Integration in Finite Terms with Special Functions: Error Functions, Logarithmic Integrals and Polylogarithmic Integrals.

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Notations and terminologies

- Throughout, a field always means a field of characteristic zero.
- Differential fields: A field F with an additive map $' : F \to F$ that satisfies the Leibnitz rule, i.e $(fg)' = fg' + f'g$ for all $f, g \in F$.
- The kernel of the map $'$ is denoted by C_F, called the field of constants.
- Differential field extension: A differential field E is said to differential field extension of F if E is a field extension of F and the derivation map of E restricted to F coincides with the derivation map of F.

Introduction

Let E be a differential field extension of F having the same field of constants as F.

Problem

When an element $\alpha \in F$ admits an antiderivative in E?

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(ii) $\theta_i' = u'\theta_i$ for $u \in F_{i-1}$ (i.e. $\theta_i = e^u$, called an exponential of u).

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(vii) $\theta_i' = \ell_2(u)u'/u$, where $u, \ell_2(u) \in F_{i-1}$ (i.e. $\theta_i = \ell_3(u)$, called trilogarithmic integral of u).

Elementary Extensions

A differential field extension $E = F(\theta_1, \ldots, \theta_n)$ of F is called an elementary extension if each θ_i is either algebraic, exponential or logarithmic over F_{i-1}. Elements of an elementary extension field are called elementary functions

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- We obtain a simpler proof of Baddoura's theorem which neither assumes that F is a liouvillian extension of C_F nor that C_F is an algebraically closed field.
- Our results contain both necessary and sufficient conditions and therefore, these results will help in formulating algorithms for integration in finite terms with special functions.

Liouville's Theorem

Theorem (Rosenlicht, 1968)

Let $E \supset F$ be an elementary field extension of F with $C_E = C_F$. If there is an element $u \in E$ with $u' \in F$ then there are $\mathbb{Q}-$linearly independent constants $c_1, \ldots, c_n$ and elements $g_1, \ldots, g_n, w$ in F such that

$$u' = \sum_{i=1}^{n} c_i \frac{g_i'}{g_i} + w'.$$

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$\mathcal{DEL}$−Extensions and $\mathcal{DEL}$−Expressions

We say that $v \in F$ admits a **$\mathcal{DEL}$−expression** over F if there are some finite indexing sets I, J, K and elements $w, r_i, g_i, u_j, \log(u_j),\, v_k, e^{-v_k^2}$ in F and constants a_j, b_k for all i, j, k in I, J, K respectively such that

$$v = \sum_{i \in I} r_i \frac{g_i'}{g_i} + \sum_{j \in J} a_j \frac{u_j'}{\log(u_j)} + \sum_{k \in K} b_k v_k' e^{-v_k^2} + w',$$

where for each i, there is an integer n_i such that $r_i' = \sum_{l=1}^{n_i} c_{il} h_{il}' / h_{il}$ for some constants c_{il} and elements $h_{il} \in F$.

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(b) a $\mathcal{D}$−expression if it is special and for all j, k, $a_j = b_k = 0$.

A differential field extension E of F will be called a logarithmic extension of F if $C_E = C_F$ and there are elements $h_1, \ldots, h_m$ in F such that $E = F(\log h_1, \ldots, \log h_m)$.

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A transcendental $\mathcal{DEL}$−extension will be called transcendental dilogarithmic-elementary extension of F if for each i, θ_i is either an exponential or logarithm or dilogarithmic integral over F_{i-1}.

Integration in Finite Terms: $\mathcal{DEL}-$Extensions

Theorem (YK-VRS, J. Symb. Comp, 94 (2019) 210-233.)

Let $E \supset F$ be a transcendental $\mathcal{DEL}-$extension of F. Suppose that there is an element u in E with u' in F then u' admits a special $\mathcal{DEL}-$expression over some logarithmic extension of F.

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The theorem follows from the next lemma.

Integration in Finite Terms: $\mathcal{DEL}-$Extensions

Lemma (YK-VRS, J. Symb. Comp, 94 (2019) 210-233.)

