Topological large fields, their generic differential expansions and transfer results.

Françoise Point (FNRS-FRS (UMons)
joint work with Pablo Cubidès (Caen, France)

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Outline of the talk

- Given a theory T of *large* topological fields with quantifier elimination, the class of existentially closed differential expansions is axiomatizable by a set of axioms T_D^*,
- *Immediate* transfer results from T to T_D^*,
- Transfer results (continued) : elimination of imaginaries, continuous definable functions, open core,
- Transfer results to the theory of *dense* pairs of models of T.

Examples of topological fields

Let $\bar{K} := (K, +, \cdot, -, 0, 1)$.
• $(\bar{K}, <)$ an ordered real-closed field, ie a model of RCF [o-minimal theory]
• $(\bar{K}, v)$ a p-adically closed valued field of rank d, respectively of ${}_pCF_d$ [p-minimal theory]
• $(\bar{K}, <, v)$ an ordered valued real-closed field, respectively of $RCVF$ respectively [weakly o-minimal theory].
• $(\bar{K}, v)$ a non-trivially valued algebraically closed field, respectively of $ACVF_{0,0}$ [C-minimal],

Large fields

All these fields share a common algebraic property: they are *large*. [Pop, 1996] A field K is large iff K is existentially closed in the field of Laurent series $K((t))$ ($K \subseteq_{ec} K((t))$) and equivalently in any iterated Laurent series field extension $K((t_1))((t_2))\cdots((t_n))$, for some natural number $n \geq 1$ (also denoted by $K((\mathbb{Z}^n))$).

(This second equivalence is straightforward using Frayne's embedding theorem: if a structure $\mathcal{A}$ is existentially closed in $\mathfrak{B}$ ($\mathcal{A} \subseteq_{ec} \mathfrak{B}$), then there is an embedding of $\mathfrak{B}$ in an ultrapower of $\mathcal{A}$, which is the identity on $\mathcal{A}$.)

They also share a common model-theoretic property, called *dp*-minimal.

Definition

A theory T is not *dp*-minimal if there is a model $\mathcal{M}$ of T, $a_{ij} \in M$ and uniformly unary definable sets $X_i,\ Y_j \subseteq M$, i, $j \in \mathbb{N}$, such that
$a_{ij} \in X_{i'} \leftrightarrow i = i'$,
$a_{ij} \in Y_{j'} \leftrightarrow j = j.'$

dp-minimal fields

[Johnson, 2016] If $\mathcal{K} := (K, +, \cdot, 0, 1, \cdots)$ is an expansion of an infinite field with a *dp*-minimal theory but not strongly minimal, then $\mathcal{K}$ can be endowed with a non-discrete Hausdorff *definable* field topology, namely $\mathcal{K}$ has a uniformly definable basis of neighbourhoods of zero compatible with the field operations. Furthermore, in this case the topology is induced either by a non-trivial valuation or an absolute value.

Moreover, any definable subset of $\mathcal{K}$ has finite boundary and every infinite definable set has non-empty interior (so $\mathcal{K}$ eliminates $\exists^\infty$, namely there is a bound on a uniformly definable family of finite sets).

Results of Simon and Walsberg on dp-minimal fields-dimension

• In a dp-minimal field $\mathcal{K}$, we have always the notion of a topological dimension:
let $X \subseteq K^n$, then t-$dim(X) := max\{\ell$: there is a projection $\pi : K^n \to K^\ell$ such that $\pi(X)$ has non-empty interior$\}$.

Let X be a definable subset of $\mathcal{K}^n$.
• One can define acl-dim as follows: acl-$dim(\bar{u}/A) := min\{\ell$: there is a subtuple $\bar{d}$ of $\bar{u}$ of length ℓ such that $\bar{u} \in acl(A, \bar{d})\}$. Then acl-$dim(X/A) := max\{$acl-$dim(\bar{u}/A) : \bar{u} \in X\}$.
Note that it is not assumed that acl has the exchange.

Theorem (Simon-Walsberg, to appear)

Then t-dim(X)=acl-dim(X).

From now on we will use *dim* for any of these dimensions.

Correspondences

Definition

Let $E,\ F$ be two definable subsets of K^n, then a correspondence f is a definable subset *graph*(f) of $E \times F$ such that

$$0 < |\{y \in F :\ (x, y) \in graph(f)\}| < \infty, \text{forall}\ \ x \in E.$$

A correspondance f is an m-correspondence if for all $x \in E$, $|\{y \in F :\ (x, y) \in graph(f)\}| = m$.

