{\alpha,t}$ for every s and t with $t \geq s \geq v_\alpha$.

Now $\hat{\nu}_1 \in \widehat{\mathcal{R}}^0_\alpha \subseteq \widehat{\mathcal{M}}^0_\alpha = \widehat{\mathcal{E}}^0_\alpha$ by Lemma 4.3.10. Therefore Lemma 4.3.0(i), yields an infinite set $\{x_i\}_{i\in\omega} \subseteq \overline{A}$ such that

$$(\forall i)(\exists s)[x_i \in S_{\alpha,s+1} - Y_{\alpha,s} \ \ \& \ \ \nu(\alpha, x_i, s+1) = \nu_1].$$

Let x be any such x_i with the corresponding $s > v_\alpha$. Now Step 0 will not apply to x at any stage $t > s+1$ because $x \in \overline{A}$. Steps 1 and 2 would both remove x from S_α, which is impossible at any stage $t > v_\alpha$. By Lemma 4.3.9, α must be $\mathcal{M}$-consistent, so Step 3 will never apply. Also, Step 5 does not apply to $\mathcal{R}^0$-inconsistent nodes such as α. Therefore, if x is to be removed from state ν_1 as required by $F(\beta, \nu_1)$, then Step 4 must act, enumerating x into some red set U_γ with $\gamma \subseteq \alpha$. Since this happens for infinitely many elements x, and there are only finitely many α-states ν with $\nu_1 <_R \nu$, one of those states ν must lie in $\mathcal{F}^0_\alpha$, hence in $\mathcal{M}^0_\alpha$, by Lemma 4.3.10(iii). This contradicts $\mathcal{R}^0$-inconsistency.

To prove $\widehat{\mathcal{R}}^0$-consistency of α, assume for a contradiction that $\alpha \subset f$ and α is not $\widehat{\mathcal{R}}^0$-consistent. Choose $\hat{\nu}_1 \in \widehat{\mathcal{R}}^0_\alpha$ such that for all $\hat{\nu}_2 \in \mathcal{M}^0_\alpha$, $\hat{\nu}_1 \not<_R \hat{\nu}_2$. Being inconsistent, α is a terminal node on T, so $\hat{S}_{\alpha,s} = \hat{R}_{\alpha,s}$ for all s. Thus, by the dual of Lemma 4.3.4(v), there exists a stage v_α such that $\hat{S}_{\alpha,s} \cap \overline{B} \subseteq \hat{S}_{\alpha,t}$ for every s and t with $t \geq s \geq v_\alpha$.

Now $\hat{\nu}_1 \in \widehat{\mathcal{R}}^0_\alpha \subseteq \widehat{\mathcal{M}}^0_\alpha = \widehat{\mathcal{E}}^0_\alpha$ by the dual of Lemma 4.3.10. Therefore there exist infinitely many elements $\hat{x}$ such that

$$(\exists s)[\hat{x} \in S_{\alpha,s+1} - (B_s \cup Y_{\alpha,s}) \ \ \& \ \ \hat{\nu}(\alpha, \hat{x}, s+1) = \hat{\nu}_1].$$

Take any such $\hat{x} > k_\alpha$ for which the corresponding $s > v_\alpha$. Step 0 does not apply to the $\widehat{\omega}$-side, and Steps $\hat{1}$ and $\hat{2}$ would both remove $\hat{x}$ from $\hat{S}_\alpha$, which is impossible at any stage $t > v_\alpha$. By the dual of Lemma 4.3.9, α must be $\mathcal{M}$-consistent, so Step $\hat{3}$ will never apply. Steps $\hat{5}$ and $\hat{0}$ do not apply to $\widehat{\mathcal{R}}^0$-inconsistent nodes such as α. Therefore, if $\hat{x}$ is to be removed from state $\hat{\nu}_1$ as required by $\hat{F}(\beta, \hat{\nu}_1)$, then Step $\hat{4}$ must act, enumerating $\hat{x}$ into some red set V_γ with $\gamma \subseteq \alpha$. Since this happens for infinitely many elements $\hat{x}$, and there are only finitely many α-states $\hat{\nu}$ with $\hat{\nu}_1 <_R \hat{\nu}$, one of those states $\hat{\nu}$ must lie in $\widehat{\mathcal{F}}^0_\alpha$, hence in $\widehat{\mathcal{M}}^0_\alpha$, by the dual of Lemma 4.3.10(iii). This contradicts $\widehat{\mathcal{R}}^0$-inconsistency. ■

