Differential transcendence and difference equations

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Classification of numbers vs functions

\[ \mathbb{Q} \leftrightarrow \mathbb{C}(z) \]

\[ \mathbb{Q} \leftrightarrow \mathbb{C}(z) \cap \text{holonomic} \]

\[ \mathbb{Q} \leftrightarrow \mathbb{C}(z) \cap \text{differentially algebraic} \]

\[ \mathbb{C} \leftrightarrow \mathbb{C}((z^{1/\ell})) = \bigcup_{\ell=1}^{\infty} \mathbb{C}((z^{1/\ell})) \]
Classification of functions

• We say that $f \in \overline{\mathbb{C}(z)}$ if $\exists 0 \neq P \in \mathbb{C}(z)[X]$ such that

$$P(f) = 0.$$  

Example: $z^{1/2}$

• We say that $f$ is holonomic if $\exists c_0, \ldots, c_n \in \mathbb{C}(z), c_n \neq 0$, such that

$$c_0 f + \cdots + c_n \partial^n_z(f) = 0.$$  

Example: $\exp(z), \log(z), \ldots$

• We say that $f$ is differentially algebraic if $\exists n \in \mathbb{N}$, $0 \neq P \in \mathbb{C}(z)[X_0, \ldots, X_n]$, such that

$$P(f, \ldots, \partial^n_z(f)) = 0.$$  

Example: $\wp(z)$, some walks in the quarter plane

• We say that $f$ is differentially transcendental otherwise

Example: $\Gamma(z), \zeta(z)$
Some functions are differentially transcendental, for instance:

- $\Gamma(z)$;
- $f_1(z) := \sum_{n=0}^{\infty} \frac{(1-a)^2(1-aq)^2\cdots(1-aq^{n-1})^2}{(1-q)^2(1-q^2)^2\cdots(1-q^n)^2} z^n$, where $q \in \mathbb{C}^*$ is not a root of unity, $a \not\in q^{\mathbb{Z}}$ and $a^2 \in q^{\mathbb{Z}}$;
- $f_2(z) = \sum_{n \geq 0} z^{2n}$.

They are solutions of difference equations $\Gamma(z + 1) = z\Gamma(z)$, $f_2(z^2) = f_2(z) - z$, and

$$f_1(q^2 z) - \frac{2az - 2}{a^2 z - 1} f_1(qz) + \frac{z - 1}{a^2 z - 1} f_1(z) = 0.$$
Differential algebraicity and difference equations

On the other hand, there are differentially algebraic functions solutions of difference equations:

- $\exp(z)$, solution of $\exp(z + 1) = e \exp(z)$;
- $\theta_q(z) = \sum_{n \in \mathbb{Z}} q^{-n(n-1)/2} z^n$, solution of $\theta_q(qz) = z \theta_q(z)$;
- $\log(z)$, solution of $\log(z^2) = 2 \log(z)$. 
Let \( y \in F \), solution of

\[
a_0 y + a_1 \rho(y) + \cdots + \rho^n(y) = 0, \quad a_i \in \mathbb{C}(z). \tag{E}
\]

**Case S**  
\( F = \mathbb{C}((z^{-1})) \),  
\( \rho : y(z) \mapsto y(z + h), \ h \in \mathbb{C}^* \).

**Case Q**  
\( F = \mathbb{C}((z^{1/\ast})) \),  
\( \rho : y(z) \mapsto y(qz), \ q \in \mathbb{C}^*, \text{ not a root of unity} \).