Now for a dp-minimal field $\mathcal{K}$, we will describe a generalisation of a cell decomposition theorem due to L. Mathews (for certain topological fields).

Results of Simon and Walsberg on dp-minimal fields-definable sets

Let X be a A-definable subset of $\mathcal{K}^n$ with A a subset of K, then:

Theorem (Proposition 4.1, Simon-Walsberg, to appear)

There a finitely many A-definable subsets X_i with $X = \bigcup X_i$ such that X_i is the graph of a A-definable continuous m-correspondance $f : U_i \rightrightarrows K^{n-d}$, where U_i is a A-definable open subset of K^d, for some $0 \leq d \leq n$.

Conventions: if $d = 0$, $f : K^0 \rightrightarrows K^{n-d}$, then *graph*$(f)$ is identified with a finite set and if $d = n$, $f : U \rightrightarrows K^0$, *graph*$(f)$ is identified with U (an open subset of K^n).

Theorem (Proposition 4.3, Simon-Walsberg, to appear)

Let $Fr(X) := closure(X) \setminus X$, then $dim(Fr(X)) < dim(X)$.

Theories of large fields

Let $\mathcal{K}$ be a topological field $(K, +, -, \cdot, 0, 1, \cdots)$ of characteristic 0 and assume that $\chi(x, \bar{y})$ be an $\mathcal{L}$-formula such that for any $\bar{a} \subset K$, $\chi(K, \bar{a})$ is an open neighbourhood of 0 in K. We put the product topology on K^n.

From now on we will always consider a language $\mathcal{L}$ which is a relational expansion of the ring (field) language and we assume that every relation and its complement is the union of an algebraic set set and an open subset.

Let T be the theory of $\mathcal{K}$. We will assume that T admits quantifier elimination in the language $\mathcal{L}$.

Examples

Let $\mathcal{L}$ be the language of fields. Let div be a binary relation.

1. Let $\mathcal{L}_< := \mathcal{L} \cup \{<\}$, then RCF admits quantifier elimination (Tarski),
2. Let $\mathcal{L}_p := \mathcal{L} \cup \{div, c_1, \cdots, c_d, P_n;\ n \geq 1\}$, then ${}_pCF_d$ admits quantifier elimination in $\mathcal{L}_p$ (Macintyre, Prestel-Roquette).
3. Let $\mathcal{L}_{<,div} := \mathcal{L}_< \cup \{div\}$, then $RCVF$ admits quantifier-elimination (Cherlin-Dickmann).
4. Let $\mathcal{L}_{div} := \mathcal{L} \cup \{div\}$, then $ACVF$ admits quantifier-elimination (Robinson).

In all the above cases, the relations and their complements satisfy the hypothesis to be the union of an open set with an algebraic set. Moreover any definable set is a finite union of an algebraic set and an open set.

Differential expansions

We consider the *generic* expansion of $\mathcal{K}$ with a derivation δ, namely we put no a priori continuity assumptions on δ. Denote by $\mathcal{L}_D := \mathcal{L} \cup \{\delta\}$ and T_D the $\mathcal{L}_D$-theory $T \cup \{\delta$ is a derivation $\}$.

Question: under which conditions, the class of existentially closed models is well-behaved?

Scheme (DL)

Let T and χ be as before. Set $T_D^* := T_D \cup (DL)$, where (DL) is the following list of axioms:

Let $\mathcal{K} \models T$. For each $n \geq 1$, let $\mathcal{V}_n := \{\chi(K, \bar{a}_1) \times \cdots \times \chi(K, \bar{a}_n) : \bar{a}_i \subset K, 1 \leq i \leq n\}$ be a (definable) basis of neighbourhoods of $\bar{0}$ in K^n.

