Extension of Gröbner-Shirshov basis of an algebra to its generating free differential algebra

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Joint work with Li Guo

May 10, 2019, Kolchin Seminar at GC/CUNY

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Introduction

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- Differential algebras are usually defined for commutative algebras and fields. Free differential algebras mean polynomial algebras over differential variables.
- In recent years, this traditional framework has been extended in two directions. One removes the commutativity condition to include large classes of naturally arisen algebras, e.g. differential Lie algebras or path algebras. Secondly, the Leibniz rule is generalized to include the difference $\frac{f(x+\lambda)-f(x)}{\lambda}$, leading to the notion of differential algebra of weight λ.

Introduction

- Further, as the left adjoint functor of the forgetful functor from the category of differential algebras to that of algebras, the free differential algebra generated by an algebra can be defined (in Poinsot's work). One expects that properties similar to the usual free differential algebras (on a set) still hold.

- Further, as the left adjoint functor of the forgetful functor from the category of differential algebras to that of algebras, the free differential algebra generated by an algebra can be defined (in Poinsot's work). One expects that properties similar to the usual free differential algebras (on a set) still hold.
- However, at this time, such free objects are only expressed as quotients modulo large differential ideals via universal algebra consideration. It is desirable to give their explicit construction, providing a canonical basis of it. We adapt the method of Gröbner-Shirshov bases.

Introduction

- In the 1960s, Buchberger provided an algorithmic approach for the Gröbner bases for commutative algebras. As their differential counterpart, characteristic sets are commonly used. Further, the theory of differential Gröbner bases was developed by Carrà Ferro, Mansfield, etc.

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- To work in broader contexts, we work in the framework of Gröbner-Shirshov (GS) bases. In fact there have been several studies via GS bases on structures related to differential algebras, for operated (Rota-Baxter) differential algebras, integro-differential algebras and differential type algebras, etc.

Introduction

- It is still challenging to establish differential Gröbner bases for any differential ideals in the classical setting (commutative, weight zero). Our goal is to pursue this direction in the non-classical setting and gain further understanding in the classical setting, with our current work as a special case.

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- It is still challenging to establish differential Gröbner bases for any differential ideals in the classical setting (commutative, weight zero). Our goal is to pursue this direction in the non-classical setting and gain further understanding in the classical setting, with our current work as a special case.
- In this talk, we start with the notion of Gröbner-Shirshov bases in the free differential algebras over sets, specializing those given by Chen and Qiu in the free objects additionally with multiple operators, then show that a GS basis of an algebra can "differentially" extend to one for its free differential algebra, except for the "classical" 0-differential commutative case.

Differential algebras

Fix a base field $\mathbf{k}$ of characteristic 0 throughout.

Definition

Let $\lambda \in \mathbf{k}$. A differential $\mathbf{k}$-algebra of weight λ (simply a differential algebra) is an associative $\mathbf{k}$-algebra R with a differential operator (or derivation) $d : R \to R$ of weight λ such that

$$d(xy) = d(x)y + xd(y) + \lambda d(x)d(y), \forall\, x, y \in R.$$

A differential algebra means a (noncommutative) λ-differential algebra. If R is unital, it further requires that $d(1_R) = 0$. A homomorphism of differential algebras is an algebra map commutating with derivations.

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A differential ideal I of (R, d) is an algebraic ideal of R s.t. $d(I) \subseteq I$. For any $U \subseteq R$,

$$\mathrm{DI}(U) := \left(d^n(u) \,\middle|\, u \in U, n \geq 0\right)$$

is the differential ideal of R generated by U.

Free differential algebras over sets

Definition

The free differential algebra generated by a set X is a differential algebra $\mathcal{D}_\lambda\langle X\rangle$ with λ-derivation d_X and a map $i_X : X \to \mathcal{D}_\lambda\langle X\rangle$ of sets satisfying the universal property as follows,

- *if (R, d_R) is a differential algebra with a map $f : X \to R$ of sets, then there exists a unique differential algebra map $\bar{f} : \mathcal{D}_\lambda\langle X\rangle \to R$ s.t. $f = \bar{f} \circ i_X$.*

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Also, the free differential commutative algebra generated by a set X is a differential commutative algebra $\mathcal{D}_\lambda[X]$ with λ-derivation d_X and a map $i_X : X \to \mathcal{D}_\lambda[X]$ of sets satisfying the universal property similar as above, but with (R, d_R) being commutative.

Construction of free differential algebras over sets

Theorem

Given a set X, let $\Delta(X) := X \times \mathbb{N} = \{x^{(n)} \mid x \in X, n \geq 0\}$.