**Case M**  
\( F = \mathbb{C}((z^{1/\ast})) \),  
\( \rho : y(z) \mapsto y(z^p), \ p \in \mathbb{N}_{\geq 2} \).
Let $y \in F$, solution of

$$a_0 y + a_1 \rho(y) + \cdots + \rho^n(y) = 0.$$  \hfill (E)

**Theorem**

*If $y$ is holonomic, then $y \in \mathbb{C}(z)$.*

$\rightarrow$ Case S: Schäfke/Singer, Case Q Ramis, Case M, Bézivin  
$\rightarrow$ See also Bézivin/Gramain
Let $y \in F$, solution of

$$\rho(y) = ay + b, \quad a, b \in \mathbb{C}(z).$$

**Theorem**

*Either $y \in \mathbb{C}(z)$, either $y$ is differentially transcendental.*

→ *Case S: Adamczewski/D/Hardouin, Case Q Ishizaki, Case M, Randé*

→ *See also Hölder, Hardouin/Singer, Moore, Nishioka, Nguyen...*
Let $y \in F$, solution of

$$a_0 y + a_1 \rho(y) + \cdots + \rho^n(y) = 0. \quad (E)$$

**Theorem**

Assume that the difference Galois group of (E) contains $\text{SL}_n(\mathbb{C})$. Either $y = 0$, either $y$ is differentially transcendental.

→ Case S: Arreche/Singer, Cases Q and M D/Hardouin/Roques

→ See also Arreche/D/Roques and Arreche/Singer
Let $y \in F$, solution of

$$a_0 y + a_1 \rho(y) + \cdots + \rho^n(y) = 0. \quad (E)$$

Theorem (Adamczewski/D/Hardouin)

*Either $y \in \mathbb{C}(z)$, either $y$ is differentially transcendental.*
1 Difference Galois theory

2 Proof in the $n = 2$ case

3 Proof in the general case
Difference Galois theory
Let $0 \neq y \in F$, solution of

$$a_0 y + a_1 \rho(y) + \cdots + \rho^n(y) = 0, \quad (E)$$

with

$$a_i \in \mathbb{C}(z), \quad a_0 \neq 0.$$

Case S \hspace{1cm} K = \mathbb{C}(z), \ F = \mathbb{C}((z^{-1})),
\rho : y(z) \mapsto y(z + h), \ h \in \mathbb{C}^*.

Case Q \hspace{1cm} K = \mathbb{C}(z^{1/\ast}), \ F = \mathbb{C}((z^{1/\ast})),
\rho : y(z) \mapsto y(qz), \ q \in \mathbb{C}^*, \text{ not a root of unity.}

Case M \hspace{1cm} K = \mathbb{C}(z^{1/\ast}), \ F = \mathbb{C}((z^{1/\ast})),
\rho : y(z) \mapsto y(z^p), \ p \in \mathbb{N}_{\geq 2}.
Let us see (E) as a system:

\[
\rho(Y) = AY,
\]

\[
A = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & 0 \\
0 & 0 & \cdots & 0 & 1 \\
-a_0 & -a_1 & \cdots & \cdots & -a_{n-1}
\end{pmatrix}
\in \text{GL}_n(\mathbb{C}(z)).
\]

**Proposition**

There exists a unique ring extension \( R | K \), such that

- \( \exists U \in \text{GL}_n(R) \) such that \( \rho(U) = AU \).
- the first column of \( U \) is \((y, \ldots, \rho^{n-1}(y))\);
- \( R = K[U, \det(U)^{-1}] \);
- the only difference ideals of \( R \) are \((0)\) and \( R \).
Let

\[ G = \{ \sigma \in \text{Aut}(R|K) \mid \sigma \rho = \rho \sigma \}. \]

**Theorem**

The image of

\[ G \rightarrow \text{GL}_n(\mathbb{C}) \]
\[ \sigma \mapsto U^{-1} \sigma(U), \]

is an algebraic subgroup of \( \text{GL}_n(\mathbb{C}) \).
A useful property

For $B, T \in \text{GL}_n(K)$, define

$$T[B] := \rho(T)BT^{-1}.$$  

We have

$$\rho(Y) = BY \iff \rho(TY) = T[B]TY.$$  

Theorem (van der Put/Singer)

- $G/G^\circ$ is cyclic, where $G^\circ$ is the identity component of $G$;
- $\exists T \in \text{GL}_n(K)$ such that $T[A] \in G(K)$. 