Let $F(\theta) \supset F$ be a transcendental $\mathcal{DEL}-$extension of F. If $v \in F$ admits a special $\mathcal{DEL}-$expression over the differential field $F(\theta)(\log y_1, \dots, \log y_n)$, where each $y_i \in F(\theta)$, having the same field of constants as F then there is a differential field $M = F(\log h_1, \dots, \log h_m, \theta)$, where each $h_i \in F$, having the same field of constants as F such that each v admits a special $\mathcal{DEL}-$expression over M.

Example

Let $F = \mathbb{C}\left(z, \log(z+1), \log(z(z-1)(z^2+z-1))\right)$ and $E = F(\log z, \ell_2(1-z), \ell_2(1-z(z+1))$ be differential fields with the derivation $' := d/dx$. Let $v = -\log(z+1)\frac{(1-z(z+1))'}{1-z(z+1)} + \log\left(z(z-1)(z^2+z-1)\right)\frac{z'}{z} + w' \in F$, where $w \in F$ is arbitrary. Then we have the following:

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- E is a dilogarithmic-elementary extension of F.
- $u := \ell_2(1-z(z+1)) + \ell_2(1-z) + v_0 \in E$, $u' = v \in F$.
- Over the field $F(\log z)$, for element $v_0 = -(1/2)\log^2(z) + \log\left(z(z-1)(z^2+z-1)\right)\log(z) + w$ in $F(\log z)$, we can rewrite v as

$$u' = -\frac{(1-z(z+1))'}{1-z(z+1)}\log(z(z+1)) - \frac{(1-z)'}{1-z}\log z + v_0'$$

which is a $\mathcal{D}$-expression over $F(\log z)$.

Example

- u' cannot be written as a $\mathcal{D}-$expression over F.

Integration in Finite Terms: $\mathcal{DEL}-$Extensions

Theorem (YK, PhD Thesis, IISERM)

Let $E \supset F$ be a transcendental $\mathcal{DEL}-$extension of F. Then there is an element $u \in E$ with $u' \in F$, if and only if u' satisfies the following $\mathcal{DEL}-$expression over F :

$$u' = \sum_{i \in I} r_i \frac{g_i'}{g_i} + \sum_{l \in L} s_l \frac{h_l'}{h_l} + \sum_{j \in J} a_j \frac{u_j'}{\log(u_j)} + \sum_{k \in K} b_k v_k' e^{-v_k^2} + w',$$

where for each $i, p \in I$ and $l, t \in L$, there are constants a_{ip}, b_{lt}, c_{ip} and d_{il} with $c_{ii} \neq 0$ such that

$$r_i' = \sum_{p \in I} c_{ip} \frac{(1 - g_p)'}{1 - g_p} + \sum_{p \in I} a_{ip} \frac{g_p'}{g_p} + \sum_{l \in L} d_{il} \frac{h_l'}{h_l} \quad \text{and}$$

$$s_l' = \sum_{i \in I} d_{il} \frac{g_i'}{g_i} + \sum_{t \in L} b_{lt} \frac{h_t'}{h_t}.$$

Example (converse of the above theorem)

Let the differential field $F := \mathbb{C}(x, \ln(x), \ln(1-x) + 5\ln(1+x))$ with derivation $' := d/dx$. Note that

$$v := (\ln(1-x) + 5\ln(1+x))\,\frac{1}{x} + 5\ln(x)\frac{1}{1+x} + w',$$

where $w \in F$ is a $\mathcal{DEL}-$expression over F of the desired form. Over the differential field $F(\log(1+x))$, we shall rewrite v as

$$v := \ln(1-x)\frac{1}{x} + 5\left(\ln(1+x)\frac{1}{x} + \ln(x)\frac{1}{1+x}\right)' + w'.$$

Thus the dilogarithmic-elementary extension field $F(\ln(1+x), \ell_2(x))$ contains an antiderivative of v, namely,

$$-\ell_2(x) + 5\ln(1+x)\ln(x) + w.$$

Generalisation of Baddoura's Theorem

The element $D(g) := l_2(g) + (1/2)\log(g)\log(1-g)$ is called the Bloch-Wigner Spence function of g and its derivative is

$$D(g)' = -\frac{1}{2}\frac{g'}{g}\log(1-g) + \frac{1}{2}\frac{(1-g)'}{(1-g)}\log(g).$$

Our theorem concerning dilogarithmic integrals leads to a simpler version of Baddoura's theorem (2006):

Theorem (YK-VRS, J. Symb. Comp, 94 (2019) 210-233.)