(1) *The free differential algebra $(\mathcal{D}_\lambda\langle X\rangle, d_X)$ on X is given by the free (noncommutative) algebra $\mathbf{k}\langle\Delta(X)\rangle$ generated by $\Delta(X)$, equipped with the λ-derivation d_X defined by $d_X(x^{(n)}) = x^{(n+1)}$, $n \geq 0$.*

(2) *The free differential commutative algebra $(\mathcal{D}_\lambda[X], d_X)$ on X is given by the free commutative algebra $\mathbf{k}[\Delta(X)]$ generated by $\Delta(X)$, equipped with the same λ-derivation d_X.*

Free differential algebras over algebras

Definition

The free differential algebra on an algebra A is a differential algebra $\mathcal{D}_\lambda(A)$ with derivation d and algebra map $i_A : A \to \mathcal{D}_\lambda(A)$ satisfying the universal property as follows,

- *if (R, d_R) is a differential algebra with algebra map $\varphi : A \to R$, then there exists a unique differential algebra map $\bar{\varphi} : \mathcal{D}_\lambda(A) \to R$ s.t. $\varphi = \bar{\varphi} \circ i_A$.*

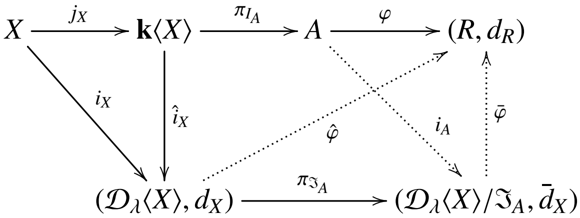
When A is commutative, its free differential commutative algebra $C\mathcal{D}_\lambda(A)$ is a differential commutative algebra with the same universal property as above, but in the category of differential commutative algebras.

Proposition

For any algebra $A = \mathbf{k}\langle X\rangle/I_A$ with I_A an ideal of the free $\mathbf{k}$-algebra $\mathbf{k}\langle X\rangle$ on X, $\mathcal{D}_\lambda(A)$ can be presented as $\mathcal{D}_\lambda\langle X\rangle/\mathfrak{I}_A$, where $\mathfrak{I}_A$ is the differential ideal of $\mathcal{D}_\lambda\langle X\rangle$ generated by I_A.

Similarly, for commutative algebra $A = \mathbf{k}[X]/I_A$, $C\mathcal{D}_\lambda(A) \cong \mathcal{D}_\lambda[X]/\mathfrak{I}_A$, where $\mathfrak{I}_A$ is the differential ideal of $\mathcal{D}_\lambda[X]$ generated by I_A.

It can be verified under the following commutative diagram,



Next we briefly recall the notion of Gröbner-Shirshov bases for associative algebras, then move to the version for free differential algebras.

Gröbner-Shirshov bases for associative algebras

Denote by $M(X)$ (resp. $S(X)$) the free monoid (resp. semigroup) generated by X. Any well order $<$ on X can induce a **monomial order** $<$ on $M(X)$, e.g. the deg-lex order, s.t.

$$1 < u \text{ and } u < v \;\Rightarrow\; wuz < wvz \text{ for any } u, v, w, z \in S(X).$$

Denote by $[X]$ (resp. $S[X]$) the free commutative monoid (resp. semigroup) on X. Any well order $<$ on X can also induce a monomial order $<$ on $[X]$ s.t.

$$1 < u \text{ and } u < v \;\Rightarrow\; uw < vw \text{ for any } u, v, w \in S[X].$$

Let $\bar{f} \in M(X)$ (resp. $[X]$) be the lead term of any $f \in \mathbf{k}\langle X\rangle$ (resp. $\mathbf{k}[X]$) with respect to this order $<$. Recall that a Gröbner-Shirshov basis in $\mathbf{k}\langle X\rangle$ (or $\mathbf{k}[X]$) is a subset of monic polynomials whose intersection and including compositions are trivial modulo it.

CD lemma for associative algebras

Lemma (Composition-Diamond lemma for (commutative) algebras)

Let $\mathrm{I}(S)$ *be the ideal of* $\mathbf{k}\langle X\rangle$ *generated by a monic subset* S*, and* $<$ *a monomial order on* $M(X)$*. TFAE:*

(1) *the set* S *is a Gröbner-Shirshov basis in* $\mathbf{k}\langle X\rangle$*;*

(2) *if* $f \in \mathrm{I}(S)\backslash\{0\}$*, then* $\bar{f} = u\bar{s}v$ *for some* $u, v \in M(X)$ *and* $s \in S$*;*

(3) *the set of* ***S-irreducible words*** $\mathrm{Irr}(S) := M(X)\backslash\{u\bar{s}v \mid u, v \in M(X),\ s \in S\}$ *is a* $\mathbf{k}$*-basis of* $\mathbf{k}\langle X \mid S\rangle = \mathbf{k}\langle X\rangle/\mathrm{I}(S)$*. In particular,* $\mathbf{k}\langle X\rangle = \mathbf{k}\mathrm{Irr}(S) \oplus \mathrm{I}(S)$*.*

Let $\mathrm{I}(S)$ *be the ideal of* $\mathbf{k}[X]$ *generated by a subset* S *of monic polynomials, and* $<$ *a monomial order on* $[X]$*. TFAE:*

(1) *the set* S *is a Gröbner(-Shirshov) basis in* $\mathbf{k}[X]$*;*

(2) *if* $f \in \mathrm{I}(S)\backslash\{0\}$*, then* $\bar{f} = \bar{s}u$ *for some* $u \in [X]$ *and* $s \in S$*;*

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GS bases in free differential algebras over sets

There exists a unique order $\prec$ on $\Delta(X)$ extending the given order $<$ on X and determined by

$$(1)\ m < n \Longrightarrow x^{(m)} \prec y^{(n)}, \quad (2)\ x < y \Longrightarrow x^{(n)} \prec y^{(n)},$$

for $x, y \in X$ and $m, n \geq 0$. It induces a **deg-lex order** $\prec$ on $M(\Delta(X))$ with the **degree** $|u|$ of $u \in M(\Delta(X))$ defined as the number of letters of u in $\Delta(X)$.