$\mathcal{K}$ satisfies (DL) if for every $n \geq 1$, for every differential polynomial $f(X) \in K\{X\}$, with $f(X) = f^*(X, X^{(1)}, \ldots, X^{(n)})$ and for every $W \in \mathcal{V}_n$, we have:

$(\exists \alpha_0, \ldots, \alpha_n \in K)(f^*(\alpha_0, \ldots, \alpha_n) = 0 \wedge s_f^*(\alpha_0, \ldots, \alpha_n) \neq 0) \Rightarrow \Big((\exists z)(f(z) = 0 \wedge s_f(z) \neq 0 \wedge (z^{(0)} - \alpha_0, \ldots, z^{(n)} - \alpha_n) \in W)\Big).$

Axiomatisation of differential t-large e.c. topological fields of characteristic 0

Under the further hypothesis, called t-large–it adapts in this topological setting the property of largeness-, we show that the theory T_D^* is consistent and axiomatize the class of existentially closed models of T_D.

Theorem (Guzy-P)

Let T be a theory of topological t-large $\mathcal{L}$-fields of characteristic 0, admitting quantifier elimination.
Then T_D^* is the model-completion of T_D and admits quantifier elimination.

t-large fields

Let K be a model of T and consider the iterated Laurent series field extension $K((\mathbb{Z}^n)) := K((t_1))\cdots((t_n))$ endowed with the valuation map v taking its values in the lexicographic product $\mathbb{Z}^n$ of n copies of $\langle \mathbb{Z}, +, -, <, 0, 1\rangle$. We endow $K((\mathbb{Z}^n))$ with the following fundamental system of neighbourhoods $\mathcal{W}$ of zero:

$$W_{V,0} := \{a \in K((\mathbb{Z}^n)) : \ a = \sum_{\gamma \geq 0} \alpha_\gamma . t^\gamma, \ \alpha_0 \in V \text{ with } V \in \mathcal{V}\},$$

$$\text{and} \quad \text{for } \gamma \in (\mathbb{Z}^n)^{\geq 0}, \ W_\gamma := \{a \in K((\mathbb{Z}^n)) : \ v(a) \geq \gamma\}.$$

We will denote the corresponding topological structure by $\langle K((\mathbb{Z}^n)), \mathcal{W}\rangle$ and let $\mathcal{W}_{K,0} := \{W_{V,0};\ V \in \mathcal{V}\}$. It is easy to see that $\langle K((\mathbb{Z}^n)), \mathcal{W}\rangle$ is a topological $\mathcal{L}_{rings}$-extension of $\langle K, \mathcal{V}\rangle$.

A model K of T is *t-large* if:

given the topological $\mathcal{L}_{\text{rings}}$-extension $\langle K((\mathbb{Z}^n)), \mathcal{W}\rangle$ of $\mathcal{K}$ and a polynomial $f(X) \in K((\mathbb{Z}^n))[X]$ with coefficients in $W_{K,0}$, if $f(a) \sim_{\mathcal{W}_{K,0}} 0$ and $f'^2(a) \not\sim_{\mathcal{W}_{K,0}} 0$ for some element $a \in W_{K,0}$, then there exists $\widetilde{L}$ a model of T extending $K((\mathbb{Z}^n))$ such that

1. $t_i \sim_K 0$, $i = 1, \ldots, n$ and
2. there exists an element b of $\widetilde{L}$ with $f(b) = 0$ and $a \sim_K b$.

Note that if $\mathcal{L}$ is the language of rings and if K is a large field, then K is t-large.

Examples

- We obtain for the theory T_D^*:

1. CODF in case $T = RCF$,
2. $RCVF_D^*$ in case $T = RCVF$ (an expansion of CODF),
3. ${}_pCF_D^*$ in case $T =_p CF$,
4. ${ACVF_{0,0}}_D^*$ in case $T = ACVF_{0,0}$ (an expansion of DCF_0),

Some notation-(prolongations)

By assumption on $\mathcal{L}$, any $\mathcal{L}_D$-term $t(x)$ with $x = (x_1, \ldots, x_n)$, is equivalent, modulo the theory of differential fields, to an $\mathcal{L}$-term $t^*(\bar{\delta}^{m_1}(x_1), \cdots, \bar{\delta}^{m_n}(x_n))$ for some $(m_1, \cdots, m_n) \in \mathbb{N}^n$.
So, we may associate with any quantifier-free $\mathcal{L}_D$-formula $\varphi(x)$ an equivalent $\mathcal{L}_D$-formula, modulo the theory of differential fields, of the form $\varphi^*(\bar{\delta}^m(x))$, $m \in \mathbb{N}$, where φ^* is a $\mathcal{L}$-quantifier-free formula which arises by uniformly replacing every occurrence of $\delta^m(x_i)$ by a new variable y_i^m in φ with the following choice for the order of variables $\varphi^*(y_1^0, \cdots, y_1^m, \cdots, y_n^0, \cdots, y_n^m)$. So we get

$$\varphi(x_1, \ldots, x_n) \Leftrightarrow \varphi^*(\bar{\delta}^m(x_1), \ldots, \bar{\delta}^m(x_n)).$$

