Algorithmic Approach to Strong Consistency Analysis of Finite Difference Approximations to PDE Systems

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Outline

- Strong consistency of finite difference approximations
- Thomas decomposition for nonlinear PDE systems
- Decomposition of nonlinear difference systems
1. Strong consistency of finite difference approximations
Strong vs. weak consistency

Approximate PDE system (differential polynomials in $u^{(1)}, \ldots, u^{(m)}$) by a difference system (difference polynomials in $\tilde{u}^{(1)}, \ldots, \tilde{u}^{(m)}$) on Cartesian grid $\{(x_1 + k_1 h, \ldots, x_n + k_n h) \mid k_1, \ldots, k_n \in \mathbb{Z}\}$, $h > 0$

e.g., $\partial_j u^{(\alpha)}(x) = \frac{\tilde{u}^{(\alpha)}_{k_1, \ldots, k_j+1, \ldots, k_n} - \tilde{u}^{(\alpha)}_{k_1, \ldots, k_j-1, \ldots, k_n}}{2h} + \mathcal{O}(h^2)$
Strong vs. weak consistency

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e.g.,
\[
\partial_j u^{(\alpha)}(x) = \frac{\tilde{u}^{(\alpha)}_{k_1 \ldots, k_j+1 \ldots, k_n} - \tilde{u}^{(\alpha)}_{k_1 \ldots, k_j-1 \ldots, k_n}}{2h} + \mathcal{O}(h^2)
\]

Def. $\tilde{f} \succ f$ if Taylor expansion of $\tilde{f}$ about grid point $x$ yields
\[
\tilde{f}(\tilde{u}) = h^d f(u) + \mathcal{O}(h^{d+1}), \quad d \in \mathbb{Z}_{\geq 0},
\]

after clearing denominators containing $h$. 

Kolchin seminar, 13/09/2019
Strong vs. weak consistency

Approximate PDE system (differential polynomials in $u^{(1)}, \ldots, u^{(m)}$) by a difference system (difference polynomials in $\tilde{u}^{(1)}, \ldots, \tilde{u}^{(m)}$) on Cartesian grid $\{(x_1 + k_1 h, \ldots, x_n + k_n h) \mid k_1, \ldots, k_n \in \mathbb{Z}\}, \quad h > 0$

e.g., $\partial_j u^{(\alpha)}(x) = \frac{\tilde{u}_{k_1, \ldots, k_j+1, \ldots, k_n}^{(\alpha)} - \tilde{u}_{k_1, \ldots, k_j-1, \ldots, k_n}^{(\alpha)}}{2h} + O(h^2)$

Def. $\tilde{f} \triangleright f$ if Taylor expansion of $\tilde{f}$ about grid point $x$ yields

$$\tilde{f}(\tilde{u}) = h^d f(u) + O(h^{d+1}), \quad d \in \mathbb{Z}_{\geq 0},$$

after clearing denominators containing $h$.

Def. FDA $\widetilde{F}$ is strongly consistent (s-consistent) with PDEs $F$ if

$$(\forall \tilde{f} \in [\widetilde{F}]) \ (\exists f \in [F]) \ [\tilde{f} \triangleright f].$$
Strong vs. weak consistency

equation-wise / weak consistency of difference schemes for PDEs
Strong vs. weak consistency

equation-wise / weak consistency of difference schemes for PDEs

linear PDEs: Janet basis $J_1$ for PDEs, $J_2$ for difference system

idea: FDA strongly consistent (s-consistent) if $J_2 \xrightarrow{h \to 0} J_1$

V. P. Gerdt, D. Robertz
Consistency of Finite Difference Approximations for Linear PDE Systems and its Algorithmic Verification,
Strong vs. weak consistency

equation-wise / weak consistency of difference schemes for PDEs

*linear* PDEs: Janet basis $J_1$ for PDEs, $J_2$ for difference system

idea: FDA *strongly consistent* (s-consistent) if $J_2 \xrightarrow{h \to 0} J_1$

V. P. Gerdt, D. Robertz

*Consistency of Finite Difference Approximations for Linear PDE Systems and its Algorithmic Verification*,

⇝ new difference scheme for 2D Navier-Stokes equations:

P. Amodio, Y. Blinkov, V. Gerdt, R. La Scala,

*Algebraic construction and numerical behavior of a new s-consistent difference scheme for the 2D Navier-Stokes equations*,

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*Algebraic construction and numerical behavior of a new s-consistent difference scheme for the 2D Navier-Stokes equations*

\[
\begin{align*}
\frac{u(n,j+1,k)-u(n,j-1,k)}{2h} + \frac{v(n,j,k+1)-v(n,j,k-1)}{2h} &= 0, \\
\frac{u(n+1,j,k)-u(n,j,k)}{\tau} + \frac{u(n,j+1,k)-u(n,j-1,k)^2}{2h} + \frac{u(n,j,k+1)v(n,j,k+1)-u(n,j,k-1)v(n,j,k-1)}{2h} + \frac{p(n,j+1,k)-p(n,j-1,k)}{2h} &= 0, \\
\frac{1}{\text{re}} \left( \frac{(u(n,j+1,k)-2u(n,j,k)+u(n,j,k-1))+(u(n,j,k+1)-2u(n,j,k)+u(n,j,k-1))}{h^2} \right) &= 0, \\
\frac{v(n+1,j,k)-v(n,j,k)}{\tau} + \frac{u(n,j+1,k)v(n,j+1,k)-u(n,j-1,k)v(n,j-1,k)}{2h} + \frac{v(n,j,k+1)^2-v(n,j,k-1)^2}{2h} + \frac{p(n,j,k+1)-p(n,j,k-1)}{2h} &= 0, \\
\frac{1}{\text{re}} \left( \frac{(v(n,j+1,k)-2v(n,j,k)+v(n,j,k-1))+(v(n,j,k+1)-2v(n,j,k)+v(n,j,k-1))}{h^2} \right) &= 0, \\
\frac{u(n,j+2,k)^2-2u(n,j,k)^2+u(n,j-2,k)^2}{4h^2} + \frac{v(n,j,k+2)^2-2v(n,j,k)^2+v(n,j,k-2)^2}{4h^2} + \frac{2u(n,j+1,k+1)v(n,j+1,k+1)-u(n,j+1,k-1)v(n,j+1,k-1)}{4h^2} + \frac{2u(n,j-1,k+1)v(n,j-1,k+1)+u(n,j-1,k-1)v(n,j-1,k-1)}{4h^2} + \frac{p(n,j+2,k)-2p(n,j,k)+p(n,j-2,k)}{4h^2} + \frac{p(n,j,k+2)-2p(n,j,k)+p(n,j,k-2)}{4h^2} &= 0
\end{align*}
\]
**Consistency, stability, convergence**

**Def.** FDA to PDEs is *stable* if the error caused by a small perturbation in the numerical solution of the difference equations stays bounded.
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In many cases (for a single PDE), consistency and stability of FDA are equivalent to *convergence*.
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In many cases (for a single PDE), consistency and stability of FDA are equivalent to \textit{convergence}.

Thm. (Lax-Richtmyer, 1956). A consistent finite difference scheme for a linear PDE, for which the initial value problem is well-posed, is convergent if it is stable.
Strong vs. weak consistency

Example.

\[
\begin{align*}
\frac{\partial u}{\partial x} - u^2 &= 0 \\
\frac{\partial u}{\partial y} + u^2 &= 0
\end{align*}
\]

\[ u = u(x, y) \]
Strong vs. weak consistency

Example.