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Let $\Delta(X)^{\star} := \Delta(X) \sqcup \{\star\}$,

$$S(\Delta(X))^{\star} := \left\{u \star v \in S(\Delta(X)^{\star}) \,\middle|\, u, v \in M(\Delta(X))\right\}$$

and the **u-word** $q|_u := q|_{\star \mapsto u} \in \mathbf{k}\langle\Delta(X)\rangle$ for $q \in S(\Delta(X))^{\star}$ and $u \in \mathbf{k}\langle\Delta(X)\rangle$.

Let $\overline{f}$ be the leading term of $f \in \mathbf{k}\langle\Delta(X)\rangle$ w.r.t. $\prec$, $\mathrm{lc}(f)$ be the leading coefficient of f, and $f^{\natural} := \mathrm{lc}(f)^{-1} f$ when $f \neq 0$.

GS bases in free differential algebras over sets

For monic polynomials f, g in $\mathbf{k}\langle\Delta(X)\rangle$, if $\exists\, u, v, w_{i,j} \in M(\Delta(X)),\ i, j \geq 0$, s.t. $w_{i,j} = \overline{d_X^i(f)}u = v\overline{d_X^j(g)}$, $|w_{i,j}| < |\bar{f}| + |\bar{g}|$, then $(f, g)_{w_{i,j}}^{u,v} := d_X^i(f)^\natural u - v d_X^j(g)^\natural$ is called an **intersection composition** of f, g w.r.t. $w_{i,j}$.

If $w_{i,j} = \overline{d_X^i(f)} = q|_{\overline{d_X^j(g)}}$, $q \in S(\Delta(X))^\star$, $i, j \geq 0$, $(f, g)_{w_{i,j}}^{q} := d_X^i(f)^\natural - q|_{d_X^j(g)^\natural}$ is called an **including composition** of f, g w.r.t. $w_{i,j}$.

GS bases in free differential algebras over sets

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Definition

Let S be a set of monic polynomials in $\mathbf{k}\langle\Delta(X)\rangle$ and $w \in M(\Delta(X))$.

(1) *For $u, v \in \mathbf{k}\langle\Delta(X)\rangle$, we call u and v* **congruent modulo** *(S, w), denoted by $u \equiv v \mod (S, w)$, if $u - v = \sum_i c_i q_i|_{d_X^{k_i}(s_i)}$ with $c_i \in \mathbf{k}$, $q_i \in S(\Delta(X))^\star$ and $s_i \in S$, s.t. $q_i|_{\overline{d_X^{k_i}(s_i)}} < w$ for any i.*

(2) *The set $S \subseteq \mathbf{k}\langle\Delta(X)\rangle$ is called a* **(differential) Gröbner-Shirshov basis**, *if for any $f, g \in S$, all compositions $(f, g)_{w_{i,j}} \equiv 0 \mod (S, w_{i,j})$.*

The CD lemma for differential algebras

Lemma

Let $u_1, \ldots, u_r \in \Delta(X)$, $i \geq 0$, *then* $d_X^i(u_1 \cdots u_r)$ *has the leading term* $d_X^i(u_1) \cdots d_X^i(u_r)$ *with coefficient* $\lambda^{(r-1)i}$ *if weight* $\lambda \neq 0$, *otherwise* $d_X^i(u_1)u_2 \cdots u_r$ *with coefficient 1 if weight* $\lambda = 0$.

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Lemma (Composition-Diamond lemma for differential algebras)

Suppose $S \subset \mathbf{k}\langle\Delta(X)\rangle$ *monic and* $<$ *a monomial order on* $M(\Delta(X))$. *TFAE:*

(1) S *is a Gröbner-Shirshov basis in* $\mathbf{k}\langle\Delta(X)\rangle$.

(2) $f \in \mathrm{DI}(S)\backslash\{0\} \Rightarrow \bar{f} = q|_{\overline{d_X^i(s)}}$ *for some* $q \in S(\Delta(X))^\star$, $s \in S$ *and* $i \geq 0$.

(3) *the set of* **differential S-irreducible words**

$$\mathrm{DIrr}(S) := \left\{ u \in M(\Delta(X)) \,\middle|\, u \neq q|_{\overline{d_X^i(s)}},\ q \in S(\Delta(X))^\star,\ s \in S \right\}$$

is a $\mathbf{k}$*-basis of* $\mathbf{k}\langle\Delta(X)|S\rangle := \mathbf{k}\langle\Delta(X)\rangle/\mathrm{DI}(S)$.

