Rota-Baxter Algebras and Quasi-symmetric Functions

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Abstract

- This talk discusses the relationship between Rota-Baxter algebras and quasi-symmetric functions.
- First introduced by Glenn Baxter, (Rota-)Baxter algebra owed its early developments mostly to Gian-Carlo Rota, from the viewpoint of algebraic combinatorics.
- In the 1960s, Rota made the first connection between Rota-Baxter algebra and symmetric functions in his construction of free commutative Rota-Baxter algebras.
- In the 1990s, Rota made the conjecture that Rota-Baxter algebra should be the suitable framework to study generalizations of symmetric functions.
- Evidences in support of Rota's conjecture appeared over the years as pieces of free Rota-Baxter algebras were realized as quasi-symmetric functions.
- In recent papers, the full free commutative nonunitary and unitary Rota-Baxter algebras were realized as generalizations of quasi-symmetric functions.

Background on Rota-Baxter algebras

- Fix λ in a base ring $\mathbf{k}$. A Rota-Baxter operator or a Baxter operator of weight λ on a $\mathbf{k}$-algebra R is a linear map $P: R \to R$ such that

$$P(x)P(y) = P(xP(y)) + P(P(x)y) + \lambda P(xy), \ \forall\, x, y \in R.$$

- **Examples.** Integration: $R = \mathbf{Cont}(\mathbb{R})$ (ring of continuous functions on $\mathbb{R}$).

$$P: R \to R, P[f](x) := \int_0^x f(t)dt.$$

- Then P is a weight 0 Rota-Baxter operator:

$$F(x) := P[f](x) = \int_0^x f(t)\,dt, \quad G(x) := P[g](x) = \int_0^x g(t)\,dt.$$

Then the integration by parts formula states

$$\int_0^x F(t)G'(t)dt = F(x)G(x) - \int_0^x F'(t)G(t)dt$$

$(F(0) = G(0) = 0)$. That is,

$$P[P[f]g](x) = P[f](x)P[g](x) - P[fP[g]](x).$$

- Summation: On a suitable class of functions, define

$$P[f](x) := \sum_{n\geq 1} f(x+n).$$

- Then P is a Rota-Baxter operator of weight 1:

$$P[f](x)\,P[g](x) = P[P[f]g](x) + P[fP[g]](x) + P[fg](x).$$

- Laurent series: Let $R = \mathbb{C}[\varepsilon^{-1}, \varepsilon]]$ be the ring of Laurent series $\sum_{n=-k}^{\infty} a_n\varepsilon^n,\ k \geq 0$. Define the pole part projection

$$P(\sum_{n=-k}^{\infty} a_n\varepsilon^n) = \sum_{n=-k}^{-1} a_n\varepsilon^n.$$

Then P is a Rota-Baxter operator of weight -1.

- **Classical Yang-Baxter equation:** Let $\mathfrak{g}$ be a Lie algebra with a perfect pairing $\mathfrak{g} \otimes \mathfrak{g} \to \mathbf{k}$. Then $\mathfrak{g}^{\otimes 2} \cong \mathfrak{g} \otimes \mathfrak{g}^* \cong \mathrm{End}(\mathfrak{g})$. Let $r_{12} \in \mathfrak{g}^{\otimes 2}$ be anti-symmetric. Then r_{12} is a solution (r-matrix) of the **classical Yang-Baxter equation (CYB)**

$$CYB(r) := [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$$

if and only if the corresponding $P \in \mathrm{End}(\mathfrak{g})$ is a (Lie algebra) Rota-Baxter operator of weight 0:

$$[P(x), P(y)] = P[P(x), y] + P[x, P(y)]$$

Others: Partial sums, scalar product, Hochschild homology ring, associative Yang-Baxter equation, dendriform algebras, rooted trees, divided powers, ...