We remark that while the two halves of the preceding proof appear quite similar, the similarity is deceptive. In fact, the proof of $\mathcal{R}^0$-consistency, depends on the lowness of A, which guided the proof of Lemma 4.3.0. On the other hand, in the proof of the dual $\widehat{\mathcal{R}}^0$-consistency, we used instead the fact that inconsistent nodes do not require any elements to be enumerated into any blue sets, including B itself. This works in the present situation because the only external requirements for the construction of B are negative requirements, namely the $\mathcal{Q}_e$ of the Sacks preservation strategy. (The positive requirements stem from the automorphism construction itself, not from any properties which we demand of B.) Herein lies the connection between lowness of A and the ability of A to avoid an upper cone.

Lemma 4.3.12 *If $\alpha \subset f$ and $\nu_1 \in \mathcal{B}^0_\alpha$, then*

$$\{x : x \in Y_\alpha \;\;\&\;\; \nu(\alpha, x) = \nu_1\} =^* \emptyset.$$

Proof. Fix $\alpha \subset f$ and $\nu_1 \in \mathcal{B}^0_\alpha$. Let v_α be as in Lemma 4.3.4(iv). Assume for a contradiction that $x \in R_{\alpha,s}$ for some $s > v_\alpha$ and that for all $t \geq s$, $\gamma = \alpha(x,t)$, and $\nu_1 = \nu(\alpha, x, t)$. Now $\gamma \supseteq \alpha$ and $\alpha \in T$, so by the Definition 4.2.2 (vi) of T we have $\nu_1' \in \mathcal{B}^0_\gamma$ for all $\nu_1' \in \mathcal{M}^0_\gamma$ such that $\nu_1' \restriction \alpha = \nu_1$.

Case 1. If γ is $\mathcal{R}$-consistent, then by Lemma 4.3.6, Step 5 Case 1 will apply to x and γ at some stage $t+1 > s$, so $\nu_1' = \nu(\gamma, x, t)$, $\nu_2' = \nu(\gamma, x, t+1)$, $\nu_1' <_B \nu_2'$, and $\nu_2' \in \mathcal{M}^0_\gamma - \mathcal{B}^0_\gamma$. Hence, $\nu_2 = \nu_2' \restriction \alpha \in \mathcal{M}^0_\alpha - \mathcal{B}^0_\alpha$, and $\nu(\alpha, x, t+1) = \nu_2 >_B \nu_1$.

Case 2. Otherwise there will be a stage $t+1 > s$ at which Step 5 Case 2 applies to x and $\delta = \gamma^- \supseteq \alpha$. Hence $\nu(\alpha, x, t+1) = \nu_2 >_B \nu_1$ as in Case 1 but with δ in place of γ.

In the dual case, we note that the state $\hat{\nu}_2'$ might possibly be a B-state. If so, then $\hat{\nu}_2$ would not lie in $\widehat{\mathcal{M}}^0_\alpha$. However, in that case $\hat{\nu}_2$ would also be a B-state, so $\hat{\nu}_2 \neq \hat{\nu}_1$. ∎

Lemma 4.3.13 *For every $\alpha \subset f$, $\mathcal{M}^0_\alpha = \widehat{\mathcal{M}}^0_\alpha$ and $\mathcal{N}^0_\alpha = \widehat{\mathcal{N}}^0_\alpha$.*

Proof. Lemma 4.3.10(i) gives the result for $\mathcal{M}$. Moreover, since $\alpha \subset f$, we know that $\mathcal{R}^0_\alpha = \widehat{\mathcal{B}}^0_\alpha$ and $\mathcal{B}^0_\alpha = \widehat{\mathcal{R}}^0_\alpha$ (see Definition 4.2.10). To prove $\mathcal{N}^0_\alpha = \widehat{\mathcal{N}}^0_\alpha$, therefore, we need only show that for each $\overline{A}$-α-state ν in $\mathcal{M}^0_\alpha$,

$$\nu \in \mathcal{B}^0_\alpha \cup \mathcal{R}^0_\alpha \iff \{x \in \omega : \nu(\alpha, x) = \nu\} \text{ is finite,}$$

and similarly for $\hat{\nu} \in \widehat{\mathcal{B}}^0_\alpha \cup \widehat{\mathcal{R}}^0_\alpha$.

