Asymptotic valued differential fields

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For the purposes of this talk, all fields in sight have characteristic 0.

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- satisfy "L'Hôpital's Rule at ∞":

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} \;=\; \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

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  - example series:

$$7e^{e^{x}+e^{x/2}+e^{x/4}+\cdots} - 3e^{x^2} + 5x^{\sqrt{2}} - (\log x)^{\pi} + 42 + x^{-1} + x^{-2} + \cdots + e^{-x}$$

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- satisfies valuation analogue of "L'Hôpital's Rule at ∞"
- introduced by Écalle in proving Dulac's conjecture and Dahn–Göring in studying models of the reals with exponentiation
- studied also by Aschenbrenner, van den Dries, and van der Hoeven:
  - axiomatization
  - model completeness in ordered valued differential field language
  - quantifier elimination in language expanded by three extra predicates

Model theory of valued fields: Ax–Kochen/Ershov

Theorem (Ax–Kochen, Ershov)

Let K_1 and K_2 be henselian valued fields. Then

$$K_1 \equiv K_2 \quad \iff \quad \boldsymbol{k}_1 \equiv \boldsymbol{k}_2 \text{ and } \Gamma_1 \equiv \Gamma_2.$$

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Tools:

1. maximal immediate extensions of K are isomorphic over K
2. K is henselian $\iff$ it is algebraically maximal
3. K has a henselization

Valued fields

A *valued field* is a field K with a surjective map $v\colon K \to \Gamma \cup \{\infty\}$, where Γ is an ordered abelian group and $\Gamma < \infty$, satisfying:

1. $v(x) = \infty \iff x = 0$;
2. $v(xy) = v(x) + v(y)$;
3. $v(x + y) \geqslant \min\{v(x), v(y)\}$.

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Notation:

- write $f \preccurlyeq g$ if $vf \geqslant vg$ and $f \prec g$ if $vf > vg$
- $\mathcal{O} := \{f : f \preccurlyeq 1\}$ is the *valuation ring*
- $\mathcal{o} := \{f : f \prec 1\}$ is the (unique) maximal ideal of $\mathcal{O}$
- $\boldsymbol{k} := \mathcal{O}/\mathcal{o}$ is the *residue field*

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Notation:

- $f' := \partial(f)$
- $C := \{f : f' = 0\}$ is the *constant field* of K
- $K\{Y\} := K[Y, Y', Y'', \ldots]$ is the *differential polynomial ring* over K

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- K is *maximal* if it has no proper immediate extensions
- K is d-*algebraically maximal* if it has no proper d-algebraic immediate extensions

Differential-henselianity

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- $\boldsymbol{k}$ is *linearly surjective* if every $1 + a_0 Y + a_1 Y' + \cdots + a_r Y^{(r)}$, $a_i \in \boldsymbol{k}$, $a_r \neq 0$, has a zero in $\boldsymbol{k}$
- Note: K is d-henselian $\Longrightarrow$ $\boldsymbol{k}$ is linearly surjective

Uniqueness of maximal immediate extensions

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If $\boldsymbol{k}$ is linearly surjective, then any two maximal immediate extensions of K are isomorphic over K.

This has been proven for monotone K by Aschenbrenner, van den Dries, and van der Hoeven, and for K whose value group has finite archimedean rank by van den Dries and PC.

Uniqueness of maximal immediate extensions for asymptotic fields

K is *asymptotic* if for all nonzero $f, g \prec 1$,

$$\frac{f}{g} \prec 1 \quad \Longleftrightarrow \quad \frac{f'}{g'} \prec 1.$$

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Theorem (PC)

Suppose K is asymptotic and $\boldsymbol{k}$ is linearly surjective. Then any two maximal immediate extensions are isomorphic over K.

Differential-henselianity and differential-algebraic maximality

Theorem (Aschenbrenner–van den Dries–van der Hoeven)

If $\boldsymbol{k}$ is linearly surjective and K is d*-algebraically maximal, then K is* d*-henselian.*

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This was first proven in the monotone case by Aschenbrenner, van den Dries, and van der Hoeven.

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L is a d-*henselization* of K if:

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Summary

Theorem (PC)

Suppose K is asymptotic and $\boldsymbol{k}$ is linearly surjective. Then:

1. *any two maximal immediate extensions of K are isomorphic over K;*
2. *if K is* d*-henselian, then it is* d*-algebraically maximal;*
3. *K has a* d*-henselization.*

Proof sketch of (2)

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Key properties of d-henselian asymptotic fields:

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- Each $P \in \mathcal{O}[Y, Y', \ldots, Y^{(r)}]$ does not have $r+2$ distinct zeroes in a certain configuration.
- If $P \in \mathcal{O}\{Y\}$ with $\deg \overline{P} = 1$, and E is an immediate extension of K, then P has the same zeroes in $\mathcal{O}_E$ as in $\mathcal{O}$.

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Proof sketch of theorem:

1. take f in an immediate extension of K, so f is the pseudolimit of a pseudocauchy sequence (f_ρ) over K;

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3. use pseudocauchy sequence to find infinitely many zeroes of P in configuration as above, contradicting the key property.

Main step

Step (3) is difficult:

Proposition

Suppose K is asymptotic and henselian, and $\boldsymbol{k}$ is linearly surjective. Let (f_ρ) be a pseudocauchy sequence in K and P is minimal with $P(f_\rho) \rightsquigarrow 0$.

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- proof is technical
- involves developing a differential newton diagram method
- problem: $v(f)$ does not really control $v(f')$

Thank you!