Rota's standard RBA

- As a motivation, we recall the first construction of free commutative Rota–Baxter algebras given by Rota, called the standard Rota–Baxter algebra, and their relationship with symmetric functions.
- Let X be a given set. Let $t_n^{(x)}, n \geq 1, x \in X$, be distinct symbols.
- Denote

$$\overline{X} = \bigcup_{x \in X} \left\{ t_n^{(x)} \mid n \geq 1 \right\}$$

and let $\mathfrak{A}(X) = \mathbf{k}[\overline{X}]^{\mathbb{P}}$ denote the algebra of sequences with entries in the polynomial algebra $\mathbf{k}[\overline{X}]$, with componentwise operations.
- Define

$$P_X^r : \mathfrak{A}(X) \to \mathfrak{A}(X), \quad (a_1, a_2, a_3, \cdots) \mapsto (0, a_1, a_1 + a_2, a_1 + a_2 + a_3, \cdots).$$

Then P_X^r defines a Rota–Baxter operator on $\mathfrak{A}(X)$.
- The standard Rota–Baxter algebra on X is the Rota–Baxter subalgebra $\mathfrak{S}(X)$ of $\mathfrak{A}(X)$ generated by the sequences $t^{(x)} := (t_1^{(x)}, \cdots, t_n^{(x)}, \cdots), \ x \in X$.
- Theorem (Rota, 1969) $(\mathfrak{S}(X), P_X^r)$ is the free commutative Rota–Baxter algebra on X.

Spitzer's Identity

- Spitzer's Identity. Let (R, P) be a unitary commutative Rota-Baxter $\mathbb{Q}$-algebra of weight 1. Then for $a \in R$, we have

$$\exp\left(P(\log(1+\lambda at))\right) = \sum_{n=0}^{\infty} t^n \underbrace{P(P(P(\cdots(P(a)a)a)a))}_{n\text{-iterations}}$$

in the ring of power series $R[[t]]$.
- With the notation $P_a(c) := P(ac)$, this becomes

$$\exp\left(-\sum_{k=1}^{\infty} \frac{(-t)^k P(a^k)}{k}\right) = \sum_{n=0}^{\infty} t^n P_a^n(1).$$

- Take $X = \{x\}, x_n := t_n^{(x)}$, $R = \mathbf{k}[x_n, n \geq 1]^{\mathbb{P}}$, P the partial sum operator and $a := (x_1, \cdots, x_n, \cdots)$.

Rota-Baxter algebras and Symmetric functions

- Then
$$P_a^n(1) = (0, e_n(x_1), e_n(x_1, x_2), e_n(x_1, x_2, x_3), \cdots)$$
where $e_n(x_1, \cdots, x_m) = \sum_{1 \le i_1 < i_2 < \cdots < i_n \le m} x_{i_1} x_{i_2} \cdots x_{i_n}$ is the elementary symmetric function of degree n in the variables $x_1, \cdots, x_m$ with the convention that $e_0(x_1, \cdots, x_m) = 1$ and $e_n(x_1, \cdots, x_m) = 0$ if $m < n$.
- Also by definition,
$$P(a^k) = (0, p_k(x_1), p_k(x_1, x_2), p_k(x_1, x_2, x_3), \cdots),$$
where $p_k(x_1, \cdots, x_m) = x_1^k + x_2^k + \cdots + x_m^k$ is the power sum symmetric function of degree k in the variables $x_1, \cdots, x_m$.
- So Spitzer's Identity becomes Waring's formula:
$$\exp\left(-\sum_{k=1}^{\infty} (-1)^k t^k p_k(x_1, x_2, \cdots, x_m)/k \right)$$
$$= \sum_{n=0}^{\infty} e_n(x_1, x_2, \cdots, x_m) t^n \text{ for all } m \geq 1.$$

Rota's Conjecture/Question

- Rota conjectured in 1995:

  a very close relationship exists between the Baxter identity and the algebra of symmetric functions.

- and concluded

  The theory of symmetric functions of vector arguments (or Gessel functions) fits nicely with Baxter operators; in fact, identities for such functions easily translate into identities for Baxter operators. $\cdots$ *In short: Baxter algebras represent the ultimate and most natural generalization of the algebra of symmetric functions.*

- As it turns out, Rota-Baxter algebras are closely relates to quasi-symmetric functions.

Free commutative Rota-Baxter algebras

- A basic question for a Rota-Baxter algebra is how to multiply its two elements.
- Integration by parts:

$$\int_0^x f(t)\,dt \int_0^x g(t)\,dt = \int_0^x f(t)\Big(\int_0^t g(s)\,ds\Big)\,dt + \int_0^x \Big(\int_0^t f(s)\,ds\Big) g(t)\,dt.$$

  So a product of two integrals is the sum of two nested integrals.
- What about the product of two double integrals:

$$\left(\int_0^x f_1(t_1)\left(\int_0^{t_1} f_2(t_2)\,dt_1\right)\right)\left(\int_0^x g_1(s_1)\left(\int_0^{s_1} g_2(s_2)\,ds_1\right)\right) = ?$$

- What about the product of any two iterated integrals?
- Such products are reduced to the construction of free Rota-Baxter algebras, since an equation in a free Rota-Baxter algebra automatically holds for every Rota-Baxter algebra.