Proof in the $n = 2$ case
Assume \( n = 2 \). Let \( G \subset \text{GL}_2(\mathbb{C}) \) be the Galois group. Then, either

- \( G \) is conjugated to a subgroup of \( \left( \begin{array}{cc}
\ast & \ast \\
0 & \ast 
\end{array} \right) \),

- \( G \) is conjugated to a subgroup of \( \left( \begin{array}{cc}
\ast & 0 \\
0 & \ast 
\end{array} \right) \cup \left( \begin{array}{cc}
0 & \ast \\
\ast & 0 
\end{array} \right) \),

- \( G \) contains \( \text{SL}_2(\mathbb{C}) \).
Case 1

Assume that $y$ is diff. alg. Then, $\exists T = (t_{i,j}) \in \text{GL}_2(K)$ such that

$$\rho(TU) = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} TU.$$  

Let $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = T \begin{pmatrix} y \\ \rho(y) \end{pmatrix}$ be the first column of $TU$. Then

- $v_2 = t_{2,1}y + t_{2,2}\rho(y)$.
- $v_2 \in F$ is diff. alg.
- $\rho(v_2) = cv_2$.
- Order one case $\Rightarrow v_2 \in K$.
- Affine order one case $\Rightarrow y \in K$. 
Assume that $y$ is diff. alg. Then, $\exists T = (t_{i,j}) \in \text{GL}_2(K)$ such that

$$\rho(TU) = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} TU.$$ 

Let \( \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = T \begin{pmatrix} y \\ \rho(y) \end{pmatrix} \) be the first column of $TU$. Then

- $v_1 \in F$ is diff. alg.
- $v_1 = t_{1,1}y + t_{1,2}\rho(y)$.
- $\rho^2(v_1) = b\rho(a)v_1$.
- Order one case with $\rho^2$ implies $v_1 \in K$.
- Affine order one case $\Rightarrow y \in K$. 
Assume that $G$ contains $SL_2(\mathbb{C})$.

By

- Arreche/Singer (Case S),
- D/Hardouin/Roques (Cases Q and M),

$y$ is diff. tr.
Proof in the general case

The case $n = 1$ is

- Adamczewski/D/Hardouin, (Case S);
- Ishizaki (Case Q);
- Randé (case M).

From now, we assume $n \geq 2$. 
Irreducibility of $G$

**Definition**

We say that $G \subset \text{GL}_n(\mathbb{C})$ is irreducible if it acts irreducibly on $\mathbb{C}^n$. We say that $G$ is reducible otherwise.

**Proposition**

The following are equivalent:

- $G$ is reducible.
- $\exists T \in \text{GL}_n(K), 0 < r < n, \text{ such that }$

$$T[A] = \begin{pmatrix} B_1 & B_2 \\ 0 & B_3 \end{pmatrix}, \quad B_1 \in \text{GL}_r(K).$$
**Definition**

When $G$ is irreducible, we say that $G$ is imprimitive if $\exists r \geq 2$, and $V_1, \ldots, V_r$, some $\mathbb{C}$-vector spaces satisfying

(i) $\mathbb{C}^n = V_1 \oplus \cdots \oplus V_r$.

(ii) $\forall g \in G$, the mapping $V_i \mapsto g(V_i)$ is a permutation of the set $\{V_1, \ldots, V_r\}$.

We say that $G$ is primitive otherwise.

**Lemma**

If $G$ is irreducible and connected then $G$ is primitive.
For $\ell \geq 1$ let

$$A[\ell] = \rho^{\ell-1}(A) \times \cdots \times A.$$ 

Note that

$$\rho(Y) = AY \Rightarrow \rho^{\ell}(Y) = A[\ell]Y.$$ 

**Lemma**

There exist $\ell \geq 1$ and a ring extension $R|K$, such that

- $\exists U \in \mathbb{GL}_n(R)$ such that $\rho^{\ell}(U) = A[\ell]U$.
- the first column of $U$ is $(y, \ldots, \rho^{n-1}(y))$;
- $R = K[U, \det(U)^{-1}]$;
- the only $\rho^{\ell}$ ideals of $R$ are $(0)$ and $R$.
- $G[\ell]$, the Galois group of $\rho^{\ell}(Y) = A[\ell]Y$ is connected.
Lemma (Singer/Ulmer)

If $G \subset SL_n(\mathbb{C})$ is irreducible and primitive, then $G$ is semi simple.

