Differential transcendence and difference equations

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Classification of numbers vs functions

$$
\begin{array}{ccl}
\mathbb{Q} & \longleftrightarrow & \mathbb{C}(z) \\
\cap & & \cap \\
\overline{\mathbb{Q}} & \longleftrightarrow & \overline{\mathbb{C}(z)} \\
 & & \cap \\
 & & \text{holonomic} \\
\cap & & \cap \\
 & & \text{differentially algebraic} \\
 & & \cap \\
\mathbb{C} & \longleftrightarrow & \mathbb{C}((z^{1/*})) = \bigcup_{\ell=1}^{\infty} \mathbb{C}((z^{1/\ell}))
\end{array}
$$

Classification of functions

- We say that $f \in \overline{\mathbb{C}(z)}$ if $\exists 0 \neq P \in \mathbb{C}(z)[X]$ such that

$$P(f) = 0.$$

Example: $z^{1/2}$

- We say that f is holonomic if $\exists\ c_0, \ldots, c_n \in \mathbb{C}(z)$, $c_n \neq 0$, such that

$$c_0 f + \cdots + c_n \partial_z^n(f) = 0.$$

Example: $\exp(z), \log(z), \ldots$

- We say that f is differentially algebraic if $\exists\ n \in \mathbb{N}$, $0 \neq P \in \mathbb{C}(z)[X_0, \ldots, X_n]$, such that

$$P(f, \ldots, \partial_z^n(f)) = 0.$$

Example: $\wp(z)$, some walks in the quarter plane

- We say that f is differentially transcendental otherwise

Example: $\Gamma(z), \zeta(z)$

Some functions are differentially transcendental, for instance:

- $\Gamma(z)$;
- $\mathfrak{f}_1(z) := \sum_{n=0}^{\infty} \frac{(1-a)^2(1-aq)^2\cdots(1-aq^{n-1})^2}{(1-q)^2(1-q^2)^2\cdots(1-q^n)^2} z^n$, where $q \in \mathbb{C}^*$ is not a root of unity, $a \notin q^{\mathbb{Z}}$ and $a^2 \in q^{\mathbb{Z}}$;
- $\mathfrak{f}_2(z) = \sum_{n\geq 0} z^{2^n}$.

They are solutions of difference equations $\Gamma(z+1) = z\Gamma(z)$, $\mathfrak{f}_2(z^2) = \mathfrak{f}_2(z) - z$, and

$$\mathfrak{f}_1(q^2 z) - \frac{2az-2}{a^2 z - 1}\mathfrak{f}_1(qz) + \frac{z-1}{a^2 z - 1}\mathfrak{f}_1(z) = 0\,.$$

Differential algebraicity and difference equations

On the other hand, there are differentially algebraic functions solutions of difference equations:

- $\exp(z)$, solution of $\exp(z+1) = e\exp(z)$;
- $\theta_q(z) = \sum_{n\in\mathbb{Z}} q^{-n(n-1)/2} z^n$, solution of $\theta_q(qz) = z\theta_q(z)$;
- $\log(z)$, solution of $\log(z^2) = 2\log(z)$.

Difference framework

Let $y \in F$, solution of

$$a_0 y + a_1 \rho(y) + \cdots + \rho^n(y) = 0, \quad a_i \in \mathbb{C}(z). \qquad \text{(E)}$$

$\boxed{\textit{Case S}}$ $F = \mathbb{C}((z^{-1}))$,
$\rho : y(z) \mapsto y(z+h)$, $h \in \mathbb{C}^*$.

$\boxed{\textit{Case Q}}$ $F = \mathbb{C}((z^{1/*}))$,
$\rho : y(z) \mapsto y(qz)$, $q \in \mathbb{C}^*$, not a root of unity.

$\boxed{\textit{Case M}}$ $F = \mathbb{C}((z^{1/*}))$,
$\rho : y(z) \mapsto y(z^p)$, $p \in \mathbb{N}_{\geq 2}$.