Order of a definable set

Let $A \subset K$, set $\mathrm{Jet}_m(A)$ for $\{\bar{\delta}^m(a) : a \in A\}$, where $\bar{\delta}^m(a) := (a, \delta(a), \cdots, \delta^m(a))$.
Likewise for $A \subset K^n$, set $\mathrm{Jet}_m(A) := \{\bar{\delta}^m(a) : a \in A\} \subset K^{(m+1)n}$, where for $a = (a_1, \ldots, a_n) \in K^n$, $\bar{\delta}^m(a) := (\bar{\delta}^m(a_1), \ldots, \bar{\delta}^m(a_n)) \in K^{(m+1)n}$.

Since T_D^* admits quantifier elimination, every $\mathcal{L}_D$-definable set $X \subseteq K^n$ is of the form $\mathrm{Jet}_m^{-1}(Y)$ for some quantifier-free $\mathcal{L}$-definable set $Y \subseteq K^{(m+1)n}$.

Definition (Order)

Let $X \subseteq K^n$ be an $\mathcal{L}_D$-definable set. The *order of* X, denoted by $o(X)$, is the smallest integer m such that $X = \mathrm{Jet}_m^{-1}(Y)$ for some $\mathcal{L}$-definable set $Y \subseteq K^{(m+1)n}$.

First properties (direct consequences of the axiomatisation)

Let $\mathcal{K} \models T_D^*$ and denote by C_K its subfield of constants.
Using the axiomatisation (respectively the geometrical axiomatisation), two observations:

- Then C_K is dense in K.

- (Brouette, Cousins, Pillay, P.–in case $\mathcal{L}$ is the language of rings–) Then $C_K \models T$.

Using the fact that T_D^* admits q.e. (and the forgetful functor), one can observe:

- (Guzy-P.)If T is NIP, then T_D^* is NIP.

- (Chernikov, 2015) If T is distal, then T_D^* is distal.

Open core

Now we wish to associate with an $\mathcal{L}_D$-definable set A, an $\mathcal{L}$-definable set where the differential points coming from A are dense.

Let $\mathcal{K}$ be a model of T_D^* and assume that it is $|K_0|^+$-saturated where K_0 be a differential subfield of K.

Property $(\star)$: For any $X \subseteq K^n$ $\mathcal{L}_D$-definable non-empty subset, there is an integer $m \geq o(X)$ and an $\mathcal{L}$-definable set $Z \subseteq K^{(m+1)n}$ such that

1. $x \in X$ if and only if $\mathrm{Jet}_m(x) \in Z$ and
2. $\overline{Z} = \overline{\mathrm{Jet}_m(X)}$.

Note that equivalently in Property $(\star)$ one can require that $m = o(X)$.

Kolchin polynomial

The $\mathcal{L}$-definable set Z decomposes as a finite disjoint union of cells C.

Let $\bar{u} \in X$ and assume that $\bar{\delta}(\bar{u})$ belongs to such cell C and assume it is a $\mathcal{L}$-generic point of C. By hypothesis there is a projection $\pi_{[m_1,\cdots,m_n]}$ such that $\pi_{[m_1,\cdots,m_n]}(C)$ is an open subset of $K^{(o(X)+1).n}$, $m_i \leq o(X)+1$, $1 \leq i \leq n$.

Let $\alpha = |\{1 \leq i \leq n : m_i = m+1\}|$ and $\beta = \sum_{i=1}^{n}(m+1-m_i)$.

Consider the subfields $K_0^{[t]} := K_0(\bar{\delta}^t(\bar{u}))$ of K, $t \in \omega$.

Theorem (Johnson, Pong)

The transcendence degree of $K_0^{[t]}$ over K_0 stabilises for t sufficiently big and is equal to $\alpha.t+\beta$

The coefficient α is the differential transcendence degree of $K_0(\bar{\delta}^\ell(\bar{u});\ \ell \in \omega)$ over K_0.

