Differential transcendence of elliptic hypergeometric functions through Galois theory[1]

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Introduction

The theory of elliptic hypergeometric functions has been studied in the mathematical physics community since the early 2000s.

These are analogues/generalizations of the classical Gauss hypergeometric functions, related to elliptic curves.

They find applications in:

- representation theory (connected to math. physics, and conjecturally to reps. of "elliptic quantum groups");
- four-dimensional sypersymmetric quantum field theories;
- exactly solvable models in statistical mechanics;
- ...

We have shown (with Dreyfus and Roques) that most of these special functions do not satisfy any algebraic differential equations with elliptic function coefficients.

Theta functions

Let $p \in \mathbb{C}^*$ such that $|p| < 1$, and denote $(z;p)_\infty = \prod_{j \geq 0} (1 - zp^j)$.

We define the *theta function*

$$\theta(z;p) = (z;p)_\infty (pz^{-1};p)_\infty \in \mathcal{M}er(\mathbb{C}^*).$$

Note that

$$\theta(z_0;p) = 0 \qquad \text{if and only if} \qquad z_0 \in p^{\mathbb{Z}} = \{p^n \mid n \in \mathbb{Z}\},$$

and we have the functional equation

$$\theta(pz;p) = \theta(z^{-1};p) = -z^{-1}\theta(z;p).$$

p-periodic functions and theta functions

We say that $f(z) \in \mathcal{M}er(\mathbb{C}^*)$ is p-periodic if $f(pz) = f(z)$.

The field of p-periodic functions is identified with the field of meromorphic functions $\mathcal{M}er(E)$ on the elliptic curve $E = \mathbb{C}^*/p^{\mathbb{Z}}$.

- If $\tau \in \mathbb{C}$ is such that $\mathrm{Im}(\tau) > 0$ and $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$, then

$$\mathbb{C} \to \mathbb{C}^* : w \mapsto \exp(2\pi i w)$$

  induces an isomorphism $\mathbb{C}/\Lambda \simeq \mathbb{C}^*/p^{\mathbb{Z}}$, where $p = \exp(2\pi i \tau)$.

If $a_1, \ldots, a_m, b_1, \ldots, b_m \in \mathbb{C}^*$ satisfy the *balancing condition*

$$\prod_{j=1}^{m} a_j = \prod_{j=1}^{m} b_j, \qquad \text{the function} \qquad c\frac{\prod_{j=1}^{m} \theta(a_j z; p)}{\prod_{j=1}^{m} \theta(b_j z; p)} \quad (c \in \mathbb{C})$$

is p-periodic. *Any* p-periodic function can be so expressed.

Elliptic gamma functions

Now letting $q \in \mathbb{C}^*$ such that $|q| < 1$ and $p^{\mathbb{Z}} \cap q^{\mathbb{Z}} = \{1\}$, we denote $(z; p, q)_\infty = \prod_{j,k \geq 0}(1 - zp^j q^k)$.

We define the *elliptic Gamma function*

$$\Gamma(z; p, q) = \frac{(pq/z; p, q)_\infty}{(z; p, q)_\infty}.$$

Note that

$$\Gamma(pz; p, q) = \theta(z; q)\Gamma(z; p, q)$$

and

$$\Gamma(qz; p, q) = \theta(z; p)\Gamma(z; p, q).$$

- Elliptic analogues of the classical Euler Gamma function $\Gamma(z)$ with $\Gamma(z + 1) = z\Gamma(z)$.
- Classical Gauss hypergeometric functions can be defined in terms of the Euler Gamma function (Barnes integral formula).

Elliptic hypergeometric functions

For $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_8)$ satisfying the *balancing condition*

$$\prod_{j=1}^{8} \varepsilon_j = p^2 q^2, \qquad (1)$$

the *elliptic hypergeometric function* $f_{\boldsymbol{\varepsilon}}(z) \in \mathcal{M}er(\mathbb{C}^*)$ is defined in terms of elliptic gamma functions (with a formula analogous to the Barnes integral formula in the classical setting).

Theorem (A.-Dreyfus-Roques)

If every multiplicative relation among $\varepsilon_1, \ldots, \varepsilon_8$, p, q is induced from (1), then $f_{\boldsymbol{\varepsilon}}(z)$ is differentially transcendental over $\mathcal{M}er(E)$.

Remark

Hypothesis: $\varepsilon_1, \ldots, \varepsilon_8, p, q$ are "as independent as possible".

$\sigma\delta$-fields of elliptic functions

As before, we let $p, q \in \mathbb{C}^*$ such that:

$$|p| < 1, \quad |q| < 1, \quad \text{and} \quad p^{\mathbb{Z}} \cap q^{\mathbb{Z}} = \{1\}.$$

The last condition means that $q \pmod{p^{\mathbb{Z}}}$ is of infinite order in the abelian group $E = \mathbb{C}^*/p^{\mathbb{Z}}$.

Base field: $K = \mathcal{M}er(E)$, the field of meromorphic functions on E.

Difference operator: The automorphism $\sigma : f(z) \mapsto f(qz)$.

Differential operator: The invariant derivation δ on E is $\delta = z\frac{d}{dz}$.

With this, K is a $\sigma\delta$-field: $\sigma \circ \delta = \delta \circ \sigma$.

Difference-differential Galois theory (Hardouin-Singer)

Let K be a $\sigma\delta$-field such that $C = K^\sigma$ is δ-closed, and consider a linear difference equation

$$a_n\sigma^n(y) + a_{n-1}\sigma^{n-1}(y) + \cdots + a_1\sigma(y) + a_0 y = 0, \qquad (2)$$

where $a_n, \ldots, a_0 \in K$ and $a_n a_0 \neq 0$.