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\]

\( u = u(x, y) \)

\[
\begin{cases}
D_1^+ \tilde{u} - \tilde{u}^2 = 0 \quad (A) \\
D_2^+ \tilde{u} + \tilde{u}^2 = 0 \quad (B)
\end{cases}
\]

\[
\leadsto \quad \sigma_2 A - \sigma_1 B + (\ldots) A + (\ldots) B = -2h^3 u_{i,j}^4
\]
Strong vs. weak consistency

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\end{align*}
\]

\[
\sigma_2 A - \sigma_1 B + (\ldots) A + (\ldots) B = -2h^3 u_{i, j}
\]

\[
\begin{align*}
D_1^+ \tilde{u} - \tilde{u}^2 &= 0 \quad (A') \\
D_2^- \tilde{u} + \tilde{u}^2 &= 0 \quad (B')
\end{align*}
\]

is s-consistent with (\(\ast\)).
2. Thomas decomposition for nonlinear PDE systems
Systems of linear PDEs

\[
\begin{align*}
\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial u}{\partial y} &= 0 \\
\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} &= 0
\end{align*}
\]

find: \( u = u(x, y) \) analytic
Systems of linear PDEs

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\end{cases}
\]

find: \( u = u(x, y) \) analytic

\[
u(x, y) = a_{0,0} + a_{1,0} x + a_{0,1} y + a_{2,0} \frac{x^2}{2!} + a_{1,1} \frac{xy}{1!1!} + a_{0,2} \frac{y^2}{2!} + \ldots
\]
Systems of linear PDEs

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\begin{cases}
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\end{align*}
\]

find: \( u = u(x, y) \) analytic

\[
\frac{\partial}{\partial x} \left( \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial u}{\partial y} \right) - \frac{\partial}{\partial y} \left( \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} \right) \rightarrow \frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial y} = 0
\]

\[
u(x, y) = a_{0,0} + a_{1,0} x + a_{0,1} y + a_{2,0} \frac{x^2}{2!} + a_{1,1} \frac{xy}{1!1!} + a_{0,2} \frac{y^2}{2!} + \cdots
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find: \( u = u(x, y) \) analytic

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\frac{\partial}{\partial x} \left( \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial u}{\partial y} \right) - \frac{\partial}{\partial y} \left( \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} \right) \rightarrow \frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial y} = 0
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\[ u(x, y) = a_{0,0} + a_{1,0} x + a_{0,1} y + a_{2,0} \frac{x^2}{2!} + a_{1,1} \frac{xy}{1!1!} + a_{0,2} \frac{y^2}{2!} + \ldots \]

*Janet’s algorithm* computes a vector space basis for power series solutions

(Maurice Janet, \( \sim 1920 \))
Janet’s Algorithm

\[
\begin{align*}
   u_{y,y} & = 0 \\
   u_{x,x} - yu_{z,z} & = 0
\end{align*}
\]

is equivalent to

\[
\begin{align*}
   u_{y,y} & = 0 \\
   u_{x,x} - yu_{z,z} & = 0 \\
   u_{y,z,z} & = 0 \\
   u_{x,y,y} & = 0 \\
   u_{z,z,z,z} & = 0 \\
   u_{x,y,z,z} & = 0 \\
   u_{x,z,z,z,z} & = 0 \\
   u_{x,z,z,z,z} & = 0
\end{align*}
\]
Janet’s Algorithm

\[
\begin{aligned}
&u_{y,y} = 0 & A \\
&u_{x,x} - yu_{z,z} = 0 & B \\
\end{aligned}
\]

is equivalent to

\[
\begin{aligned}
&u_{y,y} = 0 & A \\
&u_{x,x} - yu_{z,z} = 0 & B \\
&u_{y,z,z} = 0 & \frac{1}{2} (\partial_x^2 - y \partial_z^2)A - \frac{1}{2} \partial^2_y B \\
&u_{x,y,y} = 0 & \partial_x A \\
&u_{z,z,z,z} = 0 & \frac{1}{2} (\partial_x^4 - 2y \partial_x^2 \partial_z^2 + y^2 \partial_z^4)A - \frac{1}{2} (\partial_x^2 \partial_y^2 - y \partial_y^2 \partial_z^2 + 2 \partial_y \partial_z^2)B \\
&u_{x,y,z,z} = 0 & \frac{1}{2} (\partial_x^3 - y \partial_x \partial_z^2)A - \frac{1}{2} \partial_x \partial^2_y B \\
&u_{x,z,z,z} = 0 & \frac{1}{2} (\partial_x^5 - 2y \partial_x^3 \partial_z^2 + y^2 \partial_x \partial_z^4)A - \frac{1}{2} (\partial_x^3 \partial_y^2 + y \partial_x \partial_y^2 \partial_z^2 - 2 \partial_x \partial_y \partial_z^2)B \\
\end{aligned}
\]
Janet’s Algorithm

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  u_{x,y,y} = 0 & \partial_x A \\
  u_{z,z,z,z} = 0 & \frac{1}{2} (\partial^4_x - 2y\partial^2_x\partial^2_z + y^2\partial^4_z)A - \frac{1}{2} (\partial^2_x\partial^2_y - y\partial^2_y\partial^2_z + 2\partial_y\partial^2_z)B \\
  u_{x,y,z,z} = 0 & \frac{1}{2} (\partial^3_x - y\partial_x\partial^2_z)A - \frac{1}{2}\partial_x\partial^2_y B \\
  u_{x,z,z,z} = 0 & \frac{1}{2} (\partial^5_x - 2y\partial^3_x\partial^2_z + y^2\partial_x\partial^4_z)A - \frac{1}{2} (\partial^3_x\partial^2_y + y\partial_x\partial^2_y\partial^2_z - 2\partial_x\partial_y\partial^2_z)B
\end{cases}
\]

Taylor coeff’s for  1, z, y, x, z^2, yz, xz, xy, z^3, xz^2, xyz, xz^3  arbitrary,
all other coeff’s determined by linear equations
Differential algebraic geometry

Differential algebra (Ritt, Kolchin, Seidenberg, . . . )

\( \mathbb{Q} \subseteq K \) a differential field with commuting derivations \( \partial_1, \ldots, \partial_n \)
Differential algebraic geometry

*Differential algebra* (Ritt, Kolchin, Seidenberg, ...) 

\( \mathbb{Q} \subseteq K \) a differential field with commuting derivations \( \partial_1, ..., \partial_n \)

*Differential polynomial ring* with derivations \( \partial_1, ..., \partial_n \)

\[
K\{u\} := K[\partial_1^{i_1} \cdots \partial_n^{i_n} u \mid i \in (\mathbb{Z}_{\geq 0})^n] = K[u, u_{z_1}, ..., u_{z_n}, u_{z_1z_1}, ...]
\]
Differential algebraic geometry

*Differential algebra* (Ritt, Kolchin, Seidenberg, . . .)

$\mathbb{Q} \subseteq K$ a differential field with commuting derivations $\partial_1, \ldots, \partial_n$

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$K\{u\}$ not Noetherian (e.g., $[u'u'', u''u''', \ldots] \subseteq K\{u\}$ not fin. gen.)
Differential algebraic geometry

*Differential algebra* (Ritt, Kolchin, Seidenberg, ...)

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\( K\{u\} \) not Noetherian (e.g., \([u'u'', u''u''', \ldots] \subseteq K\{u\} \) not fin. gen.)

**Thm.** (Ritt-Raudenbush).

Every radical differential ideal of \( K\{u^{(1)}, \ldots, u^{(m)}\} \) is finitely generated and is intersection of finitely many prime differential ideals.
Differential algebraic geometry

*Differential algebra* (Ritt, Kolchin, Seidenberg, ...)

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$K\{u\}$ not Noetherian (e.g., $[u'u'', u''u''', \ldots] \subseteq K\{u\}$ not fin. gen.)

**Thm.** (Ritt-Raudenbush).
Every radical differential ideal of $K\{u^{(1)}, \ldots, u^{(m)}\}$ is finitely generated and is intersection of finitely many prime differential ideals.

**Thm.** (Differential Nullstellensatz).
Every radical diff. ideal $I \subsetneq K\{u^{(1)}, \ldots, u^{(m)}\}$ has a zero in a diff. field ext. of $K$. If $f \in K\{u^{(1)}, \ldots, u^{(m)}\}$ vanishes for all zeros of $I$, then $f \in I$. 
Thomas decomposition for nonlinear PDE systems

\[ K\{u\} = K[u, u_x, u_y, \ldots, u_{x,x}, u_{x,y}, u_{y,y}, \ldots] \quad \text{diff. polynomial ring} \]

\[ u < \ldots < u_y < u_x < \ldots < u_{y,y} < u_{x,y} < u_{x,x} < \ldots \quad \text{(ranking)} \]
Thomas decomposition for nonlinear PDE systems

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\[ u < \ldots < u_y < u_x < \ldots < u_{y,y} < u_{x,y} < u_{x,x} < \ldots \quad \text{(ranking)} \]

algebraic reduction:

\[ p = u_{x,x,y}^3 + \ldots \]

\[ q = c u_{x,x,y}^2 + \ldots \]

\[ p \rightarrow r = c \cdot p - u_{x,x,y} \cdot q \]
Thomas decomposition for nonlinear PDE systems

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**algebraic reduction:**

\[ p = u_{x,x,y}^3 + \ldots \]

\[ q = c u_{x,x,y}^2 + \ldots \]

\[ p \rightarrow r = c \cdot p - u_{x,x,y} \cdot q \]

**differential reduction:**

\[ p = u_{x,x,y,y}^3 + \ldots \]

\[ q = c u_{x,x,y}^2 + \ldots \]

\[ \partial_y q = \frac{\partial q}{\partial u_{x,x,y}} u_{x,x,y,y} + \ldots \]

\[ p \rightarrow r = \frac{\partial q}{\partial u_{x,x,y}} \cdot p - u_{x,x,y,y}^2 \cdot \partial_y q \]
Thomas decomposition for nonlinear PDE systems

\[ K\{u\} = K[u, u_x, u_y, \ldots, u_{x,x}, u_{x,y}, u_{y,y}, \ldots] \quad \text{diff. polynomial ring} \]

\[ u < \ldots < u_y < u_x < \ldots < u_{y,y} < u_{x,y} < u_{x,x} < \ldots \quad (\text{ranking}) \]

algebraic reduction:

\[ p = u_{x,x,y}^3 + \ldots \]