Open core

Definition

Let $\mathcal{K} \models T$. Then $(\mathcal{K}, D)$ has $\mathcal{L}$-open core if every $\mathcal{L}_D$-definable open subset is $\mathcal{L}$-definable.
An $\mathcal{L}_D$-expansion of T has $\mathcal{L}$-open core if every model of that expansion has $\mathcal{L}$-open core.

Lemma

Property $(\star)$ is equivalent to: T_D^* has $\mathcal{L}$-open core.

$(\Rightarrow)$ one shows that given an $\mathcal{L}_D$-definable set X, its closure $\bar{X}$ is $\mathcal{L}$-definable.
Indeed, $\bar{X} = \overline{\pi(\bar{Z})}$, where Z has the property $(\star)$ and π is the projection sending each block of $(m+1)$ coordinates to its first coordinate.

Open core-continued

($\Leftarrow$) Conversely, if the theory T_D^* has $\mathcal{L}$-open core, then:

If $X \subseteq K^n$ is a non-empty $\mathcal{L}_\delta$-definable set, there is an $\mathcal{L}$-definable set $Z \subseteq K^{(o(X)+1)n}$ such that $(\star)$

1. $x \in X$ if and only if $\mathrm{Jet}_{o(X)}(x) \in Z$ and
2. $\overline{Z} = \overline{\mathrm{Jet}_{o(X)}(X)}$.

Take $Y \subset K^{(o(X)+1)n}$ be an $\mathcal{L}$-definable set such that $X = \mathrm{Jet}^{-1}_{o(x)}(Y)$. Set $Z := Y \cap \overline{\mathrm{Jet}_{o(X)}(X)}$. Since $\overline{\mathrm{Jet}_{o(X)}(X)}$ is both closed and $\mathcal{L}_\delta$-definable, it is $\mathcal{L}$-definable since T_D^* has open core. So the set Z is $\mathcal{L}$-definable. Since $\mathrm{Jet}_{o(X)}(X) \subseteq Z \subseteq \overline{\mathrm{Jet}_{o(X)}(X)}$, both properties (1) and (2) are easily shown.

Elimination of imaginaries

Given an automorphism σ and a set X, we say that X is σ-invariant if σ fixes X setwise. We say that a theory T admits elimination of imaginaries if every definable set X has a code e, namely for any automorphism σ, X σ-invariant iff it fixes e.

Theorem

Suppose that T admits elimination of imaginaries in some expansion $\mathcal{L}^{\mathcal{G}}$ of $\mathcal{L}$ and that definable subsets in models of T are endowed with a dimension function dim *as before. Suppose that the theory T_D^* has $\mathcal{L}$-open core .Then the theory T_D^* admits elimination of imaginaries in $\mathcal{L}_D^{\mathcal{G}}$.*

We follow an idea of Marcus Tressl, associating to an $\mathcal{L}_D$-definable set X, the pair of $\mathcal{L}$-definable sets: $(Z, Jet_{o(X)}^{-1}(\overline{Z}) \setminus X)$.

Elimination of imaginaries

Let X be a non-empty $\mathcal{L}_D$-definable set.
Consider the $\mathcal{L}_D$-definable set $\tilde{X} := Jet^{-1}_{o(X)}(\overline{Z})$. Recall that Z is $\mathcal{L}$-definable. We proceed by induction on $dim(Z)$.
If $dim(Z) = 0$, X is finite.
Claim: $dim(\overline{Jet_{o(X)}(\tilde{X} \setminus X)}) < dim(\overline{Z})$.
Suppose the Claim holds, so by induction hypothesis there is e_1 a code for $\tilde{X} \setminus X$. Let e_2 be a code for Z. Then (e_1, e_2) is a code for X.
To show the Claim: we apply both properties of Z and the following property of dim:
$dim(\overline{Jet_{o(X)}(\tilde{X} \setminus X)}) \leq dim(Fr(Z)) < dim(\overline{Z})$.

Elimination of imaginaries

Fact: $CODF$ has $\mathcal{L}$-open core.
Proof: it eliminates $\exists^\infty$ and it is definably complete.

• In case $T = RCF$, we obtain yet another proof that CODF admits elimination of imaginaries (e.i.) in the language of differential fields.

• In case $T = RCVF$ and $T =_p CF$, we know which sorts to add to $\mathcal{L}$ in order to get e.i. and so it transfers to the corresponding T_D^*, modulo the proof that T_D^* has $\mathcal{L}$-open core. In those two cases, one can show that using the following property of continuous $\mathcal{L}_D$-definable functions.

