Local Definability of Holomorphic Functions

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Abstract

Local Definability of Holomorphic Functions

Given a collection $\mathcal{F}$ of complex or real analytic functions, one can ask what other functions are obtainable from them by finitary algebraic operations. If we just mean polynomial operations we get some field of functions.

If we include as algebraic operations such things as taking implicit functions, maybe in several variables, we get a much more interesting framework, which is closely related to the theory of local definability in an o-minimal setting, starting with suitable restrictions of the functions in $\mathcal{F}$.

O-minimality is a setting for tame topology of real- or complex-analytic functions which does not allow for “bad” singularities. However some more tame singularities can occur. In this talk I will explain work showing what singularities we have to consider to get a characterisation of the locally definable functions in terms of complex analytic operations.

Ax’s theorem on the differential algebra version of Schanuel’s conjecture is important to give one counterexample, and also for some applications to exponential and elliptic functions.

This is joint work with Gareth Jones, Olivier Le Gal, and Tamara Servi.
Outline

1. Motivation

2. Local Definability

3. Wilkie’s conjecture and theorem

4. Counterexamples to Wilkie’s conjecture

5. The current state of play
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Motivation 1

Motivating question
Given some complex analytic functions, what other analytic functions can be obtained from them by “local algebraic” operations? We need a robust notion of “algebraic”.

Example
Starting with \( \exp \), define \( f(z) = \exp(\exp(z)) \). \( f \) is transcendental over \( \mathbb{C}(z, \exp(z)) \), so \( f \) is not algebraic in the sense of a field of functions. Composition of functions is a different sort of algebraic operation we want to include. Likewise inverse and implicit functions. From \( \exp \) we get the elementary functions.

Idea
Use some sort of model-theoretic definability as our notion of “algebraic”. Definable means there is a (finite) formula which defines the graph of the function.
Motivation 2

Complex sine is definable from \( \exp \): \[
\sin(z) = \frac{\exp(iz) - \exp(-iz)}{2i}.
\]
However the real case is different.

Theorem (Bianconi, 1997)

*In the model-theoretic structure* \( \mathbb{R}_{\exp} = \langle \mathbb{R}; +, -, \cdot, 0, 1, \exp \rangle \), *no restriction of the real sine function to an interval is definable.*

Example (Some converse is false)

In \( \langle \mathbb{R}; +, -, \cdot, 0, 1, \sin \rangle \), \( 2\pi\mathbb{Z} \) is setwise definable, as is \( \mathbb{Z} \), and this allows us to code sums of infinite series definably. Much of analysis of real and complex functions becomes definable. In particular, real and complex \( \exp \) are definable.

Theorem (Better converse - Bianconi 1997)

*In* \( \langle \mathbb{R}; +, -, \cdot, 0, 1, \{\sin\}_{a < b} \rangle \), *no restriction of real \( \exp \) to an interval is definable.*

Bianconi’s proofs use Ax’s differential algebra version of Schanuel’s conjecture.
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Definability

In this talk, a model-theoretic structure is always of the form \( \mathbb{R}_F = \langle \mathbb{R}; +, -, \cdot, 0, 1, F \rangle \) where \( F \) is a collection of analytic functions \( f : U \rightarrow \mathbb{R} \) for open sets \( U \subseteq \mathbb{R}^n \). (Both \( U \) and \( n \) can depend on \( f \).)

Complex case

We deal with both the real and complex cases. For the complex case, we identify \( \mathbb{C} \) with \( \mathbb{R}^2 \). Then take the real and imaginary parts of holomorphic functions.

Definition (Definable sets and functions)

A subset of \( \mathbb{R}^n \) is then \textbf{definable} if it can be obtained from the graphs of the functions in \( F \), with +, \cdot, by the logical operations: intersection, union, complement, and projection. They correspond to conjunction, disjunction, negation, and existential quantification over the field.
A function is definable if its graph is definable.
Model theory begins with an audacious idea: to consider statements about mathematical structures as mathematical objects of study in their own right. While inherently important as a tool of mathematical logic, it also enjoys connections to and applications in diverse branches of mathematics, including algebra, number theory and analysis. Despite this, traditional introductions to model theory assume a graduate-level background of the reader. In this innovative textbook, Jonathan Kirby brings model theory to an undergraduate audience. The highlights of basic model theory are illustrated through examples from specific structures familiar from undergraduate mathematics, paying particular attention to definable sets throughout. With numerous exercises of varying difficulty, this is an accessible introduction to model theory and its place in mathematics.