Suppose $\nu \in \mathcal{R}^0_\alpha$. Then $F(\beta, \nu)$ must hold, where $\beta = \alpha^-$. Therefore, by (4.22), only finitely many $x \in Y_\beta$ remain permanently in the α-state ν. Since $\beta \subset \alpha \subset f$, we know that $Y_\beta =^* \omega$, so $\nu \in \mathcal{N}^0_\alpha$. The proof for $\hat{\nu} \in \widehat{\mathcal{R}}^0_\alpha$ is analogous.

Now let $\nu \in \mathcal{B}^0_\alpha$ and suppose $\nu(\alpha, x) = \nu$. Now there exists a node γ and a stage s_0 such that $x \in S_{\gamma,s}$ for all $s \geq s_0$. Since $\alpha \subset f$, $R_{\alpha,\infty}$ is cofinite, so we may assume that $\gamma \supseteq \alpha$. Let $\nu_1 = \nu(\gamma, x)$ be the permanent γ-state of x, and suppose that $s_1 \geq s_0$ is such that $\nu(\gamma, x, s) = \nu_1$ for all $s \geq s_1$. Then $\nu_1 \restriction \alpha = \nu$, and ν_1 is an $\overline{A}$-state. By part (vi) of Definition 4.2.2, $\nu_1 \in \mathcal{B}^0_\gamma$. If γ is a consistent node, then by Lemma 4.3.6, there will eventually be a stage $s \geq s_1$ at which Case 1 of Step 5 applies, so x will be moved into some other γ-state $\nu_2 >_B \nu_1$ at stage s_1. If γ is inconsistent, then again x will change γ-states at some stage $s \geq s_1$ at which Case 2 of Step 5 applies. In either case, this contradicts our assumption that $\nu(\gamma, x) = \nu_1$. Thus there are only finitely many x which reside permanently in the α-state ν, forcing $\nu \in \mathcal{N}^0_\alpha$.

For $\hat{\nu} \in \widehat{\mathcal{B}}^0_\alpha$, the dual proof holds for all $\hat{x} \in \overline{B}$. If $\hat{x} \in B$, then clearly $\hat{\nu}$ is not the final α-state of $\hat{x}$, since every state in $\widehat{\mathcal{B}}^0_\alpha$ is a $\overline{B}$-state. Therefore again $\hat{\nu} \in \widehat{\mathcal{N}}^0_\alpha$.

Now suppose $\nu \in \mathcal{N}^0_\alpha$, i.e. ν is a well-visited but non-well-resided α-state. In the construction, the only steps at which an element x may be moved out of ν are Steps 0, 1, 4, and 5. (Step 3 never applies to α, by Lemmas 4.3.9 and 4.3.11.) If Step 5_γ applies (for some $\gamma \supseteq \alpha$), then $\nu \in \mathcal{B}^0_\alpha$, by part (vi) of Definition 4.2.2. Since $\alpha \subset f$, Step 1 can only move elements in R_α to regions S_γ, where $\alpha \subset \gamma$ (except for finitely many elements), and when it does so, it enumerates them only into U_γ or $\hat{V}_\gamma$, leaving the α-state unchanged. Step 0 could move infinitely many elements into A, but by Lemma 4.3.0, there must also be infinitely many elements from $\overline{A}$ in the state ν, since $\nu \in \mathcal{M}^0_\alpha = \mathcal{E}^0_\alpha$.

Therefore, suppose Step 4 changes the α-state of cofinitely many of the elements in state ν. By definition of $k_\alpha = k^+_\beta$, the finitely many elements not moved can never enter Y_β. Hence $F(\beta, \nu)$ holds. Since $\nu \in \mathcal{M}^0_\alpha$ and $\alpha \subset f$, part (iii) of Definition 4.2.2 forces $\nu \in \mathcal{R}^0_\alpha \cup \mathcal{B}^0_\alpha$.