The CD lemma for differential commutative algebras

Suppose that X is well-ordered. Let $[\Delta(X)]$ (resp. $S[\Delta(X)]$) be the free commutative monoid (resp. semigroup) on $\Delta(X)$ with the following monomial order $\prec$ satisfying

$$u \prec v \;\Rightarrow\; uw \prec vw, \text{ for any } u, v, w \in [\Delta(X)].$$

First order elements in $\Delta(X)$ as before, then for any $\mathfrak{u} := u_1 \cdots u_p$ and $\mathfrak{v} := v_1 \cdots v_q \in [\Delta(X)]$ s.t. $u_1 \succeq \cdots \succeq u_p$, $v_1 \succeq \cdots \succeq v_q$, set

$$\mathfrak{u} \prec \mathfrak{v} \;\Leftrightarrow\; p < q \text{ or } p = q,\, u_1 = v_1, \cdots, u_{k-1} = v_{k-1}, \text{ but } u_k \prec v_k \text{ for some } k.$$

The CD lemma for differential commutative algebras

Definition

Let $\bar{f}$ *denote the* **leading** *monomial word of* $f \in \mathbf{k}[\Delta(X)]$ *with respect to* $\prec$. *Let* lc(f) *be the leading coefficient of* f, *and denote* $f^\natural :=$ lc$(f)^{-1}f$ *when* $f \neq 0$, *so that* $f^\natural$ *is a monic polynomial in* $\mathbf{k}[\Delta(X)]$.

The CD lemma for differential commutative algebras

Definition

Let $\bar{f}$ denote the **leading** *monomial word of $f \in \mathbf{k}[\Delta(X)]$ with respect to $<$. Let lc(f) be the leading coefficient of f, and denote $f^\natural :=$ lc$(f)^{-1}f$ when $f \neq 0$, so that $f^\natural$ is a monic polynomial in $\mathbf{k}[\Delta(X)]$.*

Lemma

Let $u_1, \ldots, u_r \in \Delta(X)$ satisfying $u_1 \geq \cdots \geq u_r$ and $i \geq 0$.

(1) *If weight $\lambda \neq 0$, then the leading term of $d_X^i(u_1 \cdots u_r)$ is $\overline{d_X^i(u_1 \cdots u_r)} = d_X^i(u_1) \cdots d_X^i(u_r)$ with coefficient $\lambda^{(r-1)i}$.*

(2) *If weight $\lambda = 0$, then the leading term of $d_X^i(u_1 \cdots u_r)$ is $\overline{d_X^i(u_1 \cdots u_r)} = d_X^i(u_1)u_2 \cdots u_r$ with coefficient being ${}^{\#}u_1$ in $u_1 \cdots u_r$.*

The CD lemma for differential commutative algebras

As the commutative case of $\mathbf{k}[X]$, any including compositions appear as a special pattern of intersection compositions in $\mathbf{k}[\Delta(X)]$.

Definition

For monic polynomials f, g in $\mathbf{k}[\Delta(X)]$, if $\exists\, u, v, w_{i,j} \in [\Delta(X)]$ with $i, j \geq 0$ s.t. $w_{i,j} = \overline{d_X^i(f)}u = \overline{d_X^j(g)}v$, $|w_{i,j}| < |\bar{f}| + |\bar{g}|$, then

$$[f, g]_{w_{i,j}}^{u,v} := d_X^i(f)^\natural u - d_X^j(g)^\natural v$$

called an **intersection composition** *of f, g w.r.t. $w_{i,j}$, shorten as $[f, g]_{w_{i,j}}$.*

Let S be a set of monic polynomials in $\mathbf{k}[\Delta(X)]$ and $w \in [\Delta(X)]$.

(1) *For $u, v \in \mathbf{k}[\Delta(X)]$, we call u and v* **congruent modulo** *(S, w), denoted by $u \equiv v \mod (S, w)$, if $u - v = \sum_i c_i d_X^{k_i}(s_i)u_i$ with $c_i \in \mathbf{k}$, $s_i \in S$, $u_i \in [\Delta(X)]$ and $k_i \geq 0$ s.t. $\overline{d_X^{k_i}(s_i)}u_i < w$ for any i.*

(2) *The set S is called a* **(differential) Gröbner-Shirshov basis**, *if for any $f, g \in S$, all intersection compositions $[f, g]_{w_{i,j}}$ are trivial modulo $(S, w_{i,j})$.*

The CD lemma for differential commutative algebras

Lemma (CD lemma for differential commutative algebras)

Let S be a monic subset of $\mathbf{k}[\Delta(X)]$, $\mathrm{DI}(S)$ be the differential ideal of $\mathbf{k}[\Delta(X)]$ generated by S and $<$ be a monomial order on $[\Delta(X)]$. TFAE:

(1) *the set S is a Gröbner-Shirshov basis in $\mathbf{k}[\Delta(X)]$;*

(2) *if $f \in \mathrm{DI}(S)\setminus\{0\}$, then $\bar{f} = \overline{d_X^i(s)}u$ for some $s \in S$, $u \in [\Delta(X)]$ and $i \geq 0$;*

(3) *the set of* **differential S-irreducible words**

$$\mathrm{DIrr}(S) := [\Delta(X)]\setminus\left\{\overline{d_X^i(s)}u \,\middle|\, s \in S,\ u \in [\Delta(X)],\ i \geq 0\right\}$$

is a $\mathbf{k}$-basis of $\mathbf{k}[\Delta(X) \,|\, S] := \mathbf{k}[\Delta(X)]/\mathrm{DI}(S)$.