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differential reduction:

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\[ q = c u_{x,x,y}^2 + \ldots \]

\[ \partial_y q = \frac{\partial q}{\partial u_{x,x,y}} u_{x,x,y,y} + \ldots \]

\[ p \rightarrow r = \frac{\partial q}{\partial u_{x,x,y}} \cdot p - u_{x,x,y,y}^2 \cdot \partial_y q \]

reduction requires: initial \( c \neq 0 \) and separant \( \frac{\partial q}{\partial u_{x,x,y}} \neq 0 \)
Thomas decomposition

\[ p = x^3 + (3y + 1)x^2 + (3y^2 + 2y)x + y^3 = 0 \]
Thomas decomposition

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\[ \text{disc}_x(p) = y^2(4 - 27y^2) \]
Thomas decomposition

\[ p = x^3 + (3y + 1)x^2 + (3y^2 + 2y)x + y^3 = 0 \]

\[ \text{disc}_x(p) = y^2(4 - 27y^2) \]

2 non-real points
Thomas decomposition for nonlinear PDE systems

\[ S = \{ p_1 = 0, \ldots, p_s = 0, q_1 \neq 0, \ldots, q_t \neq 0 \} \]

**Def.** *Thomas decomposition* of differential system *S* (or *Sol*(\(S\))):

\[ Sol(S) = Sol(S_1) \uplus \ldots \uplus Sol(S_r), \quad S_i \text{ simple differential system} \]
Thomas decomposition for nonlinear PDE systems

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Def. **Thomas decomposition** of differential system \( S \) (or Sol(\( S \))):

\[ \text{Sol}(S) = \text{Sol}(S_1) \uplus \ldots \uplus \text{Sol}(S_r), \quad S_i \text{ simple differential system} \]

Def. \( S \) is simple if

(a) \( p_1, \ldots, p_s, q_1, \ldots, q_t \) have pairwise distinct leaders,

(b) initials and discriminants of \( p_i \) and \( q_j \) do not vanish,

(c) \( p_1, \ldots, p_s \) form a passive PDE system,

(d) \( q_1, \ldots, q_t \) are reduced modulo \( p_1, \ldots, p_s \).

set of admissible derivations \( \mu_i \subseteq \{ \partial_1, \ldots, \partial_n \} \) for \( p_i, \quad i = 1, \ldots, s \)
Thomas decomposition for nonlinear PDE systems

\[ R = K\{u^{(1)}, \ldots, u^{(m)}\} \]

**Def.** Thomas decomposition of differential system \( S \) (or \( \text{Sol}(S) \)):

\[ \text{Sol}(S) = \text{Sol}(S_1) \uplus \ldots \uplus \text{Sol}(S_r), \quad S_i \text{ simple differential system} \]

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**Thm.** \( S = \{p_1 = 0, \ldots, p_s = 0, q_1 \neq 0, \ldots, q_t \neq 0\} \) simple diff. system

\( E \) differential ideal generated by \( p_1, \ldots, p_s \)

\( q \) product of initials and separants of all \( p_i \)
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\( E \) differential ideal generated by \( p_1, \ldots, p_s \)

\( q \) product of initials and separants of all \( p_i \)

Then

\[ E : q^\infty := \{ p \in R \mid q^r \cdot p \in E \text{ for some } r \in \mathbb{Z}_{\geq 0} \} = \mathcal{I}_R(\text{Sol}(S)) \]

consists of all differential polynomials in \( R \) vanishing on \( \text{Sol}(S) \).
Example

Navier-Stokes equations for an incompressible flow of a constant viscosity fluid:

\[
\begin{align*}
\begin{cases}
    u_t + (u \cdot \nabla) u + \nabla p - \mu \Delta u &= 0 \\
    \nabla \cdot u &= 0
\end{cases}
\end{align*}
\]

In cartesian coordinates \(x, y, z\) of \(\mathbb{R}^3\) we have, equivalently,

\[
\begin{align*}
\begin{cases}
    u_t &+ u u_x + v u_y + w u_z + p_x - \mu (u_x, x + u_y, y + u_z, z) &= 0 \\
    v_t &+ u v_x + v v_y + w v_z + p_y - \mu (v_x, x + v_y, y + v_z, z) &= 0 \\
    w_t &+ u w_x + v w_y + w w_z + p_z - \mu (w_x, x + w_y, y + w_z, z) &= 0 \\
    u_x + v_y + w_z &= 0
\end{cases}
\end{align*}
\]

where \(u, v, w\) denote the components of the velocity vector \(u\).

Poisson pressure equation

\[
\Delta p + \nabla \cdot ((u \cdot \nabla) u) = 0
\]

is a consequence of the Navier-Stokes equations.
Example

degree-reverse lexicographic ranking $>_{\text{drl}}$ on the set of partial derivatives

$$\left\{ \frac{\partial^{i+j+k+l} f}{\partial t^i \partial x^j \partial y^k \partial z^l} \bigg| f \in \{u, v, w, p\}, \ i, j, k, l \in \mathbb{Z}_{\geq 0} \right\}$$

$\Rightarrow$ one simple differential system:

$$\begin{align*}
\frac{u_x + v_y + w_z}{u_x} &= 0 \\
\mu v_{x,x} + \mu v_{y,y} + \mu v_{z,z} - uv_x - vv_y - wv_z - p_y - v_t &= 0 \\
\mu u_{y,y} + \mu u_{z,z} - \mu v_{x,y} - \mu w_{x,z} + uv_y + uw_z - vu_y - wu_z - p_x - u_t &= 0 \\
\mu w_{x,x} + \mu w_{y,y} + \mu w_{z,z} - uv_x - vw_y - wv_z - p_z - w_t &= 0 \\
p_{x,x} + p_{y,y} + p_{z,z} + 2 u_y v_x + 2 u_z w_x + 2 v_y^2 + 2 v_y w_z + 2 v_z w_y + 2 w_z^2 &= 0
\end{align*}$$
Example

degree-reverse lexicographic ranking $>_{\text{lr}}$ on the set of partial derivatives

$$\left\{ \frac{\partial^{i+j+k+l} f}{\partial t^i \partial x^j \partial y^k \partial z^l} \bigg| f \in \{u, v, w, p\}, i, j, k, l \in \mathbb{Z}_{\geq 0} \right\}$$

$\Rightarrow$ one simple differential system:

$$\begin{align*}
\mu u_x + v_y + w_z &= 0 \\
\mu v_x + \mu v_y, y + \mu v_z, z - uv_x - vv_y - wv_z - p_y - v_t &= 0 \\
\mu u_y, y + \mu u_z, z - \mu v_x, y - \mu w_x, z + uv_y + uw_z - vu_y - wu_z - p_x - u_t &= 0 \\
\mu w_x, x + \mu w_y, y + \mu w_z, z - uw_x - vw_y - wv_z - p_z - w_t &= 0 \\
p_{x, x} + p_y, y + p_{z, z} + 2u_yv_x + 2u_zw_x + 2v_y^2 + 2v_yw_z + 2v_zw_y + 2w_z^2 &= 0
\end{align*}$$

The last equation was obtained as

$$\frac{\partial}{\partial x} A_1 + \frac{\partial}{\partial y} A_2 + \frac{\partial}{\partial z} A_3 + \left[ v \Delta - \frac{\partial}{\partial t} - u \frac{\partial}{\partial x} - v \frac{\partial}{\partial y} - w \frac{\partial}{\partial z} + 2v_y + 2w_z \right] A_4,$$

where $A_1, A_2, A_3, A_4$ are the left hand sides of the Navier-Stokes equations.