Jonathan Kirby is a Senior Lecturer in Mathematics at the University of East Anglia. A student of Boris Zilber, his main research is in model theory and its interactions with algebra, number theory, and analysis, with particular interest in exponential functions. He has taught model theory at the University of Oxford, the University of Illinois, Chicago, and the University of East Anglia.
Tame topology: o-minimality

Towards an appropriate notion

If $\mathcal{F} \ni \sin$ then too many functions are definable for the notion to be interesting. However, if we restrict to the case where $\mathcal{F}$ contains only analytic functions on bounded domains, without approaching singularities, then we are in the o-minimal setting where definability is very much tamer.

Definition

Given $U \subseteq \mathbb{R}^n$, open, $f : U \to \mathbb{R}$ analytic, a proper restriction of $f$ is $f \upharpoonright \Delta$, where $\Delta$ is an open box with rational corners such that $\bar{\Delta} \subseteq U$. (No singularities on the boundary of $\Delta$.)

$\text{PR}(\mathcal{F})$ is the set of all proper restrictions of all functions in $\mathcal{F}$.

Definition

$\mathbb{R}_{\text{an}}$ is the case of $\mathbb{R}_{\text{PR}(\mathcal{F})}$ where $\mathcal{F}$ is all analytic functions.

Theorem (Essentially due to Gabrielov)

$\mathbb{R}_{\text{an}}$ is o-minimal.
Local definability

Definition

Let $\mathcal{F}$ be a collection of analytic functions.
$\text{PR}(\mathcal{F})$ is the set of all proper restrictions of functions in $\mathcal{F}$.
$\mathbb{R}_{\text{PR}(\mathcal{F})}$ is the structure consisting of the real field equipped with (the real and imaginary parts of) all the proper restrictions of functions in $\mathcal{F}$.

Definition

A function $f : U \to \mathbb{R}$ (or $\mathbb{C}$) is locally definable if and only if all (the real and imaginary parts of) its proper restrictions are definable.
Equivalently, for every $a \in U$ there is $U_a \ni a$ such that $f|_{U_a}$ is definable.

Definition

A function $g$ is locally definable from $\mathcal{F}$ if it is locally definable in $\mathbb{R}_{\text{PR}(\mathcal{F})}$.

Problem – real and complex versions

For arbitrary $\mathcal{F}$, characterise the analytic $g$ which are locally definable from $\mathcal{F}$ using analytic/algebraic operations, rather than logical ones.
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Wilkie’s conjecture

Notation

\[ \text{LD}(\mathcal{F}) = \{ \text{analytic functions which are locally definable from } \mathcal{F} \} . \]

Observations

\[ \text{LD}(\mathcal{F}) \text{ is closed under the following operations:} \]

- All polynomials in \( \mathbb{Q}[\bar{X}] \) are in \( \text{LD}(\mathcal{F}) \).
- Partial differentiation (using the usual \( \epsilon-\delta \) definition)
- Composition of functions
- Implicit definition (using the implicit function theorem)
- (In complex case) \( i \in \text{LD}(\mathcal{F}) \) and Schwarz reflection: \( f^{SR}(z) = \overline{f(\bar{z})} \)

Define \( \tilde{\mathcal{F}} \) to be the closure of \( \mathcal{F} \) under these operations.

Conjecture (Wilkie, 2005, published 2008)

These operations are enough to characterise \( \text{LD}(\mathcal{F}) \). That is, \( \text{LD}(\mathcal{F}) = \tilde{\mathcal{F}} \).
Wilkie’s theorem

Theorem (Wilkie, 2005, published 2008)

At generic points in $\mathbb{R}^n$ or $\mathbb{C}^n$ (most points), the conjecture is true. That is, if $g : U \to \mathbb{R}$ (or $\mathbb{C}$) is definable in $\mathbb{R}_{PR(\mathcal{F})}$ and $a \in U$ is generic then there is $U_a \ni a$ such that $g \upharpoonright U_a \in \tilde{\mathcal{F}}$. 
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Counterexample 1: Removable singularities

Let $\mathcal{F} = \{\exp\}$ and $g(z) = \begin{cases} \frac{e^z - 1}{z}, & z \neq 0 \\ 1, & z = 0 \end{cases}$. So $g \in \text{LD}(\mathcal{F})$.

Theorem

$g \notin \tilde{\mathcal{F}}$. However it can be obtained if we allow monomial division.

Lemma

Suppose $\mathcal{F}$ contains the polynomials and is closed under partial differentiation and (in the complex case) Schwarz reflection. Then $g \in \tilde{\mathcal{F}}$ iff $g$ is the implicit function of some system of equations using functions from $\mathcal{F}$.