Multiplication in commutative Rota-Baxter algebras

- The Rota-Baxter axiom

$$P(x)P(y) = P(xP(y)) + P(P(x)y) + \lambda P(xy)$$

indicates that any Rota-Baxter "couple" $P(x)P(y)$ can be replaced by some nested ones.
- Any element of a commutative Rota-Baxter algebra (R, P) can be rewrittne in the form $a_0 P(a_1 P(a_2 \cdots P(a_k) \cdots)) \mapsto a_0 \otimes a_1 \otimes \cdots \otimes a_m$.
- For two elements $a_0 P(a_1 \cdots P(a_m) \cdots)$ and $b_0 P(b_1 \cdots P(b_n) \cdots)$, their product

$$\begin{aligned} &\big(a_0 P(a_1 \cdots P(a_m) \cdots)\big)\big(b_0 P(b_1 \cdots P(b_n) \cdots)\big) \\ &= (a_0 b_0)\big(P(a_1 \cdots P(a_m) \cdots)\big)\big(P(b_1 \cdots P(b_n) \cdots)\big) \end{aligned}$$

is lifted to a suitable product

$$\begin{aligned} &(a_0 \otimes \cdots \otimes a_m) \diamond (b_0 \otimes \cdots \otimes b_n) \\ &= (a_0 b_0)\big(1 \otimes \cdots \otimes a_m) \diamond (1 \otimes \cdots \otimes b_n) \\ &=: (a_0 b_0)\big((a_1 \otimes \cdots \otimes a_m) ш_\lambda (b_1 \otimes \cdots \otimes b_n)\big). \end{aligned}$$

- We next determine the product $ш_\lambda$.

Mixable Shuffle Product

- Let A be a commutative $\mathbf{k}$-algebra. Let $Ш^+(A)(= QS(A)) = \bigoplus_{n\geq 0} A^{\otimes n}(= T(A))$. Consider the following products on $Ш^+(A)$. Define $\mathbf{1_k} \in \mathbf{k}$ to be the unit. Let $\mathfrak{a} = a_1 \otimes \cdots \otimes a_m \in A^{\otimes m}$ and $\mathfrak{b} = b_1 \otimes \cdots \otimes b_n \in A^{\otimes n}$.
- **Mixable shuffle product:** Guo-Keigher (2000) on Rota-Baxter algebras, Goncharov (2002) on motivic shuffle relations and Hazewinckle on overlapping shuffle products.
- A shuffle of $\mathfrak{a} = a_1 \otimes \ldots \otimes a_m$ and $\mathfrak{b} = b_1 \otimes \ldots \otimes b_n$ is a tensor list of a_i and b_j without change the order of the a_is and b_js.
- A mixable shuffle is a shuffle in which some pairs $a_i \otimes b_j$ are merged into $\lambda a_i b_j$.
  Define $(a_1 \otimes \ldots \otimes a_m)Ш_\lambda(b_1 \otimes \ldots \otimes b_n)$ to be the sum of mixable shuffles of $a_1 \otimes \ldots \otimes a_m$ and $b_1 \otimes \ldots \otimes b_n$.
- **Example:**

$$\begin{aligned} & a_1 Ш_\lambda (b_1 \otimes b_2) \\ & = a_1 \otimes b_1 \otimes b_2 + b_1 \otimes a_1 \otimes b_2 + b_1 \otimes b_2 \otimes a_1 \quad \text{(shuffles)} \\ & + \lambda a_1 b_1 \otimes b_2 + b_1 \otimes \lambda a_1 b_2 \quad \text{(merged shuffles)}. \end{aligned}$$