Finally, for the dual case $\hat{\nu} \in \widehat{\mathcal{N}}^0_\alpha$, the same argument holds, except that Step $\hat{0}$ could move an element out of $\hat{\nu}$. If cofinitely many of the elements which enter state $\hat{\nu}$ are so moved, then according to Step 0, cofinitely many elements in the corresponding state ν on the ω-side must have entered A. This contradicts Lemma 4.3.0. so there must be infinitely many elements in $\hat{\nu}$ which are not moved into B by Step $\hat{0}$. ∎

Lemma 4.3.14 *$\{U_\alpha : \alpha \subset f\}$ and $\{V_\alpha : \alpha \subset f\}$ each forms a skeleton for the collection of all c.e. sets. (That is, for every e there exist $\gamma \subset f$ and $\delta \subset f$ such that $W_e =^* U_\gamma =^* V_\delta$.)*

Proof. Steps 4 and $\hat{4}$ accomplish this, since $R_\alpha =^* \omega$ and $\hat{R}_\alpha =^* \widehat{\omega}$ for all $\alpha \subset f$. The only exception is the set $A = U_0$, which is covered by Substep (0.1). ∎

4.3.3 Verification that $\mathcal{G}^A = \widehat{\mathcal{G}}^B$.

Our proof that $\mathcal{G}^A = \widehat{\mathcal{G}}^B$ follows the same ideas as in [51], section 1.3.3. First, however, we need to show that all requirements are satisfied.

Lemma 4.3.15 *Every requirement $\mathcal{Q}_e$ is satisfied. (Hence $C \not\leq_T B$.)*

Proof. Each positive requirement $\mathcal{P}_{\langle \alpha', \nu', j \rangle}$ puts at most one element into B, so (by induction) there exists a stage s_0 so large that no $\mathcal{P}_{\langle \alpha', \nu', j \rangle}$ with $\langle \alpha', \nu', j \rangle \leq e$ puts any elements into B at any stage $\geq s_0$. Notice also, by the remark at the end of the construction, that only Step $\hat{0}$ ever puts any elements into B, and that it respects all higher-priority negative requirements $\mathcal{Q}_i$ when doing so.

Now suppose that $\mathcal{Q}_e$ fails, i.e. $C = \{e\}^B$. Then $\lim_s l(e, s) = \infty$, and we can use this fact to compute C. Given x, find a stage $s \geq s_0$ such that $l(e, s) > x$. As in [47], Theorem VII.3.1, we must then have

$$\{e\}_s^{B_s}(x) = \{e\}^B(x) = C(x),$$

since by our choice of s_0, the initial segment of B_s used in this computation will never again be changed.

This contradicts the noncomputability of C. Hence $\mathcal{Q}_e$ must be satisfied. ∎

Lemma 4.3.16 *For every e, $\lim_s r(e, s)$ exists and is finite.*

Proof. The proof follows the proof of Lemma 2 in [47] VII.3.1 exactly. Lemma 4.3.15 yields an x such that $C(x) \neq \{e\}^B(x)$. Taking the least such x, we choose a stage s_0 so large that:

- The functions $\{e\}_s^{B_s}$ and C_s converge to their correct values on every argument $< x$, for every $s \geq s_0$;
- $C_{s_0}(x) = C(x)$; and
- No higher-priority requirement $\mathcal{P}_{\langle \alpha, \nu, i \rangle}$ puts any element into B at any stage $s \geq s_0$.

If $\{e\}_t^{B_t}(x)\downarrow$ for some $t \geq s_0$, then the same computation converges to the same value at all stages $s > t$, so $r(e,t) = r(e,s)$ for all $s > t$. Otherwise $\{e\}_t^{B_t}(x)\uparrow$ for all $t \geq s_0$, leaving $r(e,s) = r(e,s_0)$ for all $s \geq s_0$. ■

To show that $\mathcal{G}^A = \widehat{\mathcal{G}}^B$, we will prove the following two lemmas:

Lemma 4.3.17 *For any node α and α-state ν_1, $\mathcal{L}^{\mathcal{G}}$ contains infinitely many pairs $\langle \alpha, \hat{\nu} \rangle$ if and only if $\nu \in \mathcal{G}_\alpha^A$.*

Proof. Such a pair is added to $\mathcal{L}^{\mathcal{G}}$ exactly when Step 0 enumerates some $x \in \nu_1$ into A. Moreover, no step except Step 0 ever puts any elements into A. Thus, $\mathcal{L}^{\mathcal{G}}$ contains infinitely many such pairs if and only if infinitely many $x \in \nu_1$ are enumerated into A; that is, if and only if $\nu \in \mathcal{G}^A$. ■