Continuous functions

Let T be either one of the following $\mathcal{L}$-theories $RCF, RCVF,_p CF$, then:

Theorem

Let $\mathcal{K}$ be a model of T_D^*, let $X \subset K^n$ be an $\mathcal{L}$-definable subset and let f be a continuous $\mathcal{L}_D$-definable function from X to K, then f is $\mathcal{L}$-definable.

Corollary

T_D^* has $\mathcal{L}$-open core.

Applications to dense pairs

Let $\mathcal{L}^2 := \mathcal{L} \cup \{P\}$ where P is a new unary predicate P and let T^2 be the $\mathcal{L}^2$-theory of the pairs (K, F) (*i.e.*, P is interpreted in K by F) with $F \preccurlyeq_{\mathcal{L}} K$, $F \neq K$ and F dense in K.

Theorem (van den Dries, Fornasiero)

The theory T^2 is complete.

Fact

Let $\mathcal{K}$ be a model of T_D^*. Then (K, C_K) is a model of T^2.

Observation Every model (K, F) of T^2 has an $\mathcal{L}^2$-elementary extension (K^*, F^*) such that K^* is a model of T_D^* with constant field $C_{K^*} = F^*$.

So we get another proof of:

Theorem (Boxall and Hieronymi)

T^2 has $\mathcal{L}$-open core.

Lemma

Let (K, F) be a pair of real-closed fields with F a dense subfield. Then (K, F) has an elementary extension (K^*, F^*) which has a distal expansion.

Theorem (Hieronymi, Nell, 2017)

Let T be an o-minimal theory extending the theory of ordered abelian groups. Then the theory T^2 is not distal.

Theorem (Nell, 2018)

Consider the pair (A, B) with A an ordered vector space and B a dense subspace. Then it has a distal expansion, namely $(A, B, A/B, +, 0, <)$.

Let $\mathcal{M}$ be a saturated model of T_D^*.

Definition (Hieronymi, Nell (2017))

Let $\varphi(x_1, \cdots, x_n; y)$ be a partitioned $\mathcal{L}$-formula, where x_i, $1 \leq i \leq n$ is a p-tuple of variables and y is a q-tuple of variables, $p,\ q > 0$. Then φ is distal (in T) if for every $b \in M^q$, and every indiscernible sequence $(a_i)_{i \in I}$ in M^p such that

1. $I = I_1 + c + I_2$, where both I_1, I_2 are (countable) infinite dense linear orders without end points and c is a single element with $I_1 < c < I_2$,
2. the sequence $(a_i)_{i \in I_1 + I_2}$ in M^p is indiscernible over b,

then $\mathcal{M} \models \varphi(a_{i_1}, \cdots, a_{i_n}; b) \leftrightarrow \varphi(a_{j_1}, \cdots, a_{j_n}; b)$ with $i_1 < \cdots < i_n$, $j_1 < \cdots < j_n$ in I.

Theorem (Chernikov)

Assume that T is a distal theory of topological $\mathcal{L}$-fields and that T admits quantifier elimination. Then T_D^* is distal.

Erdős-Hajnal property

Definition

Let $\mathcal{A}$ be a first-order structure and $R \subset A^m \times A^n$ a definable relation.

1. A pair of subsets $E_1 \subset A^m, E_2 \subset A^n$ are R-homogeneous if either $E_1 \times E_2 \subset R$, or $E_1 \times E_2 \cap R = \emptyset$.
2. the relation R has the strong Erdős-Hajnal property if there is a constant $c(R)$ such that for every finite subsets $E_1 \subset A^m, E_2 \subset A^n$ there are subsets $E_1^0 \subset E_1,\ E_2^0 \subset E_2$ such that $|E_1^0| \geq c.|E_1|$, $|E_2^0| \geq c.|E_2|$ and the pair $E_1^0,\ E_2^0$ is R-homogeneous.

Erdős-Hajnal property

Theorem (Chernikov and Starchenko)

Definable relations in an arbitrary differentially closed field of characteristic 0 *satisfy the strong Erdős-Hajnal property.*

One uses the fact that any model of $CODF$ interprets a model of DCF_0 (Singer).

Corollary (to Chernikov's theorem)

Definable relations in a dense pair of real-closed fields satisfy the strong Erdős-Hajnal property.