Main theorem on extension of GS bases for DAs

Recall the following natural algebra embedding

$$\hat{i}_X : \mathbf{k}\langle X\rangle \to \mathbf{k}\langle \Delta(X)\rangle,\ x \mapsto x^{(0)}, x \in X.$$

Here further defines algebra embeddings

$$\hat{i}_X^{(n)} : \mathbf{k}\langle X\rangle \to \mathbf{k}\langle \Delta(X)\rangle,\ x \mapsto x^{(n)}, x \in X$$

for all $n \geq 0$. In particular, $\hat{i}_X^{(0)} = \hat{i}_X$.

Theorem

For an algebra $A = \mathbf{k}\langle X\rangle / I_A$ with its defining ideal I_A generated by a Gröbner-Shirshov basis S in $\mathbf{k}\langle X\rangle$,

$$\hat{S} := \hat{i}_X(S) = \left\{ s^{(0)} \,\middle|\, s \in S \right\}$$

becomes a Gröbner-Shirshov basis in $\mathbf{k}\langle \Delta(X)\rangle$ for arbitrary weight λ.

Sketch of proof

Comparing leading terms of polynomials in $\mathbf{k}\langle\Delta(X)\rangle$, we find that

(a) when $\lambda \neq 0$, the ambiguities of possible compositions in $\hat{S}$ are:

(i) $w_{n,n} = \overline{d_X^n(s)}u = v\overline{d_X^n(t)}$ for some $s, t \in \hat{S}$, $u, v \in M(\Delta(X))$ and $n \geq 0$ s.t. $|\bar{s}| + |\bar{t}| > |w_{n,n}|$.

(ii) $w_{n,n} = \overline{d_X^n(s)} = q|_{\overline{d_X^n(t)}}$ for some $s, t \in \hat{S}$, $q \in S(\Delta(X))^\star$ and $n \geq 0$.

(b) when $\lambda = 0$, the ambiguities of possible compositions come from:

(i) $w_{n,n} = \overline{d_X^n(s)}u = d_X^n(v)v'\bar{t}$ for some $s, t \in \hat{S}$, $u \in \hat{i}_X(M(X))$, $v \in \hat{i}_X(X)$ and $v' \in M(\Delta(X))$ with $n \geq 0$.

(ii) $w_{n,n} = \overline{d_X^n(s)} = \overline{d_X^n(t)}u$ for some $s, t \in \hat{S}$, $u \in \hat{i}_X(M(X))$ and $n \geq 0$.

All the compositions involved are checked to be trivial.

Linear bases of the free DAs on algebras

For $n \geq 0$ and $\mathfrak{x} = x_1 \cdots x_k \in S(X)$ with $x_1, \cdots, x_k \in X$, denote

$$\mathfrak{x}^{[n]} := \begin{cases} x_1^{(n)} \cdots x_k^{(n)}, & \text{if } \lambda \neq 0, \\ x_1^{(n)} x_2^{(0)} \cdots x_k^{(0)}, & \text{if } \lambda = 0. \end{cases}$$

For a GS basis S of I_A, the embedding $\hat{S} = \hat{i}_X(S) = \left\{ s^{(0)} \,\middle|\, s \in S \right\}$ is a (differential) GS basis of $\mathrm{DI}(S)$. Then the CD Lemma indicates

Proposition

Let A be an algebra with presentation $\mathbf{k}\langle X \rangle / I_A$ and S be a GS basis of the ideal I_A in $\mathbf{k}\langle X \rangle$. Let $\overline{S} := \{\bar{s} \in M(X) \,|\, s \in S\}$ be the set of leading terms from S. The set of differential $\hat{S}$-irreducible elements

$$\mathrm{DIrr}(\hat{S}) = \left\{ \mathfrak{x} \in M(\Delta(X)) \,\middle|\, \mathfrak{s}^{[n]} \nmid \mathfrak{x} \text{ for any } \mathfrak{s} \in \overline{S}, n \geq 0 \right\},$$

giving a linear basis of the free differential algebra $\mathcal{D}_\lambda(A)$ on A, where $\mathfrak{s}^{[n]} \nmid \mathfrak{x}$ means $\mathfrak{x} \neq \mathfrak{a}\mathfrak{s}^{[n]}\mathfrak{b}$ for any $\mathfrak{a}, \mathfrak{b} \in M(\Delta(X))$.

The commutative version for extension of GS bases

By contrast, the commutative version of the main theorem holds only if $\lambda \neq 0$. Otherwise, a GS basis in $\mathbf{k}[X]$ may fails extending to one in $\mathbf{k}[\Delta(X)]$.

Proposition

For commutative algebra $A = \mathbf{k}[X]/I_A$ with defining ideal I_A generated by a Gröbner-Shirshov basis S in $\mathbf{k}[X]$, $\hat{S} := \hat{i}_X(S)$ is a Gröbner-Shirshov basis in $\mathbf{k}[\Delta(X)]$ when $\lambda \neq 0$.

The commutative version for extension of GS bases

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Indeed, for the commutative case of weight $\lambda \neq 0$, the ambiguities of all possible compositions in $\hat{S}$ are as follows:

$w_{n,n} = \overline{d_X^n(s)}u = \overline{d_X^n(t)}v$ for some $s, t \in \hat{S}$, $u, v \in [\Delta(X)], n \geq 0$ s.t. $|\bar{s}|+|\bar{t}| > |w_{n,n}|$,

and checked to be trivial.