Proof idea for theorem

Replace $\mathcal{F}$ by $\mathcal{F}_1$, the set of all polynomials in variables $\bar{z}$ and their exponentials. Then $\mathcal{F}_1$ is closed under partial differentiation and Schwarz reflection. We use Ax’s theorem to show that at 0, $f$ cannot be the implicit function of any system of equations using functions from $\mathcal{F}_1$. 
Counterexample 2: Deramification

Let $g(z)$ be holomorphic, $f(z) = g(z^2)$, and $\mathcal{F} = \{f(z)\}$. $g$ is the square deramification of $f$. We have $g \in \text{LD}(f)$, defined by

$$g(z) = w \iff \exists x [x^2 = z \land f(x) = w]$$

which is well-defined for this $f$.

Theorem

If $g$ is a sufficiently generic function, then no restriction of $g$ to a neighbourhood of 0 is obtainable from $f$ by the previous operators, including monomial division.

Proof ideas.

1. The operator of square deramification is not equal to any operator built from the previous operators.
2. If $g$ is strongly transcendental then distinct operators applied to it cannot give rise to the same function.
3. Strongly transcendental holomorphic functions exist.
Germs of holomorphic functions at 0 are determined by their Taylor expansions. The operators can be considered as operators on germs, hence on formal power series.

Suppose \( g = \mathcal{L}(f_1, \ldots, f_n) \) with the \( f_i \) in \( \mathcal{F} \), and \( \mathcal{L} \) is some operator built from polynomials, partial differentiation, composition, implicit definition, Schwarz reflection, and monomial division.

Define \( d_\mathcal{L} : \mathbb{N} \to \mathbb{N} \) by, for any power series \( h_1, \ldots, h_n \), \( d_\mathcal{L}(n) \) is the greatest number such that to compute the coefficients of degree \( \leq n \) in the Taylor series of \( \mathcal{L}(h_1, \ldots, h_n) \), it is enough to know the coefficients of degree \( \leq d_\mathcal{L}(n) \) of \( h_1, \ldots, h_n \).

If \( N \) is the number of times partial differentiation and monomial division are used, we have \( d_\mathcal{L}(n) \leq n + N \).

However, if \( \mathcal{L} \) is the square deramification operator \( \mathcal{L}(f)(z) = f(z^{1/2}) \), we have \( d_\mathcal{L}(n) = 2n \).
Strongly transcendental functions (due to Le Gal)

**Definition**

Suppose $U \subseteq \mathbb{C}$, $f : U \to \mathbb{C}$, $a = (a_1, \ldots, a_n) \in U^n$, $k \in \mathbb{N}$.

The multi-jet $j^k_n f(a)$ is the $n(k + 1)$-tuple of all derivatives $\frac{d^i}{dz^i} f(a_l)$ for $i = 0, \ldots, k$, $l = 1, \ldots, n$.

The function $f$ is strongly transcendental if for every $n, k \in \mathbb{N}$ and every $a \in U^n$ we have

$$\text{td}_\mathbb{Q} \mathbb{Q}(j^k_n f(a), j^k_n f(a)) \geq 2nk$$

Note: we could take all $a_l \in \mathbb{Q}$, so this is the strongest transcendence condition we could ask for on the derivatives of $f$.

Strongly transcendental implies differentially transcendental, but is a much stronger condition, also with number-theoretic transcendence content.

**Lemma**

One can show that in a suitable topology on a space of holomorphic functions, the strongly transcendental functions are residual. In particular, by the Baire category theorem, they exist.
Counterexample 3: Blowing down

**Definition**

The blow-up of $0 \in \mathbb{C}^2$ is the map $\pi : V \to \mathbb{C}^2$, where $V = \{(z, p) \in \mathbb{C}^2 \times \mathbb{P}_1(\mathbb{C}) \mid z \in p\}$ and $\pi(z, p) = z$.

If $U \subseteq \mathbb{C}^2$ and $f : U \to \mathbb{C}$, then $f \circ \pi : \pi^{-1}(U) \to \mathbb{C}$ is the blow-up of $f$ and $f$ is the blow-down of $f \circ \pi$.

Thanks to the compactness of the fibres of $\pi$, blow-downs preserve local definability.

**Key idea**

Blow-downs are not local operators, but all previous operators are (including monomial division and deramification).

That is, if $L$ is one of the previous operators, and $g = L(f)$, then to obtain the germ of $g$ at 0 it is enough to have the germ of $f$ at finitely many points $a_1, \ldots, a_k$.