Modulo the other equations in the system, last equation $\Leftrightarrow$ Poisson pressure equation.
Example

Taylor coefficients of $u(t, x, y, z)$, $v(t, x, y, z)$, $w(t, x, y, z)$, $p(t, x, y, z)$ whose values can be chosen arbitrarily:

$u(t, x, t, z)$:

$$
\frac{1}{(1 - \partial_t)(1 - \partial_y)(1 - \partial_z)}
$$

$v(t, x, y, z)$:

$$
\frac{1}{(1 - \partial_t)(1 - \partial_y)(1 - \partial_z)} + \frac{\partial_x}{(1 - \partial_t)(1 - \partial_y)(1 - \partial_z)}
$$

$w(t, x, y, z)$:

$$
\frac{1}{(1 - \partial_t)(1 - \partial_y)(1 - \partial_z)} + \frac{\partial_x}{(1 - \partial_t)(1 - \partial_y)(1 - \partial_z)}
$$

$p(t, x, y, z)$:

$$
\frac{1}{(1 - \partial_t)(1 - \partial_y)(1 - \partial_z)} + \frac{\partial_x}{(1 - \partial_t)(1 - \partial_y)(1 - \partial_z)}
$$

generalized Hilbert series
Example

Extending the Cauchy-Kovaleskaya Theorem we may pose the Cauchy problem for the Navier-Stokes equations around an arbitrary point \((t_0, x_0, y_0, z_0)\) as follows:

\[
\begin{align*}
  u(t, x_0, y, z) &= f_1(t, y, z) \\
  v(t, x_0, y, z) &= f_2(t, y, z) \\
  \frac{\partial v}{\partial x}(t, x_0, y_0, z) &= f_3(t, z) \\
  w(t, x_0, y, z) &= f_4(t, y, z) \\
  \frac{\partial w}{\partial x}(t, x_0, y, z) &= f_5(t, y, z) \\
  p(t, x_0, y, z) &= f_6(t, y, z) \\
  \frac{\partial p}{\partial x}(t, x_0, y, z) &= f_7(t, y, z)
\end{align*}
\]

where \(f_1, f_2, \ldots, f_7\) are arbitrary functions of their arguments which are analytic around the point \((t_0, x_0, y_0, z_0)\). The arbitrariness of analytic solutions to the Navier-Stokes equations is determined by \(f_1, f_2, \ldots, f_7\).
Singular solutions

\[ p = y^2 - 4ty - 4y + 8t^2 = 0 \quad p \in \mathbb{Q}(t)\{y\} \]

Separant of \( p \):

\[ \frac{\partial p}{\partial y} = 2y - 4t \]
Singular solutions

\[ p = \dot{y}^2 - 4t\dot{y} - 4y + 8t^2 = 0 \quad p \in \mathbb{Q}(t)\{y\} \]

Separant of \( p \):

\[ \frac{\partial p}{\partial \dot{y}} = 2\dot{y} - 4t \]

\[ \text{res}(p, \frac{\partial p}{\partial \dot{y}}, \dot{y}) = -16y + 16t^2 \]

Thomas decomposition:

\[
\begin{align*}
p &= 0 \\
y - t^2 &\neq 0 \\
y - t^2 &= 0
\end{align*}
\]
Singular solutions

\[ p = y^2 - 4t\dot{y} - 4y + 8t^2 = 0 \quad p \in \mathbb{Q}(t)\{y\} \]

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Thomas decomposition:
\[
\begin{array}{c|c}
  p & 0 \\
  y - t^2 & \neq 0 \\
\end{array} \quad \begin{array}{c|c}
  y - t^2 & = 0 \\
\end{array}
\]

general solution:
\[ y(t) = 2((t + c)^2 + c^2), \quad c \in \mathbb{R} \]

essential singular solution:
\[ y(t) = t^2 \]
Thomas decomposition

Example.

Thomas decomposition of \( \{ u_t - 6uu_x + u_{x,x,x} = 0, \ u_{u_t,x} - u_tu_x = 0 \} \):
Example.

Thomas decomposition of \[ \{ u_t - 6uu_x + u_{x,x,x} = 0, \quad uu_{t,x} - u_t u_x = 0 \} : \]

<table>
<thead>
<tr>
<th>Equation</th>
<th>Variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>( u_t - 6uu_x = 0 )</td>
<td>( \frac{\partial}{\partial t}, \frac{\partial}{\partial x} )</td>
</tr>
<tr>
<td>( u_{x,x} = 0 )</td>
<td>( \frac{\partial}{\partial x} )</td>
</tr>
<tr>
<td>( u \neq 0 )</td>
<td></td>
</tr>
<tr>
<td>( u_t = 0 )</td>
<td>( \frac{\partial}{\partial t}, \frac{\partial}{\partial x} )</td>
</tr>
<tr>
<td>( u_{x,x,x} - 6uu_x = 0 )</td>
<td>( \frac{\partial}{\partial x} )</td>
</tr>
<tr>
<td>( u_{x,x} \neq 0 )</td>
<td></td>
</tr>
<tr>
<td>( u \neq 0 )</td>
<td></td>
</tr>
</tbody>
</table>
Thomas Decomposition

- 1998: D. Wang: implementation for algebraic systems
- Since 2009: implementations in Maple for
  - algebraic systems (T. Bächler)
  - systems of PDEs (M. Lange-Hegermann)

T. Bächler, V. P. Gerdt, M. Lange-Hegermann, D. R.,
Algorithmic Thomas decomposition of algebraic and differential systems,
Implementation

Maple package DifferentialThomas (M. Lange-Hegermann)
http://www.mathb.rwth-aachen.de/go/id/rnab/lidx/1
GNU LPGL license

V. P. Gerdt, M. Lange-Hegermann, D. R.
The MAPLE package TDDS for computing Thomas decompositions of systems of nonlinear PDEs
Computer Physics Communications 234:202–215, 2019
arXiv:1801.09942

DifferentialThomas in Maple 2018 (interface by E. S. Cheb-Terrab)
Overview of the DifferentialThomas Package

Description

The **DifferentialThomas** package implements algebraic and differential elimination algorithms to perform a disjoint decomposition of a system of differential equations and inequations into so-called simple and integrable systems. The definition of simple and integrable systems and the algorithm are derived from Joseph Miller Thomas' work. This decomposition is key for simplifying systems of polynomial differential equations and computing formal power series solutions for them. (See References.)

The main functionality of the package is provided by the `ThomasDecomposition` function, which permits triangularizing a differential equation system so that it can be solved eliminating one variable at a time, simplifying the system with respect to its integrability conditions, or determining its singular cases. Commands are also provided to solve related problems, such as `intersectDecompositions` for computing the intersection of two decompositions, possibly performed on the same differential system but using different rankings, `NormalForm` for deciding membership to a radical differential ideal, and `ReducedForm` for reducing a system with respect to another one. The command for computing formal power series solutions to differential equation systems is `PowerSeriesSolution`. Other commands for analyzing mathematical properties of differential systems or performing algebraic manipulation and related programming are listed below.

For examples illustrating the use of the package’s commands, see the `Examples` section of the `ThomasDecomposition` command and the `DifferentialThomas Examples` page.

For more information about the mathematical terminology used in the help pages of this package, see the `Glossary` page of the `DifferentialAlgebra` package.

**NOTE:** To have any of the `doflow`, `pdsolve` and `PDFtools-cas stalled` commands performing their computations using the `DifferentialThomas` package for differential elimination purposes, pass the keyword `DifferentialThomas` as an extra argument.

Each command in the `DifferentialThomas` package can be accessed by using either the `long form` or the `short form` of the command name in the command calling sequence.

The `DifferentialThomas` package is based on software developed by Markus Lange-Hegermann. The redesign of the interface of `DifferentialThomas` was done by E. S. Cheb-Tenab in Maple 2018.

List of DifferentialThomas Package Commands

- `ComplementOfDecomposition`
- `IntersectDecompositions`
- `PowerSeriesSolution`
- `ThomasDecomposition`
- `Equations`
- `LinearCombination`
- `Ranking`
- `Tools`
- `Inequations`
- `NormalForm`
- `ReducedForm`

List of commands of the Tools subpackage of DifferentialThomas

- `CheckResults`
- `Differentiate`
- `Display`
- `Leader`
- `Reset`
- `RunEncapsulated`
- `Separant`
- `ThomasOptions`
- `Tolet`

Description of the DifferentialThomas package commands

- `ComplementOfDecomposition` returns a Thomas decomposition of the complement of the solution sets given by the input.
- `Equations` returns the equations of a differential system decomposition returned by `ThomasDecomposition`. 