Let $E \supset F$ be a transcendental dilogarithmic-elementary extension of F. Suppose that there is an element $u \in E$ with $u' \in F$, then

$$u = \sum_{j=1}^{m} c_j D(g_j) + \sum_{i=1}^{n} f_i \log(h_i) + w,$$

where each $f_i, h_i, g_j, w \in F$, c_j are constants and $\log(h_i)$ and $D(g_j)$ belong to some dilogarithmic-elementary extension of F.

Example

Let $F = \mathbb{C}\left(z, \log(z+1), \log(z(z-1)(z^2+z-1))\right)$ and $E = F(\log z, \ell_2(1-z), \ell_2(1-z(z+1)))$ be differential fields with the derivation $'$:= d/dx. Let $v = -\log(z+1)\frac{(1-z(z+1))'}{1-z(z+1)} + \log\left(z(z-1)(z^2+z-1)\right)\frac{z'}{z} + w' \in F$, where $w \in F$ is arbitrary.

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- For $u := \ell_2(1-z(z+1)) + \ell_2(1-z) + v_0 \in E$, $u' = v \in F$, the Bloch Wigner Spence function representation is given by

$$\begin{aligned} u &= D(1-z(z+1)) + D(1-z) \\ &+ \frac{1}{2}\log z \log\left(z(z-1)(z^2+z-1)\right) + \frac{1}{2}c\log z \\ &- \frac{1}{2}\log(z+1)\log(1-z(z+1)) + v_0. \end{aligned}$$

Trilogarithmic Integrals

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Integration with Trilogarithmic Integrals

We say that $v \in F$ admits a $\mathcal{TEL}-$expression over F if there are finite indexing sets I, J, K, L and elements $r_i, g_i, s_l, h_l, u_j, \log(u_j), v_k, e^{-v_k^2}, w \in F$ and constants a_j, b_k for all i, j, k in I, J, K respectively such that

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(a) a special $\mathcal{TEL}-$expression if for each i, $r_i' = c_i \log(1 - g_i) g_i'/g_i$ for some constant c_i and for each l, $s_l' = d_l (1 - h_l)'/(1 - h_l)$ for some constant d_l.

(b) a $\mathcal{T}-$expression if it is special and for all j, k, $a_j = b_k = 0$.

We call a differential field extension E of F to be a dilogarithmic extension of F if they have same field of constants and there are elements $y_1, \ldots, y_n, z_1, \ldots, z_m \in F$ such that $E = F(\log(y_1), \ldots, \log(y_n), \ell_2(z_1), \ldots, \ell_2(z_m))$.

Integration with Trilogarithmic Integrals

Lemma

Let $F(\theta) \supset F$ *be a transcendental trilogarithmic-elementary extension of* F. *Suppose there is an element* $v \in F$ *such that* v *admits a special* $\mathcal{TEL}-$*expression over a dilogarithmic extension* $E = F(\theta)(\log(y_1), \ldots, \log(y_n), \ell_2(z_1), \ldots, \ell_2(z_m))$ *where each* $y_i, z_i \in F(\theta)$. *Then there exists a field* $M = F(\log(p_1), \ldots, \log(p_l), \ \ell_2(q_1), \ldots, \ell_2(q_t), \theta)$, *where each* $p_i, q_i \in F$, *having the same field of constants as* F *such that* v *admits a special* $\mathcal{TEL}-$*expression over* M.