However, if $g$ is strongly transcendental and $f = g \circ \pi$ then to obtain the germ of $g$ at 0, one needs the germ of $f$ at all points of $\pi^{-1}(0)$.

Here the strongly transcendental function is a function of 2 variables, but the definition is easily modified.
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Recall: for $\mathcal{F}$ be a collection of analytic functions

$\tilde{\mathcal{F}}$ is the closure of $\mathcal{F}$ under the following operations:
- All polynomials in $\mathbb{Q}[\tilde{X}]$ are in $\text{LD}(\mathcal{F})$.
- partial differentiation
- implicit definition
- (in complex case) $i \in \tilde{\mathcal{F}}$ and Schwarz reflection: $f^{SR}(z) = \overline{f(\overline{z})}$

New operations

Let $\tilde{\tilde{\mathcal{F}}}$ be the closure of $\mathcal{F}$ under the above operations and:
- Monomial division: $f(z) \mapsto f(z)/z^n$
- Deramification: $f \mapsto f(z^{1/r})$
- Blow downs

Apply each operation only when the result is a well-defined analytic function.

Conjecture (Revised Conjecture)

For any set $\mathcal{F}$ of analytic functions, $\tilde{\tilde{\mathcal{F}}} = \text{LD}(\mathcal{F})$. 
A revised conjecture

New operations

Let $\tilde{F}$ be the closure of $F$ under the above operations and:

- Monomial division: $f(z) \mapsto f(z)/z_1$
- Deramification: $f \mapsto f(z^{1/r})$
- Blow downs

Apply each operation only when the result is a well-defined analytic function. These operations are those needed for resolution of singularities in algebraic/tame analytic settings.

Conjecture (Revised Conjecture)

For any set $F$ of analytic functions, $\tilde{F} = LD(F)$.

Work in progress by Le Gal, Servi, Vieillard Baron to prove the conjecture in the real case. The complex case is open.
References

James Ax.
On Schanuel’s conjectures.

Gareth Jones, Jonathan Kirby, and Tamara Servi.
Local interdefinability of Weierstrass elliptic functions.

Gareth Jones, Jonathan Kirby, Olivier Le Gal, and Tamara Servi.
On local definability of holomorphic functions.

A.J. Wilkie
Some local definability theory for holomorphic functions
Thank you for your attention!
Application to exponential and elliptic functions
Elliptic curves and $\wp$-functions

Weierstrass $\wp$-functions

Take $\Lambda = \mathbb{Z} + \tau \mathbb{Z} \subseteq \mathbb{G}_a(\mathbb{C})$ with $\tau \in \mathbb{C} \setminus \mathbb{R}$. The quotient $\mathbb{C}/\Lambda$ is a complex Lie group homeomorphic to a torus. As a complex manifold it is isomorphic to the projective algebraic variety (elliptic curve)

$$E(\mathbb{C}) = \{ [X : Y : Z] \in \mathbb{P}_2(\mathbb{C}) \mid Y^2 Z = 4X^3 - g_2 XZ^2 - g_3 Z^3 \}$$

for suitable $g_2, g_3 \in \mathbb{C}$.

The quotient map $\exp_E : \mathbb{C} \to E(\mathbb{C})$ is given as $z \mapsto [\wp(z) : \wp'(z) : 1]$ where

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left( \frac{1}{(z - \lambda)^2} - \frac{1}{\lambda^2} \right)$$

is a meromorphic function.

diff eq
Interdefinability theorems

We work in the complex setting.

Theorem (Jones, K., Servi 2016)

- No Weierstrass $\wp$ is in $\text{LD}(\exp)$.
- $\exp$ is not in $\text{LD}(\text{all } \wp\text{-functions})$.
- More generally: let $\mathcal{F} = \{\exp, \wp_1, \ldots, \wp_n\}$, with $\wp_i$ corresponding to a lattice $\Lambda_i$. Then some $\wp_{n+1}$ is in $\text{LD}(\mathcal{F})$ if and only if the lattice $\Lambda_{n+1}$ is isogenous to one of $\Lambda_1, \ldots, \Lambda_n$ or their complex conjugates.
- Also stronger results of similar type.

The proof uses the Ax-Schanuel differential algebra theorem explaining linear dependences between the differential forms associated with these functions in terms of isogenies / algebraic subgroups of algebraic groups.

Current work with Jones and Schmidt extends these results to other elliptic functions: Weierstrass-$\zeta$ functions and $\wp$-functions. The Weierstrass $\sigma$ is more difficult but once the relevant Ax-Schanuel theorem is understood it should be doable.