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Examples using the DifferentialThomas Package

- **Consistency check**

- **Computation of Lagrangian constraints** (Eq.13, V.P. Gerdt, D. Robertz. Lagrangian constraints and differential Thomas decomposition. Advances in Applied Mathematics 72, 113-138, 2016)


- **Painleve test for Burgers equations** (Ex.1, Fuding Xie, Yong Chen. An algorithmic method in Painleve analysis of PDE. Computer Physics Communications 154, 197-204, 2004)


- **Singular solutions of ODEs**

- **An example of an ODE not solved using DifferentialAlgebra or DEtools:-rifsimp**


Maple 2018

> restart;
> with(DifferentialThomas);

(1) [ComplementOfDecomposition, Display, Equations, Inequations, IntersectDecompositions, LinearCombination, NormalForm, PowerSeriesSolution, Ranking, ReducedForm, ThomasDecomposition, Tools]

> ivar := [t, x];

(2) ivar := [t, x]

> dvar := [u];

(3) dvar := [u]

> Ranking(ivar, dvar);

ranking

(4)

> L := [diff(u(t, x), t) - 6*u(t, x)*diff(u(t, x), x)+diff(u(t, x), x, x, x), u(t, x)*diff(u(t, x), t, x)-diff(u(t, x), t)*diff(u(t, x), x)];

(5)

> L := \[ \frac{\partial}{\partial t} u(t, x) - 6 u(t, x) \left( \frac{\partial}{\partial x} u(t, x) \right) + \frac{\partial^2}{\partial x^2} u(t, x), u(t, x) \left( \frac{\partial}{\partial t} u(t, x) \right) - \left( \frac{\partial}{\partial x} u(t, x) \right) \left( \frac{\partial}{\partial x} u(t, x) \right) \]\n
> T := ThomasDecomposition(L, [u]);

(6)

T := [DifferentialSystem, DifferentialSystem, DifferentialSystem]

> Display(T[1]);

(7)

\[ 6 u(t, x) \left( \frac{\partial}{\partial x} u(t, x) \right) - \frac{\partial}{\partial x} u(t, x) = 0, \frac{\partial^2}{\partial x^2} u(t, x) = 0, u(t, x) \neq 0 \]

> Display(T[2]);

(8)

\[ \frac{\partial}{\partial t} u(t, x) = 0, 6 u(t, x) \left( \frac{\partial}{\partial x} u(t, x) \right) - \frac{\partial^3}{\partial x^3} u(t, x) = 0, u(t, x) \neq 0, \frac{\partial^2}{\partial x^2} u(t, x) \neq 0 \]

> Display(T[3]);

(9)

\[ u(t, x) = 0 \]
Example

\[ \text{det} \left( \begin{array}{ccc} u_{x,x} & u_{x,y} & u_{y,y} \\ u_{x,y} & u_{y,y} & u_{y,z} \\ u_{x,z} & u_{y,z} & u_{z,z} \end{array} \right) = 0, \quad \mathbb{Q}\{u\} \text{ with degrevlex ranking} \]
Example

\[
\det \begin{pmatrix}
  u_{x,x} & u_{x,y} & u_{y,y} \\
  u_{x,y} & u_{y,y} & u_{y,z} \\
  u_{x,z} & u_{y,z} & u_{z,z}
\end{pmatrix} = 0, \quad \mathbb{Q}\{u\} \text{ with degrevlex ranking}
\]

Thomas decomposition:

\[
\det(\ldots) = 0
\]

\[
u_{z,z} u_{y,y} - u_{y,z}^2 \neq 0
\]

\[
u_{z,z} \neq 0
\]

\[
u_{z,z} u_{x,y} - u_{y,z} u_{x,z} = 0
\]

\[
u_{z,z} u_{y,y} - u_{y,z}^2 = 0
\]

\[
u_{z,z} \neq 0
\]

\[-u_{y,z}^2 u_{x,x} + 2u_{y,z} u_{x,z} u_{x,y} - u_{y,y} u_{x,z}^2 = 0
\]

\[
u_{y,z} \neq 0
\]

\[
u_{z,z} = 0
\]

\[
u_{x,z} = 0
\]

\[
u_{y,y} = 0
\]

\[
u_{y,z} = 0
\]

\[
u_{z,z} = 0
\]

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Example

$$\det \begin{pmatrix} u_{x,x} & u_{x,y} & u_{y,y} \\ u_{x,y} & u_{y,y} & u_{y,z} \\ u_{x,z} & u_{y,z} & u_{z,z} \end{pmatrix} = 0,$$

$\mathbb{Q}\{u\}$ with degrevlex ranking
Example

\[
\det \begin{pmatrix}
  u_{x,x} & u_{x,y} & u_{y,y} \\
  u_{x,y} & u_{y,y} & u_{y,z} \\
  u_{x,z} & u_{y,z} & u_{z,z}
\end{pmatrix} = 0,
\]

\( \mathbb{Q}\{u\} \) with degrevelx ranking

\[
\begin{align*}
u_{z,z} u_{y,y} - u_{y,z}^2 &\neq 0 \\
u_{z,z} &\neq 0 & \Rightarrow (T_1) \\
u_{z,z} & = 0 & \Rightarrow (T_2)
\end{align*}
\]

\[
\begin{align*}
u_{z,z} u_{y,y} - u_{y,z}^2 & = 0 \\
u_{z,z} &\neq 0 & \Rightarrow (T_3) \\
u_{z,z} & = 0 & \Rightarrow (T_5)
\end{align*}
\]

\[
\begin{align*}
u_{y,y} &\neq 0 & \Rightarrow (T_4) \\
u_{y,y} & = 0 & \Rightarrow (T_5)
\end{align*}
\]
Example

\[
\det \begin{pmatrix}
 u_{x,x} & u_{x,y} & u_{y,y} \\
 u_{x,y} & u_{y,y} & u_{y,z} \\
 u_{x,z} & u_{y,z} & u_{z,z}
\end{pmatrix} = 0,
\]

\( \mathbb{Q}\{u\} \) with degrevlex ranking
Example

$$\det \begin{pmatrix} u_{x,x} & u_{x,y} & u_{y,y} \\ u_{x,y} & u_{y,y} & u_{y,z} \\ u_{x,z} & u_{y,z} & u_{z,z} \end{pmatrix} = 0,$$

$\mathbb{Q}\{u\}$ with degrevlex ranking

Differential Algebra (Maple 17):

\begin{align*}
\det(\ldots) &= 0 \\
 u_{z,z}u_{y,y} - u_{y,z}^2 &\neq 0 \\
 u_{z,z}u_{x,y} - u_{y,z}u_{x,z} &= 0 \\
 u_{z,z}u_{y,y} - u_{y,z}^2 &= 0 \\
 u_{z,z} &\neq 0
\end{align*}

\begin{align*}
 u_{x,z} &= 0 \\
 u_{y,z} &= 0 \\
 u_{z,z} &= 0
\end{align*}

\begin{align*}
 u_{x,z} &= 0 \\
 u_{y,y} &= 0 \\
 u_{y,z} &= 0 \\
 u_{z,z} &= 0
\end{align*}
3. Decomposition of nonlinear difference systems
Difference algebra

\textit{Difference algebra} (Ritt, Cohn, Levin, \ldots)

\( \mathbb{Q} \subseteq \tilde{\mathbb{K}} \) a difference field with commuting automorphisms \( \sigma_1, \ldots, \sigma_n \)
Difference algebra

**Difference algebra** (Ritt, Cohn, Levin, ...)

$\mathbb{Q} \subseteq \tilde{K}$ a difference field with commuting automorphisms $\sigma_1, \ldots, \sigma_n$

**Difference polynomial ring** with comm. endomorphisms $\sigma_1, \ldots, \sigma_n$

$\tilde{K}\{\tilde{u}\} := \tilde{K}[\sigma_1^{i_1} \cdots \sigma_n^{i_n} \tilde{u} | i \in (\mathbb{Z}_{\geq 0})^n]$

$\tilde{u}(i_1,\ldots,i_n) := \sigma_1^{i_1} \cdots \sigma_n^{i_n} \tilde{u}$
Difference algebra

*Difference algebra* (Ritt, Cohn, Levin, . . .)

\( \mathbb{Q} \subseteq \tilde{K} \) a difference field with commuting automorphisms \( \sigma_1, \ldots, \sigma_n \)

*Difference polynomial ring* with comm. endomorphisms \( \sigma_1, \ldots, \sigma_n \)

\[ \tilde{K}\{\tilde{u}\} := \tilde{K}[\sigma_1^{i_1} \cdots \sigma_n^{i_n}\tilde{u} \mid i \in (\mathbb{Z}_{\geq 0})^n], \quad \tilde{u}(i_1, \ldots, i_n) := \sigma_1^{i_1} \cdots \sigma_n^{i_n}\tilde{u} \]

\( \tilde{K}\{\tilde{u}\} \) not Noetherian (e.g., \( [\tilde{u}\tilde{u}_1, \tilde{u}\tilde{u}_2, \tilde{u}\tilde{u}_3, \ldots] \subseteq \tilde{K}\{\tilde{u}\} \) not fin. gen.)
Difference algebra (Ritt, Cohn, Levin, . . . )

\[ \mathbb{Q} \subseteq \tilde{K} \] a difference field with commuting automorphisms \( \sigma_1, \ldots, \sigma_n \)

**Difference polynomial ring** with comm. endomorphisms \( \sigma_1, \ldots, \sigma_n \)

\[ \tilde{K}\{\tilde{u}\} := \tilde{K}[\sigma_1^{i_1} \cdots \sigma_n^{i_n}\tilde{u} \mid i \in (\mathbb{Z}_{\geq 0})^n], \quad \tilde{u}(i_1,\ldots,i_n) := \sigma_1^{i_1} \cdots \sigma_n^{i_n}\tilde{u} \]

\( \tilde{K}\{\tilde{u}\} \) not Noetherian (e.g., \([\tilde{u}\tilde{u}_1, \tilde{u}\tilde{u}_2, \tilde{u}\tilde{u}_3, \ldots] \subseteq \tilde{K}\{\tilde{u}\} \) not fin. gen.)