To (2) is associated a $\sigma\delta$-PV extension R, generated as K-algebra by a C-basis of solutions $y_1, \ldots, y_n \in R$ together with[3] their iterates under σ and δ.

The $\sigma\delta$-Galois group is

$$\mathrm{Gal}_{\sigma\delta}(R/K) := \{\gamma \in \mathrm{Aut}_{K\text{-alg}}(R) \mid \gamma\circ\sigma = \sigma\circ\gamma,\ \gamma\circ\delta = \delta\circ\gamma\};$$

gets identified with a linear differential algebraic group in $\mathrm{GL}_n(C)$.

[3] And also $\det(\sigma^{i-1}(y_j))^{-1}$, where $1 \leq i, j \leq n$.

Linear differential algebraic groups

Definition

If C is a δ-field, we write $C^\delta := \{c \in C \mid \delta(c) = 0\}$.
A *linear differential algebraic group* is a subgroup of $\mathrm{GL}_n(C)$ defined by polynomial differential equations in the matrix entries.

Examples:

- algebraic groups over C;
- algebraic groups over C^δ;

Let $\mathcal{L} = \sum_{i=0}^{n} c_i \delta^i$ with $c_n, \ldots, c_0 \in C$.

- $\{\alpha \in \mathbb{G}_a(C) \mid \mathcal{L}(\alpha) = 0\}$;
- $\{\alpha \in \mathbb{G}_m(C) \mid \mathcal{L}(\frac{\delta(\alpha)}{\alpha}) = 0\}$.

Theorem (Cassidy)

Every δ-algebraic subgroup of $\mathbb{G}_a(C)$ or $\mathbb{G}_m(C)$ is as above.

Main Result

[Under mild conditions on the otherwise arbitrary $\sigma\delta$-field K.]

Theorem (A.-Dreyfus-Roques)

Let $f \neq 0$ be a solution of

$$\sigma^2(f) + a\sigma(f) + bf = 0,$$

where $a, b \in K$ and $b \neq 0$. Assume that:

- *There is no $u \in K$ such that $\sigma(u)u + au + b = 0$.*
- *There are no $c_0, \ldots, c_n \in C$ with $c_n \neq 0$ and $h \in K$, such that*

$$c_n \delta^n \left(\frac{\delta b}{b} \right) + \cdots + c_0 \frac{\delta b}{b} = \sigma(h) - h.$$

Then f is differentially transcendental over K.

Difference equation for elliptic hypergeometric functions

Theorem (Spiridonov)

The elliptic hypergeometric function $f_\varepsilon(z)$ satisfies

$$A(z)(\sigma(f_\varepsilon) - f_\varepsilon) + A(z^{-1})(\sigma^{-1}(f_\varepsilon) - f_\varepsilon) + \nu f_\varepsilon = 0, \qquad (3)$$

where

$$A(z) = \frac{1}{\theta(z^2;p)\theta(qz^2;p)} \prod_{j=1}^{8} \theta(\varepsilon_j z;p), \quad \nu = \prod_{j=1}^{6} \theta(\varepsilon_j \varepsilon_8/q;p).$$

- It follows from the *balancing condition* $\prod_{j=1}^{8} \varepsilon_j = p^2 q^2$ that the coefficients $\underline{A(z), A(z^{-1}) \in \mathcal{M}er(E) = K}$.
- Hence, (3) is equivalent to a second-order linear difference equation over K.

Proving differential transcendence of $f_\varepsilon(z)$

To prove differential transcendence of the elliptic hypergeometric function $f_\varepsilon(z)$, we verified the conditions of our Main Result assuming that $\varepsilon_1, \ldots, \varepsilon_8, p, q$ are "as independent as possible".

- Earlier work of Dreyfus-Roques provides criteria to decide existence of solutions $u \in \mathcal{M}er(E)$ to the Riccati equation

$$\sigma(u)u + au + b = 0,$$

  depending on the divisors of $a, b \in \mathcal{M}er(E)$.

- The non-existence of a telescoper $0 \neq \mathcal{L} \in C[\delta]$ and certificate $h \in \mathcal{M}er(E)$ such that

$$\mathcal{L}\left(\frac{\delta(b)}{b}\right) = \sigma(h) - h$$

  is also proved by analyzing the divisor of $b \in \mathcal{M}er(E)$.

Sketch of proof: Main Result (1/2)

One of the following three cases occurs for the $\sigma\delta$-Galois group G.

1. G is conjugate to a group of upper triangular matrices. This happens if and only if there exists a solution $u \in K$ to the Riccati equation $\sigma(u)u + au + b = 0$.
2. G is conjugate to a subgroup of
$$\left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \,\middle|\, \alpha, \beta \in C^\times \right\} \bigcup \left\{ \begin{pmatrix} 0 & \gamma \\ \mu & 0 \end{pmatrix} \,\middle|\, \gamma, \mu \in C^\times \right\}.$$
3. G contains $\mathrm{SL}_2(C)$.

No solutions to Riccati equation $\Rightarrow$ G is irreducible.

Sketch of proof: Main Result (2/2)

No telescoper $\Rightarrow \det(G) = \mathbb{G}_m(C) \Rightarrow G$ is either

- $\left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \;\middle|\; \alpha, \beta \in C^\times \right\} \bigcup \left\{ \begin{pmatrix} 0 & \gamma \\ \mu & 0 \end{pmatrix} \;\middle|\; \gamma, \mu \in C^\times \right\}$;
- $\mathrm{GL}_2(C)$.

In either case, G is sufficiently large to guarantee that any one solution $f \neq 0$ is differentially transcendental over K.