Linear bases of the free commutative DAs

For $n \geq 0$ and $\mathfrak{x} = x_1 \cdots x_k \in [X]$ with $x_1 \geq \cdots \geq x_k \in X$, define

$$\mathfrak{x}^{[n]} := \begin{cases} x_1^{(n)} \cdots x_k^{(n)} & \text{if } \lambda \neq 0, \\ x_1^{(n)} x_2^{(0)} \cdots x_k^{(0)} & \text{if } \lambda = 0. \end{cases}$$

For a GS basis S of I_A, the embedding $\hat{S} = \hat{i}_X(S) = \left\{ s^{(0)} \,\middle|\, s \in S \right\}$ is a (differential) GS basis of $\mathrm{DI}(S)$. Then the CD Lemma implies

Proposition

Let A be a commutative algebra with presentation $\mathbf{k}[X]/I_A$ and S be a Gröbner(-Shirshov) basis of I_A in $\mathbf{k}[X]$. Let $\overline{S} := \{\overline{s} \in [X] \,|\, s \in S\}$. When $\lambda \neq 0$, the set of differential $\hat{S}$-irreducible words

$$\mathrm{DIrr}(\hat{S}) := \left\{ \mathfrak{x} \in [\Delta(X)] \,\middle|\, \mathfrak{s}^{[n]} \nmid \mathfrak{x} \text{ for any } \mathfrak{s} \in \overline{S},\ n \geq 0 \right\},$$

is a $\mathbf{k}$-linear basis of $\mathcal{CD}_\lambda(A)$, where $\mathfrak{s}^{[n]} \nmid \mathfrak{x}$ means $\mathfrak{x} \neq \mathfrak{s}^{[n]}\mathfrak{a}$, $\forall \mathfrak{a} \in [\Delta(X)]$.

The "classical" commutative case of weight 0

Remark

If weight $\lambda = 0$, such an extension of GS bases does not exist in general. Indeed, the ambiguities of all possible compositions in $\hat{S}$ have the form,

$$w_{m,n} := (s,t)^{u,v}_{w_{m,n}} = \overline{d^m_X(s)}u = \overline{d^n_X(t)}v \text{ for } s,t \in \hat{S},\ u,v \in [\Delta(X)],\ m,n \geq 0$$

s.t. $|\bar{s}| + |\bar{t}| > |w_{m,n}|$. These extra compositions, with different $m,n \geq 0$, are not necessarily trivial.

Based on the previous discussion, we next study some concrete examples of free differential algebras.

A positive example for the commutative case

As previously pointed out, a GS bases of an algebra might fail to be extended to one for the free differential commutative algebras of weight 0 on the algebra. Here we first give a positive example.

Proposition

Let $A = \mathbf{k}[x, y, z]/(x + y + z + 1)$*, and* $X = \{x, y, z\}$ *with* $x > y > z$*. The free differential commutative algebra* $\mathcal{CD}_\lambda(A)$ *on* A *of weight* λ *is*

$$\mathbf{k}[\Delta(X)]/\left(x^{(m)} + y^{(m)} + z^{(m)} + \delta_{m,0} \,\middle|\, m \geq 0\right).$$

$\hat{S} := \left\{x^{(0)} + y^{(0)} + z^{(0)} + 1\right\}$ *is a GS basis in* $\mathbf{k}[\Delta(X)]$ *for arbitrary weight* λ *(including* $\lambda = 0$*), and*

$$\mathrm{DIrr}(\hat{S}) = \left\{y^{(m)}z^{(n)} \,\middle|\, m, n \geq 0\right\} \cup \{1\}$$

is a $\mathbf{k}$*-basis of* $\mathcal{CD}_\lambda(A)$*.*

Free differential algebras over one-variable algebras

For the free differential algebra on $\mathbf{k}[x]/(f(x))$ with $f(x)$ as a nonconstant polynomial, our main theorem provides

Proposition

Let $A = \mathbf{k}[x]/(f(x))$ *with* $f \in \mathbf{k}[x]$ *of degree* $n > 0$*, and* $X = \{x\}$.

$\hat{S} := \left\{ f(x^{(0)}) \right\}$ *is a GS basis in* $\mathbf{k}\langle \Delta(X) \rangle$ *for arbitrary weight* λ*, s.t. the set*

$$\mathrm{DIrr}(\hat{S}) = \left\{ \mathfrak{x} \in M(\Delta(X)) \,\middle|\, x^{(m)} (x^{(m(1-\delta_{\lambda,0}))})^{n-1} \nmid \mathfrak{x},\ m \geq 0 \right\}$$

is a $\mathbf{k}$*-basis of* $\mathcal{D}_\lambda(A)$*;*

When $\lambda \neq 0$, $\hat{S} := \left\{ f(x^{(0)}) \right\}$ *is a GS basis in* $\mathbf{k}[\Delta(X)]$*. In this case, the* $\mathbf{k}$*-basis* $\mathrm{DIrr}(\hat{S})$ *of* $\mathcal{CD}_\lambda(A)$ *can be written more explicitly as*

$$\left\{ \prod_{i \geq 0} (x^{(i)})^{n_i} \,\middle|\, \sum_{i \geq 0} n_i < \infty \text{ with all } n_i = 0, 1, \ldots, n-1 \right\}.$$