A difference ideal \( \tilde{I} \) of \( \tilde{K}\{\tilde{u}\} \) is said to be

- **reflexive** if \( \sigma_i(\tilde{f}) \in \tilde{I} \) implies \( \tilde{f} \in \tilde{I} \) \( (\tilde{f} \in \tilde{K}\{\tilde{u}\}) \)

- **perfect** if the iteration of the process of forming the difference ideal generated by \( \tilde{f} \) such that \( \sigma_1^{i_1}(\tilde{f})^{k_1} \cdots \sigma_r^{i_r}(\tilde{f})^{k_r} \in \tilde{I} \) yields \( \tilde{I} \)
Thm. (Ritt-Raudenbush).
Every perfect difference ideal of $\tilde{K}\{\tilde{u}^{(1)}, \ldots, \tilde{u}^{(m)}\}$ is finitely generated and is intersection of finitely many prime difference ideals.
Thm. (Ritt-Raudenbush). Every perfect difference ideal of \( \tilde{K}\{\tilde{u}^{(1)}, \ldots, \tilde{u}^{(m)}\} \) is finitely generated and is intersection of finitely many prime difference ideals.

Prop. The vanishing ideal in \( \tilde{K}\{\tilde{u}^{(1)}, \ldots, \tilde{u}^{(m)}\} \) of a set of \( m \)-tuples of functions is a perfect difference ideal.
Difference algebra

**Thm.** (Ritt-Raudenbush).
Every perfect difference ideal of \( \tilde{K}\{\tilde{u}^{(1)}, \ldots, \tilde{u}^{(m)}\} \) is finitely generated and is intersection of finitely many prime difference ideals.

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**Thm.** (Cohn)
Existence of generic zeros of reflexive prime difference ideals in piecewise analytic functions on \( \mathbb{R}_+ \).
Decomposition of nonlinear difference systems

Notation.

Recall

\[ \partial_j u^{(\alpha)}(x) = \frac{\tilde{u}^{(\alpha)}_{k_1,\ldots,k_j+1,\ldots,k_n} - \tilde{u}^{(\alpha)}_{k_1,\ldots,k_j-1,\ldots,k_n}}{2h} + \mathcal{O}(h^2) \]

grid functions \( \tilde{u}^{(\alpha)}_{k_1,\ldots,k_n} \)
Decomposition of nonlinear difference systems

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grid functions \[ \tilde{u}^{(\alpha)}_{k_1,\ldots,k_n} \]

\[ \tilde{K} = \mathbb{Q}(a, h), \text{ a difference field of constants} \]

\[ \tilde{R} = \tilde{K}\{\tilde{u}^{(1)}, \ldots, \tilde{u}^{(m)}\} \text{ difference polynomial ring over } \tilde{K}, \]

with automorphisms \[ \Sigma = \{\sigma_1, \ldots, \sigma_n\} \]
Decomposition of nonlinear difference systems

Notation.

Recall \[ \partial_j u^{(\alpha)}(x) = \frac{\tilde{u}^{(\alpha)}_{k_1,\ldots,k_{j+1},\ldots,k_n} - \tilde{u}^{(\alpha)}_{k_1,\ldots,k_{j-1},\ldots,k_n}}{2h} + \mathcal{O}(h^2) \]

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\( \tilde{K} = \mathbb{Q}(a, h) \), a difference field of constants

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with automorphisms \( \Sigma = \{\sigma_1, \ldots, \sigma_n\} \)

\( \text{Mon}(\Sigma) = \) monomials in \( \sigma_1, \ldots, \sigma_n \)

leader, initial, discriminant, \ldots are defined accordingly w.r.t. a ranking
Decomposition of nonlinear difference systems

\[ \tilde{S} = \{ \tilde{f}_1 = 0, \ldots, \tilde{f}_s = 0, \tilde{f}_{s+1} \neq 0, \ldots, \tilde{f}_{s+t} \neq 0 \} \]

difference system
Decomposition of nonlinear difference systems

\[ \tilde{S} = \{ \tilde{f}_1 = 0, \ldots, \tilde{f}_s = 0, \tilde{f}_{s+1} \neq 0, \ldots, \tilde{f}_{s+t} \neq 0 \} \] difference system

\( \Omega \subseteq \mathbb{R}^n \) open and connected, \( x \in \Omega \)

\( \Gamma_{x,h} := \{ (x_1 + k_1 h, \ldots, x_n + k_n h) \mid k_1, \ldots, k_n \in \mathbb{Z} \}, \quad h > 0 \)

Kolchin seminar, 13/09/2019
Decomposition of nonlinear difference systems

\[ \tilde{S} = \{ \tilde{f}_1 = 0, \ldots, \tilde{f}_s = 0, \tilde{f}_{s+1} \neq 0, \ldots, \tilde{f}_{s+t} \neq 0 \} \quad \text{difference system} \]

\( \Omega \subseteq \mathbb{R}^n \) open and connected, \( \mathbf{x} \in \Omega \)

\[ \Gamma_{\mathbf{x},h} := \{ (x_1 + k_1 h, \ldots, x_n + k_n h) \mid k_1, \ldots, k_n \in \mathbb{Z} \}, \quad h > 0 \]

\[ \mathcal{F}_{\Omega,\mathbf{x},h} := \{ \tilde{u} : \Gamma_{\mathbf{x},h} \rightarrow \mathbb{C} \mid \text{\( \tilde{u} \) is restriction to \( \Gamma_{\mathbf{x},h} \) of some locally analytic function \( u \) on \( \Omega \)} \} \]
Decomposition of nonlinear difference systems

\[ \tilde{S} = \{ \tilde{f}_1 = 0, \ldots, \tilde{f}_s = 0, \tilde{f}_{s+1} \neq 0, \ldots, \tilde{f}_{s+t} \neq 0 \} \text{ difference system} \]

\( \Omega \subseteq \mathbb{R}^n \) open and connected, \( x \in \Omega \)

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\[ \mathcal{F}_{\Omega,x,h} := \{ \tilde{u} : \Gamma_{x,h} \to \mathbb{C} \ | \ \tilde{u} \text{ is restriction to } \Gamma_{x,h} \text{ of some locally analytic function } u \text{ on } \Omega \} \]

\[ \text{Sol}_{\Omega,x,h} := \{ (\tilde{u}_1, \ldots, \tilde{u}_m) \in \mathcal{F}_{\Omega,x,h}^m \ | \ \tilde{f}_1(\tilde{u}) = 0, \ldots, \tilde{f}_s(\tilde{u}) = 0, \]
\[ \tilde{f}_{s+1}(\tilde{u}) \neq 0, \ldots, \tilde{f}_{s+t}(\tilde{u}) \neq 0 \} \]
Decomposition of nonlinear difference systems

\[ \tilde{S} = \{ \tilde{f}_1 = 0, \ldots, \tilde{f}_s = 0, \tilde{f}_{s+1} \neq 0, \ldots, \tilde{f}_{s+t} \neq 0 \} \]

Def. *Difference decomposition* of difference system \( \tilde{S} \)

\[ \text{Sol}_{\Omega, x, h}(\tilde{S}) = \text{Sol}_{\Omega, x, h}(\tilde{S}_1) \uplus \ldots \uplus \text{Sol}_{\Omega, x, h}(\tilde{S}_r), \quad \tilde{S}_i \quad \text{quasi-simple} \]
Decomposition of nonlinear difference systems

\[ \tilde{S} = \{ \tilde{f}_1 = 0, \ldots, \tilde{f}_s = 0, \tilde{f}_{s+1} \neq 0, \ldots, \tilde{f}_{s+t} \neq 0 \} \]

Def. Difference decomposition of difference system \( \tilde{S} \)

\[ \text{Sol}_{\Omega, x, h}(\tilde{S}) = \text{Sol}_{\Omega, x, h}(\tilde{S}_1) \cup \ldots \cup \text{Sol}_{\Omega, x, h}(\tilde{S}_r), \quad \tilde{S}_i \text{ quasi-simple} \]

Def. \( \tilde{S} \) is (quasi-) simple if

(a) \( \tilde{f}_1, \ldots, \tilde{f}_s, \tilde{f}_{s+1}, \ldots, \tilde{f}_{s+t} \) have pairwise distinct leaders,

(b) initials and (not nec.) discriminants of \( \tilde{f}_i \) do not vanish, \( 1 \leq i \leq s + t \),