Quasi-shuffle product

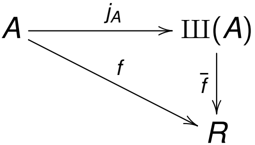
- **Quasi-shuffle product:** Hoffman (2000) on multiple zeta values and quasi-symmetric functions.
  Write $\mathfrak{a} = a_1 \otimes \mathfrak{a}'$, $\mathfrak{b} = b_1 \otimes \mathfrak{b}'$. Recursively define

  $$(a_1 \otimes \mathfrak{a}') * (b_1 \otimes \mathfrak{b}') = a_1 \otimes (\mathfrak{a}' * (b_1 \otimes \mathfrak{b}'))) + b_1 \otimes ((a_1 \otimes \mathfrak{a}') * \mathfrak{b}') + \lambda a_1 b_1 \otimes (\mathfrak{a}' * \mathfrak{b}'),$$

  with the convention that if $\mathfrak{a} = a_1$, then $\mathfrak{a}'$ multiples as the identity. It defines the shuffle product without the third term.
- **Example.**
  $a_1 * (b_1 \otimes b_2) = a_1 \otimes (\mathfrak{a}' * (b_1 \otimes b_2)) + b_1 \otimes (a_1 * b_2) + (\lambda a_1 b_1) \otimes (\mathfrak{a}' * b_2)$
  $= a_1 \otimes (b_1 \otimes b_2) + b_1 \otimes (a_1 * b_2) + (a_1 b_1) \otimes b_2.$
  $= a_1 \otimes b_1 \otimes b_2 + b_1 \otimes a_1 \otimes b_2 + b_1 \otimes b_2 \otimes a_1 + b_1 \otimes \lambda a_1 b_2 + \lambda a_1 b_1 \otimes b_2.$
- In general,

  $$* = ш_\lambda.$$

- A **free Rota-Baxter algebra over another algebra A** is a Rota-Baxter algebra $\text{Ш}(A)$ with an algebra homomorphism $j_A : A \to \text{Ш}(A)$ such that for any Rota-Baxter algebra R and algebra homomorphism $f : A \to R$, there is a unique Rota-Baxter algebra homomorphism making the diagram commute



- When $A = \mathbf{k}[X]$, we have the free Rota-Baxter algebra over X.
- Recall that $(\text{Ш}^+(A), \diamond)$ is an associative algebra. Then the tensor product algebra (scalar extension) $\text{Ш}(A) := A \otimes \text{Ш}^+(A)$ is an A-algebra.
  Theorem (Guo-Keigher, 2000) $\text{Ш}(A)$ with the shift operator $P(\mathfrak{a}) := 1 \otimes \mathfrak{a}$ is the free commutative Rota-Baxter algebra over A.
- Let $A = \mathbf{k}\,1 \oplus A^+$. The restriction to $\text{Ш}(A)^0 := \oplus_{k \geq 0}(A^{\otimes k} \otimes A^+)$ is the free commutative nonunitary Rota-Baxter algebra on A.

Previous progresses on Rota's Conjecture

- The quasi-shuffle algebra on $A := x\mathbb{Q}[x]$ is identified with the algebra $QS(A)$ of quasi-symmetric functions, spanned by **monomial quasi-symmetric functions**

$$M_{(a_1,\cdots,a_k)} := \sum_{1\leq i_1<\cdots<i_k} x_{i_1}^{a_1}\cdots x_{i_k}^{a_k} \in \mathbb{Q}[x_1,\cdots,x_n,\cdots],$$

  for compositions (vectors) $\alpha := (a_1,\cdots,a_k), a_i \geq 1$. (It is called a composition of $n \geq 1$ if $a_1+\cdots+a_k = n$.)
- At the same time, $QS(x\mathbb{Q}[x])$ is the main part of the free nonunitary Rota-Baxter algebra $Ш(x\mathbb{Q}[x])^0$. Thus to pursue Rota's Conjecture, it is desirable to identity the whole commutative Rota-Baxter algebra $Ш(\mathbb{Q}[x])$ with some generalized quasi-symmetric functions.
- We achieved this in two steps, first for nonunitary Rota-Baxter algebras, next for unitary Rota-Baxter algebras.

Step one: the nonunitary case

- $QS(x\mathbf{k}[x]) \cong QSym \subseteq LWCQSym \subseteq WCQSym$.
- A vector $\alpha := (a_1, \cdots, a_k) \in \mathbb{Z}^k_{\geq 0}$ is called a left weak composition if $a_k > 0$.
- For a left weak comp composition α, define a monomial quasi-symmetric function

$$M_\alpha := \sum_{1 \leq i_1 < \cdots < i_k} x_{i_1}^{a_1} \cdots x_{i_k}^{a_k} \in \mathbb{Q}[[x_1, \cdots, x_n, \cdots]].$$

- Let LWCQSym be the subalgebra of $\mathbb{Q}[[x_1, \cdots, x_n, \cdots]]$ spanned by M_α.
- Theorem (L. Guo-H. Yu-J. Zhao, 2016) $\mathbb{Q}[x]LWCQSym$ is the free commutative nonunitary Rota-Baxter algebra on x .