Free differential algebras over one-variable algebras

Corollary

For $A = \mathbf{k}[x]/(x^2)$ *of dual numbers and* $X = \{x\}$, *the free differential commutative algebra* $\mathcal{CD}_\lambda(A)$ *on* A *is*

$$\mathbf{k}[\Delta(X)]/\left(d_X^m((x^{(0)})^2)\,\middle|\,m \geq 0\right).$$

Moreover, $\hat{S} = \left\{(x^{(0)})^2\right\}$ *is a GS basis in* $\mathbf{k}[\Delta(X)]$ *when* $\lambda \neq 0$, *s.t.*

$$\mathrm{DIrr}(\hat{S}) = \left\{\prod\nolimits_{i\geq 0}(x^{(i)})^{n_i}\,\middle|\,\sum\nolimits_{i\geq 1} n_i < \infty \text{ with all } n_i = 0, 1\right\}$$

as a $\mathbf{k}$*-basis of* $\mathcal{CD}_\lambda(A)$.

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as a $\mathbf{k}$-basis of $\mathcal{CD}_\lambda(A)$.

When $\lambda = 0$, we particularly find the following nontrivial composition

$$((x^{(0)})^2, (x^{(0)})^2)_{w_{2,1}} = 2^{-1}\left(d_X^2((x^{(0)})^2)x^{(1)} - d_X((x^{(0)})^2)x^{(2)}\right) = (x^{(1)})^3,$$

with $w_{2,1} = x^{(2)}x^{(1)}x^{(0)}$, as it can not be any linear combination of $\hat{S}$-words, whose leading terms $\prec w_{2,1}$. Hence, $\hat{S}$ is not a GS basis in $\mathbf{k}[\Delta(X)]$, but might be suitably enlarged to become one.

Free differential algebras over one-variable algebras

Corollary

For the cyclic group C_n of order $n \geq 2$, the free differential commutative algebra $\mathcal{CD}_\lambda(C_n)$ is isomorphic to

$$\mathbf{k}[\Delta(X)]/\left(d_X^m((x^{(0)})^n) - \delta_{m,0} \,\middle|\, m \geq 0\right),$$

while $\hat{S} = \left\{(x^{(0)})^n - 1\right\}$ is a GS basis in $\mathbf{k}[\Delta(X)]$ when $\lambda \neq 0$, s.t.

$$\mathrm{DIrr}(\hat{S}) = \left\{\prod_{i\geq 0}(x^{(i)})^{n_i} \,\middle|\, \sum_{i\geq 0} n_i < \infty \text{ with all } n_i = 0, 1, \ldots, n-1\right\}$$

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is a $\mathbf{k}$-basis of $\mathcal{CD}_\lambda(C_n)$.

When $\lambda = 0$, we particularly have the nontrivial composition

$$((x^{(0)})^n - 1, (x^{(0)})^n - 1)_{w_{1,0}} = n^{-1} d_X((x^{(0)})^n - 1)x^{(0)} - ((x^{(0)})^n - 1)x^{(1)} = x^{(1)},$$

with $w_{1,0} = x^{(1)}(x^{(0)})^n$. Thus $\hat{S}$ is not a GS basis in $\mathbf{k}[\Delta(X)]$, but $\hat{S}^+ = \hat{S} \cup \{x^{(1)}\}$ makes it so that $\mathcal{CD}_\lambda(C_n) \cong \mathbf{k}[\Delta(X)\,|\,\hat{S}^+] \cong \mathbf{k}C_n$.

In the end, we construct the free differential Lie algebra on a Lie algebra from the free differential algebra over its universal enveloping algebra.

The UEDA on a Lie algebra

Given a Lie algebra $\mathfrak{g}$, the **universal enveloping differential algebra** on $\mathfrak{g}$ is a differential algebra $\mathcal{D}_\lambda(\mathfrak{g})$ with λ-derivation $d_\mathfrak{g}$ and Lie algebra map $i_\mathfrak{g} : \mathfrak{g} \to \mathcal{D}_\lambda(\mathfrak{g})$ satisfying the following universal property,

- for any differential algebra (R, d_R) and Lie algebra map $f : \mathfrak{g} \to R$, there exists a unique differential algebra map $\bar{f} : \mathcal{D}_\lambda(\mathfrak{g}) \to R$, s.t. $f = \bar{f} \circ i_\mathfrak{g}$.

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Lemma

The free differential algebra $\mathcal{D}_\lambda(U(\mathfrak{g}))$ on $U(\mathfrak{g})$ with $i_\mathfrak{g} := i_{U(\mathfrak{g})} \circ j_\mathfrak{g}$ is the universal enveloping differential algebra $\mathcal{D}_\lambda(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$, with the following commutative diagram,

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\quad f \quad} & R \\ \downarrow{\scriptstyle j_\mathfrak{g}} & \overset{\exists!\hat{f}}{\nearrow} & \uparrow{\scriptstyle \exists!\bar{f}} \\ U(\mathfrak{g}) & \xrightarrow{\; i_{U(\mathfrak{g})} \;} & \mathcal{D}_\lambda\langle U(\mathfrak{g})\rangle \, . \end{array}$$