(c) \( \tilde{f}_1, \ldots, \tilde{f}_s \) form a passive difference system,

(d) \( \tilde{f}_{s+1}, \ldots, \tilde{f}_{s+t} \) are reduced modulo \( \tilde{f}_1, \ldots, \tilde{f}_s \).

set of admissible automorphisms \( \mu_i \subseteq \{ \sigma_1, \ldots, \sigma_n \} \) for \( \tilde{f}_i \), \( i = 1, \ldots, s \)
Decomposition of nonlinear difference systems

\[ \tilde{R} = \tilde{K}\{\tilde{u}^{(1)}, \ldots, \tilde{u}^{(m)}\} \]

Prop. 6.3  \[ \tilde{S} = \{ \tilde{f}_1 = 0, \ldots, \tilde{f}_s = 0, \tilde{f}_{s+1} \neq 0, \ldots, \tilde{f}_{s+t} \neq 0 \} \] quasi-simple

\[ E \] difference ideal generated by \[ \tilde{f}_1, \ldots, \tilde{f}_s \]

\[ Q \] smallest subset of \[ \tilde{R} \] containing \[ q_1 := \text{init}(\tilde{f}_1), \ldots, q_s := \text{init}(\tilde{f}_s) \], which is multiplicatively closed and closed under \[ \sigma_1, \ldots, \sigma_n \]
Decomposition of nonlinear difference systems

\[ \tilde{R} = \tilde{K}\{ \tilde{u}^{(1)}, \ldots, \tilde{u}^{(m)} \} \]

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\( E \) difference ideal generated by \( \tilde{f}_1, \ldots, \tilde{f}_s \)

\( Q \) smallest subset of \( \tilde{R} \) containing \( q_1 := \text{init}(\tilde{f}_1), \ldots, q_s := \text{init}(\tilde{f}_s) \), which is multiplicatively closed and closed under \( \sigma_1, \ldots, \sigma_n \)

Then the elements of

\[ E : Q := \{ \tilde{f} \in \tilde{R} \mid (\theta_1(q_1))^{r_1} \cdots (\theta_s(q_s))^{r_s} \tilde{f} \in E \text{ for some } \theta_1, \ldots, \theta_s \in \text{Mon}(\Sigma), r_1, \ldots, r_s \in \mathbb{Z}_{\geq 0} \} \]

are the difference polynomials reducing to zero modulo \( \tilde{f}_1, \ldots, \tilde{f}_s \).
Input: $L \subset \tilde{R} \setminus \tilde{K}$ finite and a ranking $\succ$ on $\tilde{R}$ such that $L = \tilde{S} = \text{for some finite difference system } \tilde{S}$ which is quasi-simple as an algebraic system (in the finitely many indeterminates $(u_k)_J$ which occur in it, totally ordered by $\succ$, not necessarily square-free)

Output: $a \in \{\text{true, false}\}$ and $L' \subset \tilde{R} \setminus \tilde{K}$ finite such that

$$\langle L' \rangle : Q = \langle L \rangle : Q$$

where $Q$ is the smallest multiplicatively closed subset of $\tilde{R}$ containing all $\text{init}(\theta \tilde{f})$, where $\tilde{f} \in L$ and $\theta \in \text{ld}(L \setminus \{\tilde{f}\}) : \text{ld}(\tilde{f})$, and which is closed under $\sigma_1, \ldots, \sigma_s$; and, in case $a = \text{true}$, there exist no $\tilde{f}_1, \tilde{f}_2 \in L'$, $\tilde{f}_1 \neq \tilde{f}_2$, such that we have $v := \text{ld}(\tilde{f}_1) = \theta \text{ld}(\tilde{f}_2)$ for some $\theta \in \text{Mon}(\Sigma)$ and $\deg_v(\tilde{f}_1) \geq \deg_v(\theta \tilde{f}_2)$

Algorithm:

1. $L' \leftarrow L$
2. while there exist $\tilde{f}_1, \tilde{f}_2 \in L'$, $\tilde{f}_1 \neq \tilde{f}_2$ and $\theta \in \text{Mon}(\Sigma)$ such that we have $v := \text{ld}(\tilde{f}_1) = \theta \text{ld}(\tilde{f}_2)$ and $\deg_v(\tilde{f}_1) \geq \deg_v(\theta \tilde{f}_2)$ do
3. $L' \leftarrow L' \setminus \{\tilde{f}_1\}; \quad v \leftarrow \text{ld}(\tilde{f}_1)$
4. $\tilde{r} \leftarrow \text{init}(\theta \tilde{f}_2) \cdot \tilde{f}_1 - \text{init}(\tilde{f}_1) \cdot v^d \cdot \theta \tilde{f}_2$, where $d := \deg_v(\tilde{f}_1) - \deg_v(\theta \tilde{f}_2)$
5. if $\tilde{r} \neq 0$ then
6. return (false, $L' \cup \{\tilde{r}\}$)
7. end if
8. end while
9. return (true, $L'$)
Input: $\tilde{r} \in \tilde{R}$, $T = \{ (\tilde{f}_1, \mu_1), (\tilde{f}_2, \mu_2), \ldots, (\tilde{f}_s, \mu_s) \}$, and a ranking $\succ$ on $\tilde{R}$, where $T$ is Janet complete (with respect to $\succ$)

Output: $\tilde{r}' \in \tilde{R}$ and an element $b$ of the multiplicatively closed set generated by

$$\bigcup_{i=1}^{s} \{ \theta \text{ init}(\tilde{f}_i) \mid \theta \in \text{Mon}(\Sigma), \text{ld}(\tilde{r}) \succ \theta \text{ ld}(\tilde{f}_i) \} \cup \{1\}$$

such that $\tilde{r}'$ is Janet reduced modulo $T$, and such that $\tilde{r}' = \tilde{r}$, $b = 1$ if $T = \emptyset$, and

$$\tilde{r}' + \langle \tilde{f}_1, \ldots, \tilde{f}_s \rangle = b \cdot \tilde{r} + \langle \tilde{f}_1, \ldots, \tilde{f}_s \rangle$$

otherwise

Algorithm:

1: $\tilde{r}' \leftarrow \tilde{r}$; $b \leftarrow 1$
2: if $\tilde{r}' \not\in \tilde{K}$ then
3: \hspace{1em} $v \leftarrow \text{ld}(\tilde{r}')$
4: \hspace{1em} while $\tilde{r}' \not\in \tilde{K}$, $\exists (f, \mu) \in T$, $\theta \in \text{Mon}(\mu) : v = \theta \text{ ld}(f)$, $\text{deg}_v(\tilde{r}') \geq \text{deg}_v(\theta \tilde{f})$ do
5: \hspace{2em} $\tilde{r}' \leftarrow \text{init}(\theta \tilde{f}) \cdot \tilde{r}' - \text{init}(\tilde{r}') \cdot v^d \cdot \theta \tilde{f}$, where $d := \text{deg}_v(\tilde{r}') - \text{deg}_v(\theta \tilde{f})$
6: \hspace{1em} $b \leftarrow \text{init}(\theta \tilde{f}) \cdot b$
7: \hspace{1em} end while
8: \hspace{1em} for each coefficient $\tilde{c}$ of $\tilde{r}'$ (as a polynomial in $v$) do
9: \hspace{2em} ($\tilde{r}''$, $b'$) $\leftarrow$ Janet-reduce($\tilde{c}$, $T$, $\succ$)
10: \hspace{2em} replace the coefficient $b' \cdot \tilde{c}$ in $b' \cdot \tilde{r}'$ with $\tilde{r}''$ and replace $\tilde{r}'$ with this result
11: \hspace{1em} $b \leftarrow b' \cdot b$
12: \hspace{1em} end for
13: end if
14: return $(\tilde{r}', b)$
**Input:** A finite difference system $\tilde{S}$ over $\tilde{R}$, a ranking $\succ$ on $\tilde{R}$, and a total ordering on $\Sigma$ (used by Decompose)

**Output:** A difference decomposition of $\tilde{S}$

**Algorithm:**

1. $Q \leftarrow \{\tilde{S}\}; \ T \leftarrow \emptyset$
2. repeat
3. choose $L \in Q$ and remove $L$ from $Q$
4. compute a decomposition $\{A_1, \ldots, A_r\}$ of $L$, considered as an algebraic system, into quasi-simple systems
5. for $i = 1, \ldots, r$ do
6. if $A_i = \emptyset$ then // no equation and no inequation
7. return $\{\emptyset\}$
8. else
9. $(a, G) \leftarrow$ Auto-reduce($A_i =\succ$)
10. if $a = \text{true}$ then
11. ... 
12. else
13. insert $\{\tilde{f} = 0 \mid \tilde{f} \in G\} \cup \{\tilde{g} \neq 0 \mid \tilde{g} \in A_i \neq\}$ into $Q$
14. end if
15. end if
16. end for
17. until $Q = \emptyset$
18. return $T$
**Input:** A finite difference system $\tilde{S}$ over $\tilde{R}$, a ranking $\succ$ on $\tilde{R}$, and a total ordering on $\Sigma$ (used by Decompose)