Lemma 4.3.18 *For any node $\alpha \subset f$ and α-state ν_1, $\mathcal{L}^{\mathcal{G}}$ contains infinitely many pairs $\langle \alpha, \hat{\nu}_1 \rangle$ if and only if $\hat{\nu}_1 \in \widehat{\mathcal{G}}^B$.*

Proof. To show the "if" part of this statement, we observe that

1. We do not move any element $\hat{x}$ in α-state $\hat{\nu}_1$ into B except when required to do so in Step $\hat{0}$ by some pair $\langle \gamma, \hat{\nu}_1' \rangle$ in $\mathcal{L}^{\mathcal{G}}$ with $\alpha \subseteq \gamma$ and $\hat{\nu}_1 = \hat{\nu}_1' \upharpoonright \alpha$, and that

2. If there are infinitely many such pairs in $\mathcal{L}^{\mathcal{G}}$ then there are infinitely many pairs $\langle \alpha, \hat{\nu}_1 \rangle$ in $\mathcal{L}^{\mathcal{G}}$, since whenever we add one of the former we also add one of the latter.

To show the "only if" part, suppose that for a given α and ν, $\mathcal{L}^{\mathcal{G}}$ contains infinitely many pairs $\langle \alpha, \hat{\nu} \rangle$. We claim that for every i, the requirement $\mathcal{P}_{\langle \alpha, \nu, i \rangle}$ is satisfied.

To see this, assume by induction that $\mathcal{P}_{\langle \alpha, \nu, i-1 \rangle}$ is satisfied, and notice that we can find a stage s_0 so large that $\mathcal{L}^{\mathcal{G}}$ contains at least i pairs $\langle \alpha, \hat{\nu} \rangle$ at stage s_0 and that for all $s \geq s_0$ and all $e \leq \langle \alpha, \nu, i \rangle$, $r(e,s) = r(e,s_0)$. By Lemma 4.3.17, $\nu \in \mathcal{G}_\alpha^A$. Therefore $\nu \in \mathcal{M}_\alpha^0$, and by Lemmas 4.3.13 and 4.3.10(iii), $\hat{\nu} \in \widehat{\mathcal{M}}_\alpha^0 = \widehat{\mathcal{F}}_\alpha^0$.

If $\mathcal{P}_{\langle\alpha,\nu,i-1\rangle}$ remained unsatisfied forever, then the definition of $\widehat{\mathcal{F}}^0_\alpha$ would guarantee that there must exist distinct elements $\hat{y}_0, \hat{y}_1, \hat{y}_2, \cdots \hat{y}_{2k}$ and a stage $s > s_0$ at which these elements satisfy conditions $(\hat{0}.4)$–$(\hat{0}.6)$. Now α is consistent by Lemmas 4.3.9 and 4.3.11, and $\mathcal{P}_{\langle\alpha,\nu,i\rangle}$ would not be satisfied at stage s, so by Step $\hat{0}$ of the construction, the element $\hat{y}_{2k}$ would have to enter B from state $\hat{\nu}$ at stage $s+1$.

Since $\nu \in \mathcal{G}^A_\alpha$, we know that the hypothesis of $\mathcal{P}_{\langle\alpha,\nu,i\rangle}$ is satisfied for every i. Since the requirements themselves are all satisfied, we conclude that $\hat{\nu} \in \widehat{\mathcal{G}}^B_\alpha$. ■

With this result we can finally extend Properties 4.3.2 and 4.3.3 to B-states. Since A is infinite, $\mathcal{G}^A_\alpha$ is non-empty for each $\alpha \subset f$, so $\widehat{\mathcal{G}}^B_\alpha$ is also non-empty, forcing B to be infinite. Therefore the ρ-state $\langle \rho, \{0\}, \emptyset\rangle$ is well-resided, so $\widehat{\mathcal{F}}^+_\lambda = \widehat{\mathcal{M}}_\rho$. Also, since every well-visited ρ-state is well-resided, the guess $k_\rho = -1$ is correct.

Lemmas 4.3.17 and 4.3.18 together show that $\mathcal{G}^A = \widehat{\mathcal{G}}^B$. Lemma 4.3.15 shows that $C \not\leq_T B$. Along with Lemmas 4.3.13 and 4.3.14 and Theorem 4.2.9, this completes the proof of Theorem 4.1.1.

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