Theorem (Arreche/Singer)

Assume that $G$ is semi simple. Then, $y$ is diff. tr.
Proof in the irreducible case

Let $\ell \geq 1$, such that $G[\ell]$ is connected.

**Proposition (Adamczewski/D/Hardouin)**

*If $G[\ell]$ is irreducible, then $y$ is differentially transcendental.*

**Sketch of proof.**

$G[\ell]$ is primitive. If $G[\ell] \subset SL_n(\mathbb{C})$ then it is semi simple. If not, consider the system $\rho^\ell(Y) = \det(A[\ell])^{-1/n} A[\ell] Y$. Its Galois group is

- irreducible,
- primitive,
- inside $SL_n(\mathbb{C})$.

It is then semi simple. Semi simple implies $y$ diff. tr.
Proof in the general case

Let us prove the result by an induction on $n$.

The case $n = 1$ is already treated.

Fix $n \geq 2$ and assume the result is proved for order $r$ equations with $r < n$.

Consider an order $n$ equation. Let $\ell \geq 1$, such that $G[\ell]$ is connected.

If $G[\ell] \subset \text{GL}_n(\mathbb{C})$ is irreducible, then $y$ is diff. tr.
Assume that $G_{[\ell]}$ is reducible. Assume that $y$ is diff. alg. and let us prove that $y \in K$.

Let $T \in \text{GL}_n(K)$, $0 < r < n$ minimal, such that

$$T[A_{[\ell]}] = \begin{pmatrix} B_1 & B_2 \\ 0 & B_3 \end{pmatrix}, \quad B_1 \in \text{GL}_r(K).$$

Then, $TU$ is solution of

$$\rho^{\ell}(TU) = \begin{pmatrix} B_1 & B_2 \\ 0 & B_3 \end{pmatrix} TU.$$

Let $(v_1, \ldots, v_n)^\top = T(y, \ldots, \rho^{n-1}(y))^\top \in F^n$. Every $v_i$ is diff alg.
Sketch of proof in the reducible case (2/3)

\[ \rho^\ell(TU) = \begin{pmatrix} B_1 & B_2 \\ 0 & B_3 \end{pmatrix} TU. \]

Induction hypothesis \( \Rightarrow v_{r+1}, \ldots, v_n \in K. \)

**Lemma**

\( r = 1. \)

**Sketch of proof.**

- We have \( \rho(v_1, \ldots, v_r)^\top - B_1(v_1, \ldots, v_r)^\top \in K^r. \)
- \( v_1, \ldots, v_r \in F \) are diff. alg.
- Parametrized diff. Galois theory \( \Rightarrow \exists (w_1, \ldots, w_r)^\top \) diff. alg. such that \( \rho(w_1, \ldots, w_r)^\top = B_1(w_1, \ldots, w_r)^\top. \)
- The Galois group of \( \rho^\ell(Y) = B_1 Y \) is irreducible and connected.
- Irreducible case \( \Rightarrow r = 1. \)
Sketch of proof in the reducible case (3/3)\[ \rho^\ell(TU) = \begin{pmatrix} B_1 & B_2 \\ 0 & B_3 \end{pmatrix} TU. \]

- Remind that \( v_2, \ldots, v_n \in K \) and \( B_1 \in \mathbb{C}^* \).
- Then, \( \rho^\ell(v_1) - B_1 v_1 \in K \).
- Affine order one case implies \( v_1 \in K \).
- Then, \( T^{-1}(v_1, \ldots, v_n)^\top = (y, \ldots, \rho^{n-1}(y))^\top \in K^n \). \( \square \)