**Output:** A decomposition of $\tilde{S}$

**Algorithm:**

1: ... 
9: $(a, G) \leftarrow \text{Auto-reduce}(A_i^=, \succ)$
10: if $a = \text{true}$ then
11: $J \leftarrow \text{Decompose}(G)$
12: $P \leftarrow \{ \text{NF}(\sigma \tilde{f}, J, \succ) \mid (\tilde{f}, \mu) \in J, \sigma \in \overline{\mu} \}$
13: if $P \subseteq \{0\}$ then // $J$ is passive
14: replace each inequation $\tilde{g} \neq 0$ in $A_i$ with $\text{NF}(\tilde{g}, J, \succ) \neq 0$
15: if $0 \not\in A_i^\neq$ then
16: insert $\{ \tilde{f} = 0 \mid (\tilde{f}, \mu) \in J \} \cup \{ \tilde{g} \neq 0 \mid \tilde{g} \in A_i^\neq \}$ into $T$
17: end if
18: else if $P \cap \tilde{K} \subseteq \{0\}$ then
19: insert $\{ \tilde{f} = 0 \mid (\tilde{f}, \mu) \in J \} \cup \{ \tilde{f} = 0 \mid \tilde{f} \in P \setminus \{0\} \} \cup \{ \tilde{g} \neq 0 \mid \tilde{g} \in A_i^\neq \}$ into $Q$
20: end if
21: else
22: insert $\{ \tilde{f} = 0 \mid \tilde{f} \in G \} \cup \{ \tilde{g} \neq 0 \mid \tilde{g} \in A_i^\neq \}$ into $Q$
23: end if
24: ...
**Input:** A simple differential system $S$ over $R$, a ranking $\succ$ on $R$, a ranking $\succ$ on $\tilde{R}$, a total ordering on $\Sigma$ (used by Decompose) and a difference system $\tilde{S}$ consisting of equations that are $w$-consistent with $S$

**Output:** $\tilde{L} = \{ (\tilde{L}_1, b_1), \ldots, (\tilde{L}_r, b_r) \}$, where $\tilde{L}_i$ is $s$-consistent (resp. $w$-consistent) with $L_i \leftarrow h \to 0 \tilde{L}_i$ if $b_i = true$ (resp. $false$)

**Algorithm:**
1. $\tilde{L} = \{\tilde{L}_1, \ldots, \tilde{L}_k\} \leftarrow \text{DifferenceDecomposition}(\tilde{S}, \succ)$
2. for $i = 1, \ldots, k$ do
3. if $\exists \tilde{f} \in \tilde{L}^\neq_i$ such that $\tilde{f} \succ f \in [S^\neq]$ then
4. $\tilde{L} \leftarrow \tilde{L} \setminus \{\tilde{L}_i\}$
5. else
6. $b_i \leftarrow true$
7. for $\tilde{f} \in \tilde{L}^= \text{ do}$
8. compute $f \in R$ such that $\tilde{f} \succ f$
9. if $\text{NF}(f, S^=, \succ) \neq 0$ then
10. $b_i \leftarrow false$; break
11. end if
12. end for
13. end if
14. end for
15. return $\{ (\tilde{L}_i, b_i) \mid \tilde{L}_i \in \tilde{L} \}$
Example

Example.

\[(\ast)\left\{\begin{array}{l}
\frac{\partial u}{\partial x} - u^2 = 0 \\
\frac{\partial u}{\partial y} + u^2 = 0
\end{array}\right., \quad u = u(x, y)\]

\[
\left\{\begin{array}{l}
D_1^+ \tilde{u} - \tilde{u}^2 = 0 \quad (A) \\
D_2^+ \tilde{u} + \tilde{u}^2 = 0 \quad (B)
\end{array}\right. \quad D_1^+, \ D_2^+ \text{ forward differences}
\]

\[\sigma_2 A - \sigma_1 B + (\ldots) A + (\ldots) B = -2h^3 u_{i,j}^4\]

\[
\left\{\begin{array}{l}
D_1^+ \tilde{u} - \tilde{u}^2 = 0 \quad (A') \\
D_2^- \tilde{u} + \tilde{u}^2 = 0 \quad (B')
\end{array}\right. \quad D_2^- \text{ backward difference}
\]

is s-consistent with (\ast).
Example

Navier-Stokes equations for an incompressible flow of a constant viscosity fluid:

\[
\begin{align*}
\mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p - \mu \Delta \mathbf{u} &= 0 \\
\nabla \cdot \mathbf{u} &= 0
\end{align*}
\]
Example

Navier-Stokes equations for an incompressible flow of a constant viscosity fluid:

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{du}{dt} + (u \cdot \nabla) u + \nabla p - \mu \Delta u &= 0 \\
\nabla \cdot u &= 0
\end{array} \right.
\end{align*}
\]

ranking TOP-lex with \( \partial_t > \partial_1 > \partial_2 > \partial_3, \quad p > u > v > w; \)

passivity:

\[\Delta p + \nabla \cdot (u \cdot \nabla) u = 0\]
Navier-Stokes equations for an incompressible flow of a constant viscosity fluid:

\[
\begin{align*}
\begin{cases}
\mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p - \mu \Delta \mathbf{u} &= 0 \\
\nabla \cdot \mathbf{u} &= 0
\end{cases}
\end{align*}
\]

ranking TOP-lex with \( \partial_t > \partial_1 > \partial_2 > \partial_3, \quad p > u > v > w \); passivity:

\[
(*) \quad \Delta p + \nabla \cdot (\mathbf{u} \cdot \nabla) \mathbf{u} = 0
\]

FDA:

\[
D_t \tilde{\mathbf{u}} + (\tilde{\mathbf{u}} \cdot \mathbf{D}) \tilde{\mathbf{u}} + \mathbf{D} \tilde{p} - \mu \tilde{\Delta} \tilde{\mathbf{u}} = 0,
\]

where \( D_t \approx \partial_t, \quad \mathbf{D} = (D_1, D_2, D_3) \approx \nabla, \quad \tilde{\Delta} \approx \Delta. \)
Example

Navier-Stokes equations for an incompressible flow of a constant viscosity fluid:

\[
\begin{align*}
\{ & u_t + (u \cdot \nabla) u + \nabla p - \mu \Delta u = 0 \\
& \nabla \cdot u = 0
\end{align*}
\]

ranking TOP-lex with \( \partial_t > \partial_1 > \partial_2 > \partial_3, \quad p > u > v > w \); passivity:

\[(*) \quad \Delta p + \nabla \cdot (u \cdot \nabla) u = 0\]

FDA:

\[
D_t \tilde{u} + (\tilde{u} \cdot D) \tilde{u} + D \tilde{p} - \mu \tilde{\Delta} \tilde{u} = 0,
\]

where \( D_t \approx \partial_t, \quad D = (D_1, D_2, D_3) \approx \nabla, \quad \tilde{\Delta} \approx \Delta \).

ranking TOP-lex with \( \sigma_t \succ \sigma_1 \succ \sigma_2 \succ \sigma_3, \quad \tilde{u} \succ \tilde{u} \succ \tilde{v} \succ \tilde{w} \); passivity:

\[(**) \quad (D \cdot D) \tilde{p} + D \cdot (\tilde{u} \cdot D) \tilde{u} = 0\]
**Example**

Navier-Stokes equations for an incompressible flow of a constant viscosity fluid:

\[
\begin{align*}
\{ & u_t + (u \cdot \nabla) u + \nabla p - \mu \Delta u = 0 \\
\n& \nabla \cdot u = 0
\end{align*}
\]

ranking TOP-lex with \( \partial_t > \partial_1 > \partial_2 > \partial_3 \), \( p > u > v > w \); passivity:

\[(*) \quad \Delta p + \nabla \cdot (u \cdot \nabla) u = 0\]

FDA:

\[
D_t \tilde{u} + (\tilde{u} \cdot D) \tilde{u} + D \tilde{p} - \mu \Delta \tilde{u} = 0,
\]

where \( D_t \) approx. \( \partial_t \), \( D = (D_1, D_2, D_3) \) approx. \( \nabla \), \( \tilde{\Delta} \) approx. \( \Delta \).

ranking TOP-lex with \( \sigma_t \succ \sigma_1 \succ \sigma_2 \succ \sigma_3 \), \( \tilde{u} \succ \tilde{u} \succ \tilde{v} \succ \tilde{w} \); passivity:

\[(**) \quad (D \cdot D) \tilde{p} + D \cdot (\tilde{u} \cdot D) \tilde{u} = 0\]

We have \( (** \succ *) \) modulo PDE system; FDA is s-consistent.
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