The GS bases for the UEDA on a Lie algebra

Suppose that $\mathfrak{X} = \{x_i\}_{i \in I}$ is a $\mathbf{k}$-basis of $\mathfrak{g}$ with a well-order $<$ on I. Let $[\ ,\ ]_{\mathfrak{g}}$ be the Lie bracket of $\mathfrak{g}$, and $|x_i, x_j| := \sum_{k \in I} c_{ij}^k x_k$ as the linear expansion of $[x_i, x_j]_{\mathfrak{g}}$ in $\mathfrak{X}$. Denote

$$S_{\mathfrak{X}} := \{[x_i, x_j] - |x_i, x_j| \mid i, j \in I, i > j\}.$$

Theorem (PBW theorem via GS bases)

With the induced deg-lex order on $M(\mathfrak{X})$, $S_{\mathfrak{X}}$ is a Gröbner-Shirshov basis in $\mathbf{k}\langle \mathfrak{X} \rangle$, and the $\mathbf{k}$-basis $\mathrm{Irr}(S_{\mathfrak{X}})$ of $U(\mathfrak{g})$ consists of all monomials

$$x_{i_1} \cdots x_{i_n} \text{ with } i_1 \leq \cdots \leq i_n,\ i_1 \cdots i_n \in I, n \geq 0.$$

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Theorem (GS bases for the UEDA on a Lie algebra)

Under the previous notation, $\hat{S}_\mathfrak{X} = \hat{i}_\mathfrak{X}(S_\mathfrak{X})$ is a Gröbner-Shirshov basis in $\mathbf{k}\langle\Delta(\mathfrak{X})\rangle$ and the $\mathbf{k}$-basis $\mathrm{Irr}(\hat{S}_\mathfrak{X})$ of $\mathcal{D}_\lambda(\mathfrak{g}) = \mathbf{k}\langle\Delta(\mathfrak{X})\rangle/\mathrm{DI}(\hat{S}_\mathfrak{X})$ has the form

$$\left\{u \in M(\Delta(\mathfrak{X})) \,\middle|\, u \ne a x_i^{(n)} x_j^{(n(1-\delta_{\lambda,0}))} b,\ a, b \in M(\Delta(\mathfrak{X})), i, j \in I, i > j, n \ge 0\right\}.$$

The FDLA on a Lie algebra

Definition

Let $\lambda \in \mathbf{k}$. A **differential $\mathbf{k}$-Lie algebra of weight λ** *is a $\mathbf{k}$-Lie algebra $\mathfrak{g}$ with Lie bracket $[\cdot, \cdot]$ and also a λ-derivation $d : \mathfrak{g} \to \mathfrak{g}$, s.t.*

$$d([x, y]) = [d(x), y] + [x, d(y)] + \lambda[d(x), d(y)], \forall\, x, y \in \mathfrak{g}.$$

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$$d([x,y]) = [d(x),y] + [x,d(y)] + \lambda[d(x),d(y)], \forall\, x,y \in \mathfrak{g}.$$

The **free λ-differential Lie algebra** *on a Lie algebra $\mathfrak{g}$ is a differential Lie algebra $\mathcal{L}_\lambda(\mathfrak{g})$ with λ-derivation $d_\mathfrak{g}$ and Lie algebra map $l_\mathfrak{g} : \mathfrak{g} \to \mathcal{L}_\lambda(\mathfrak{g})$ satisfying the following universal property,*

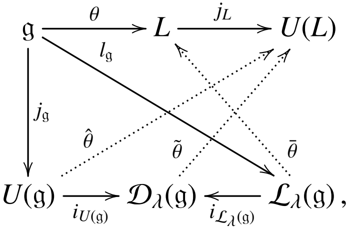
- *for any λ-differential Lie algebra (L, d_L) and Lie algebra map $\theta : \mathfrak{g} \to L$, there exists a unique differential Lie algebra map $\bar{\theta} : \mathcal{L}_\lambda(\mathfrak{g}) \to L$, s.t. $\theta = \bar{\theta} \circ l_\mathfrak{g}$.*

The FDLA on a Lie algebra from its UEDA

Proposition

The free differential Lie algebra $(\mathcal{L}_\lambda(\mathfrak{g}), d_\mathfrak{g})$ on a Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g})$ can serve as the differential Lie subalgebra of $(\mathcal{D}_\lambda(\mathfrak{g}), d_\mathfrak{g})$ under its commutator, and $\mathcal{D}_\lambda(\mathfrak{g})$ becomes the universal enveloping differential algebra of $\mathcal{L}_\lambda(\mathfrak{g})$.

This result can be illustrated as follows.



Some further discussions around the topic

- General study is needed for Gröbner-Shirshov bases in free differential algebras, especially in the noncommutative and nonzero weight cases.

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- General study is needed for Gröbner-Shirshov bases in free differential algebras, especially in the noncommutative and nonzero weight cases.
- It needs more discussion about Gröbner-Shirshov bases in free differential Lie algebras via the theory of GS bases for Lie algebras.
- Figure out how to extend GS bases of commutative algebras to their free 0-differential commutative algebras and design an algorithm.
- The **Wronskian envelope** of a Lie algebra is a free object in the category of differential commutative algebras, previously studied by Poinsot. However, the aspect of its GS bases seems unknown and is beyond our framework of extension of GS bases.

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Thank You!