Holonomy and difference equations

Let $y \in F$, solution of

$$a_0 y + a_1 \rho(y) + \cdots + \rho^n(y) = 0. \tag{E}$$

Theorem

If y is holonomic, then $y \in \mathbb{C}(z)$.

$\rightarrow$ *Case S: Schäfke/Singer, Case Q Ramis, Case M, Bézivin*

$\rightarrow$ *See also Bézivin/Gramain*

Let $y \in F$, solution of

$$\rho(y) = ay + b, \quad a, b \in \mathbb{C}(z).$$

Theorem

Either $y \in \mathbb{C}(z)$, either y is differentially transcendental.

$\rightarrow$ *Case S: Adamczewski/D/Hardouin, Case Q Ishizaki, Case M, Randé*

$\rightarrow$ *See also Hölder, Hardouin/Singer, Moore, Nishioka, Nguyen...*

Let $y \in F$, solution of

$$a_0 y + a_1 \rho(y) + \cdots + \rho^n(y) = 0. \tag{E}$$

Theorem

Assume that the difference Galois group of (E) *contains* $\mathrm{SL}_n(\mathbb{C})$. *Either* $y = 0$, *either* y *is differentially transcendental.*

$\rightarrow$ *Case S: Arreche/Singer, Cases Q and M D/Hardouin/ Roques*

$\rightarrow$ *See also Arreche/D/Roques and Arreche/Singer*

Main result

Let $y \in F$, solution of

$$a_0 y + a_1 \rho(y) + \cdots + \rho^n(y) = 0. \quad \text{(E)}$$

Theorem (Adamczewski/D/Hardouin)

Either $y \in \mathbb{C}(z)$, either y is differentially transcendental.

1. Difference Galois theory
2. Proof in the $n = 2$ case
3. Proof in the general case

Difference Galois theory

Difference framework

Let $0 \neq y \in F$, solution of

$$a_0 y + a_1 \rho(y) + \cdots + \rho^n(y) = 0, \qquad \text{(E)}$$

with

$$a_i \in \mathbb{C}(z), \quad a_0 \neq 0.$$

Case S $K = \mathbb{C}(z)$, $F = \mathbb{C}((z^{-1}))$,
$\rho : y(z) \mapsto y(z+h)$, $h \in \mathbb{C}^*$.

Case Q $K = \mathbb{C}(z^{1/*})$, $F = \mathbb{C}((z^{1/*}))$,
$\rho : y(z) \mapsto y(qz)$, $q \in \mathbb{C}^*$, not a root of unity.

Case M $K = \mathbb{C}(z^{1/*})$, $F = \mathbb{C}((z^{1/*}))$,
$\rho : y(z) \mapsto y(z^p)$, $p \in \mathbb{N}_{\geq 2}$.

Picard-Vessiot extension

Let us see (E) as a system:

$$\rho(Y) = AY, \quad A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & \cdots & -a_{n-1} \end{pmatrix} \in \mathrm{GL}_n(\mathbb{C}(z)).$$

Proposition

There exists a unique ring extension $R|K$, such that

- $\exists U \in \mathrm{GL}_n(R)$ *such that* $\rho(U) = AU$.
- *the first column of U is* $(y, \dots, \rho^{n-1}(y))$;
- $R = K[U, \det(U)^{-1}]$;
- *the only difference ideals of R are* (0) *and R.*

Difference Galois group

Let

$$G = \{\sigma \in \mathrm{Aut}(R|K) | \sigma\rho = \rho\sigma\}.$$

Theorem

The image of

$$\begin{array}{rcl} G & \to & \mathrm{GL}_n(\mathbb{C}) \\ \sigma & \mapsto & U^{-1}\sigma(U), \end{array}$$

is an algebraic subgroup of $\mathrm{GL}_n(\mathbb{C})$*.*

A useful property

For $B, T \in \mathrm{GL}_n(K)$, define

$$T[B] := \rho(T)BT^{-1}.$$

We have

$$\rho(Y) = BY \Leftrightarrow \rho(TY) = T[B]TY.$$

Theorem (van der Put/Singer)