Integration with Trilogarithmic Integrals

Proposition

Let $F(\theta) \supset F$ be a transcendental field extension with $C_{F(\theta)} = C_F$. Let f be any element in $F(\theta)$ and $\{\alpha_j; j = 1, \ldots, t\}$ be the set of all zeroes and poles of f and $1 - f$ in an algebraic closure of F. Also, for some integers a_j, b_j, let $f = \eta \prod_{j=1}^{t} (\theta - \alpha_j)^{a_j}$ and $1 - f = \xi \prod_{j=1}^{t} (\theta - \alpha_j)^{b_j}$ then

$$\ell_2(f) = \ell_2(\eta) - \sum_{\substack{j,k=1 \\ k \neq j}}^{t} a_j b_k \ell_2\left(\frac{\theta - \alpha_j}{\theta - \alpha_k}\right) - \frac{1}{2}\sum_{j,k=1}^{t} a_j b_k \log^2(\theta - \alpha_k)$$

$$- \sum_{k=1}^{t} a_k \log(\theta - \alpha_k) \log \xi - \sum_{\substack{j,k=1 \\ k \neq j}}^{t} a_j b_k \log\left(\frac{\theta - \alpha_j}{\theta - \alpha_k}\right) \log(\alpha_j - \alpha_k).$$

Integration with Trilogarithmic Integrals

Proposition

Let $F(\theta) \supset F$ be a transcendental field extension with $C_{F(\theta)} = C_F$. Let f be any element in $F(\theta)$ and $\{\alpha_j; j = 1, \dots, t\}$ be the set of all zeroes and poles of f and $1 - f$ in an algebraic closure of F. Also, for some integers a_j, b_j, let $f = \eta \prod_{j=1}^{t} (\theta - \alpha_j)^{a_j}$ and $1 - f = \xi \prod_{j=1}^{t} (\theta - \alpha_j)^{b_j}$ then

$$\ell_2(f) = \ell_2(\eta) - \sum_{\substack{j,k=1 \\ k \neq j}}^{t} a_j b_k \ell_2\left(\frac{\theta - \alpha_j}{\theta - \alpha_k}\right) - \frac{1}{2} \sum_{j,k=1}^{t} a_j b_k \log^2(\theta - \alpha_k)$$

$$- \sum_{k=1}^{t} a_k \log(\theta - \alpha_k) \log \xi - \sum_{\substack{j,k=1 \\ k \neq j}}^{t} a_j b_k \log\left(\frac{\theta - \alpha_j}{\theta - \alpha_k}\right) \log(\alpha_j - \alpha_k).$$

J. Baddoura (2006) proved a similar identity for Bloch-Wigner Spence function $D(g)$.

Integration with Trilogarithmic Integrals

Theorem

Let E be a transcendental $\mathcal{TEL}-$extension of F. Then an element u in E have u' in F if and only if there are finite indexing sets I, J, K, L and $w, g_i, h_l, r_i, s_l, u_j, \log(u_j), v_k, e^{-v_k^2}$ in F such that

$$u' = \sum_{i\in I} r_i \frac{g_i'}{g_i} + \sum_{l\in L} s_l \frac{h_l'}{h_l} + \sum_{j\in J} a_j \frac{u_j'}{\log(u_j)} + \sum_{k\in K} b_k v_k' e^{-v_k^2} + w',$$

$$r_i' = -c_i t_i \frac{g_i'}{g_i} - \sum_{l\in L} r_{il} \frac{h_l'}{h_l}, \quad s_l' = -\sum_{i\in I} r_{il} \frac{g_i'}{g_i} - \sum_{p\in L} s_{lp} \frac{h_p'}{h_p},$$

$$t_i' = \frac{(1-g_i)'}{1-g_i} - \sum_{l\in L} c_{il} \frac{h_l'}{h_l}, \quad r_{il}' = -c_i c_{il} \frac{g_i'}{g_i} - \sum_{p\in L} e_{ilp} \frac{h_p'}{h_p}$$

Theorem (Cont.)

$$\text{and} \quad s'_{lp} = -\sum_{i \in I} e_{ilp} \frac{g'_i}{g_i} - \sum_{q \in L} f_{lpq} \frac{h'_q}{h_q},$$

where each c_i is a non-zero constant, each c_{il}, e_{ilp}, f_{lpq} are some constants and each t_i, r_{il} and s_{lp} are elements in some dilogarithmic extension of F with $e_{ilp} = e_{ipl}$ and $s_{lp} = s_{pl}$ for every l and p.

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Thank You!