Step two: the unitary case

- In order to apply this approach to free commutative unitary Rota-Baxter algebras, we need to consider weak compositions, not just left weak compositions.
- For a weak composition $\alpha := (a_1, \cdots, a_k), a_i \geq 0$, the expression M_α might not make sense.
- Example: $\alpha = (0)$ gives $M_\alpha = \sum_{n\geq 1} x_n^0 = \sum_{n\geq 1} 1$.
- To fix this problem, we "modify" the rule $x^0 = 1$ by considering formal power series and quasi-symmetric functions with semigroup exponents.

Power series with semigroup exponents

- In a formal power series, a monomials $x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}$ can be regarded as the locus of the map from $X := \{x_n \mid n \geq 1\}$ to $\mathbb{N}$ sending x_{i_j} to α_j, $1 \leq j \leq k$, and everything else in X to zero.
- Our generalization of the formal power series algebra is simply to replace $\mathbb{N}$ by a suitable additive monoid with a zero element.
- Let B be a commutative additive monoid with zero 0 such that $B \backslash \{0\}$ is a subsemigroup. Let X be a set. The set of ***B*-valued maps** is defined to be $B^X := \{f : X \to B \mid \mathcal{S}(f) \text{ is finite }\}$, where $\mathcal{S}(f) := \{x \in X \mid f(x) \neq 0\}$ denotes the support of f.
- The addition on B equips B^X with an additive monoid by

$$(f+g)(x) := f(x) + g(x) \quad \text{for all } f, g \in B^X \text{ and } x \in X.$$

- As with formal power series, we identify $f \in B^X$ with its locus $\{(x, f(x)) \mid x \in \mathcal{S}(f)\}$ expressed in the form of a formal product

$$X^f := \prod_{x \in X} x^{f(x)} = \prod_{x \in \mathcal{S}(f)} x^{f(x)},$$

called a ***B*-exponent monomial**, with the convention $x^0 = 1$.

- By abuse of notation, the addition on B^X becomes

$$X^f X^g = X^{f+g} \quad \text{for all } f, g \in B^X.$$

- We then form the semigroup algebra $\mathbf{k}[X]_B := \mathbf{k}B^X$ consisting of linear combinations of B^X, called the algebra of B**-exponent polynomials.**
- Similarly, we can define the free $\mathbf{k}$-module $\mathbf{k}[[X]]_B$ consisting of possibly infinite linear combinations of B^X, called B**-exponent formal power series.**
- If B is **additively finite** in the sense that for any $a \in B$ there are finite number of pairs $(b, c) \in B^2$ such that $b + c = a$, then the multiplication above extends by bilinearity to a multiplication on $\mathbf{k}[[X]]_B$, making it into a $\mathbf{k}$-algebra, called the algebra of formal power series with B-exponents.

Back to weak compositions

- Let B be a finitely generated free commutative additively finite monoid with generating set $\{b_1, b_2, \cdots, b_t\}$. Then

$$\mathbf{k}[X]_B = \mathbf{k}[x^{b_i} | 1 \leq i \leq t, x \in X].$$

- For example, taking B as the additive monoid $\mathbb{N}$ of nonnegative integers, then B^X is simply the free monoid generated by X and $\mathbf{k}[X]_B$ is the free commutative algebra $\mathbf{k}[X]$.
- Now taking $B := \tilde{\mathbb{N}} := \mathbb{N} \cup \{\varepsilon\}$, with $0 < \varepsilon < 1$, we obtain quasi-symmetric functions for weak compositions WCQSym. Further WCQSym is a Hopf algebra with contains QSym as both a sub and quotient Hopf algebra.
- Theorem (Yu-Guo-Thibon, 2017) $\mathbb{Q}[x]WCQSym$ is isomorphic to the free commutative unitary Rota-Baxter algebra $Ш(x)$.
- This equips $Ш(x)$ with a natural Hopf algebra structure.

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- **Thank You!**