- *G/G° is cyclic, where G° is the identity component of G;*
- *$\exists T \in \mathrm{GL}_n(K)$ such that $T[A] \in G(K)$.*

Proof in the $n = 2$ case

3 possibilities for the Galois group

Assume $n = 2$. Let $G \subset \mathrm{GL}_2(\mathbb{C})$ be the Galois group. Then, either

- G is conjugated to a subgroup of
$$\begin{pmatrix} \star & \star \\ 0 & \star \end{pmatrix},$$
- G is conjugated to a subgroup of
$$\begin{pmatrix} \star & 0 \\ 0 & \star \end{pmatrix} \bigcup \begin{pmatrix} 0 & \star \\ \star & 0 \end{pmatrix},$$
- G contains $\mathrm{SL}_2(\mathbb{C})$.

Case 1

Assume that y is diff. alg. Then, $\exists T = (t_{i,j}) \in \mathrm{GL}_2(K)$ such that

$$\rho(TU) = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} TU.$$

Let $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = T \begin{pmatrix} y \\ \rho(y) \end{pmatrix}$ be the first column of TU. Then

- $v_2 = t_{2,1} y + t_{2,2} \rho(y)$.
- $v_2 \in F$ is diff. alg.
- $\rho(v_2) = c v_2$.
- Order one case $\Rightarrow v_2 \in K$.
- Affine order one case $\Rightarrow y \in K$.

Case 2

Assume that y is diff. alg. Then, $\exists T = (t_{i,j}) \in \mathrm{GL}_2(K)$ such that

$$\rho(TU) = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} TU.$$

Let $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = T \begin{pmatrix} y \\ \rho(y) \end{pmatrix}$ be the first column of TU. Then

- $v_1 \in F$ is diff. alg.
- $v_1 = t_{1,1}y + t_{1,2}\rho(y)$.
- $\rho^2(v_1) = b\rho(a)v_1$.
- Order one case with ρ^2 implies $v_1 \in K$.
- Affine order one case $\Rightarrow y \in K$.

Case 3

Assume that G contains $\mathrm{SL}_2(\mathbb{C})$.
By

- Arreche/Singer (Case S),
- D/Hardouin/Roques (Cases Q and M),

y is diff. tr.

Proof in the general case

The case $n = 1$ is

- Adamczewski/D/Hardouin, (Case S);
- Ishizaki (Case Q);
- Randé (case M).

From now, we assume $n \geq 2$.

Irreducibility of G

Definition

We say that $G \subset \mathrm{GL}_n(\mathbb{C})$ is irreducible if it acts irreducibly on $\mathbb{C}^n$. We say that G is reducible otherwise.

Proposition

The following are equivalent:

- *G is reducible.*
- *$\exists T \in \mathrm{GL}_n(K)$, $0 < r < n$, such that*

$$T[A] = \begin{pmatrix} B_1 & B_2 \\ 0 & B_3 \end{pmatrix}, \quad B_1 \in \mathrm{GL}_r(K).$$

Imprimitivity of G

Definition

When G is irreducible, we say that G is imprimitive if $\exists r \geq 2$, and $V_1, \ldots, V_r$, some $\mathbb{C}$-vector spaces satisfying

(i) $\mathbb{C}^n = V_1 \oplus \cdots \oplus V_r$.

(ii) $\forall g \in G$, *the mapping* $V_i \mapsto g(V_i)$ *is a permutation of the set* $\{V_1, \ldots, V_r\}$.

We say that G is primitive otherwise.

Lemma

If G is irreducible and connected then G is primitive.

Iteration and Galois group

For $\ell \geq 1$ let

$$A_{[\ell]} = \rho^{\ell-1}(A) \times \cdots \times A.$$

Note that

$$\rho(Y) = AY \Rightarrow \rho^{\ell}(Y) = A_{[\ell]}Y.$$

Lemma

There exist $\ell \geq 1$ and a ring extension $R|K$, such that

- $\exists U \in \mathrm{GL}_n(R)$ *such that* $\rho^{\ell}(U) = A_{[\ell]}U$.
- *the first column of U is* $(y, \ldots, \rho^{n-1}(y))$;
- $R = K[U, \det(U)^{-1}]$;
- *the only ρ^{ℓ} ideals of R are* (0) *and R.*
- $G_{[\ell]}$, *the Galois group of* $\rho^{\ell}(Y) = A_{[\ell]}Y$ *is connected.*

Semi simple case

Lemma (Singer/Ulmer)

If $G \subset \mathrm{SL}_n(\mathbb{C})$ is irreducible and primitive, then G is semi simple.

Theorem (Arreche/Singer)

Assume that G is semi simple. Then, y is diff. tr.

Let $\ell \geq 1$, such that $G_{[\ell]}$ is connected.

Proposition (Adamczewski/D/Hardouin)

If $G_{[\ell]}$ is irreducible, then y is differentially transcendental.

Sketch of proof.

$G_{[\ell]}$ is primitive. If $G_{[\ell]} \subset \mathrm{SL}_n(\mathbb{C})$ then it is semi simple.
If not, consider the system $\rho^{\ell}(Y) = \det(A_{[\ell]})^{-1/n} A_{[\ell]} Y$. Its Galois group is

- irreducible,
- primitive,
- inside $\mathrm{SL}_n(\mathbb{C})$.

It is then semi simple.
Semi simple implies y diff. tr. □

Proof in the general case

Let us prove the result by an induction on n.

The case $n = 1$ is already treated.

Fix $n \geq 2$ and assume the result is proved for order r equations with $r < n$.

Consider an order n equation. Let $\ell \geq 1$, such that $G_{[\ell]}$ is connected.

If $G_{[\ell]} \subset \mathrm{GL}_n(\mathbb{C})$ is irreducible, then y is diff. tr.

Sketch of proof in the reducible case (1/3)

Assume that $G_{[\ell]}$ is reducible. Assume that y is diff. alg. and let us prove that $y \in K$.
Let $T \in \mathrm{GL}_n(K)$, $0 < r < n$ minimal, such that

$$T[A_{[\ell]}] = \begin{pmatrix} B_1 & B_2 \\ 0 & B_3 \end{pmatrix}, \quad B_1 \in \mathrm{GL}_r(K).$$

Then, TU is solution of

$$\rho^{\ell}(TU) = \begin{pmatrix} B_1 & B_2 \\ 0 & B_3 \end{pmatrix} TU.$$

Let $(v_1, \ldots, v_n)^\top = T(y, \ldots, \rho^{n-1}(y))^\top \in F^n$. Every v_i is diff alg.

Sketch of proof in the reducible case (2/3)

$$\rho^{\ell}(TU) = \begin{pmatrix} B_1 & B_2 \\ 0 & B_3 \end{pmatrix} TU.$$

Induction hypothesis $\Rightarrow v_{r+1}, \ldots, v_n \in K$.

Lemma

$r = 1$.

Sketch of proof.

- We have $\rho(v_1, \ldots, v_r)^\top - B_1(v_1, \ldots, v_r)^\top \in K^r$.
- $v_1, \ldots, v_r \in F$ are diff. alg.
- Parametrized diff. Galois theory $\Rightarrow \exists (w_1, \ldots, w_r)^\top$ diff. alg. such that $\rho(w_1, \ldots, w_r)^\top = B_1(w_1, \ldots, w_r)^\top$.
- The Galois group of $\rho^{\ell}(Y) = B_1 Y$ is irreducible and connected.
- Irreducible case $\Rightarrow r = 1$. □

Sketch of proof in the reducible case (3/3)

$$\rho^\ell(TU) = \begin{pmatrix} B_1 & B_2 \\ 0 & B_3 \end{pmatrix} TU.$$

- Remind that $v_2, \ldots, v_n \in K$ and $B_1 \in \mathbb{C}^*$.
- Then, $\rho^\ell(v_1) - B_1 v_1 \in K$.
- Affine order one case implies $v_1 \in K$.
- Then, $T^{-1}(v_1, \ldots, v_n)^\top = (y, \ldots, \rho^{n-1}(y))^\top \in K^n$. □