# A truly universal ordinary differential equation

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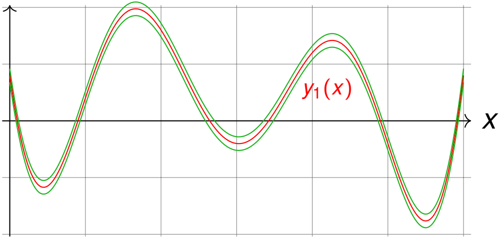
# Universal differential algebraic equation (Rubel)




**Theorem (Rubel, 1981)**

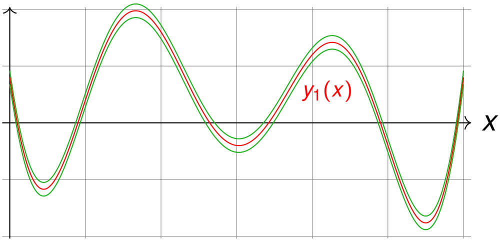
For any $f \in C^0(\mathbb{R})$ and $\varepsilon \in C^0(\mathbb{R}, \mathbb{R}_{>0})$, there exists a solution $y : \mathbb{R} \to \mathbb{R}$ to

$$\begin{aligned} 3y'^4 y'' y''''^2 \quad & -4y'^4 y'''^2 y'''' + 6y'^3 y''^2 y''' y'''' + 24y'^2 y''^4 y'''' \\ & -12y'^3 y'' y'''^3 - 29y'^2 y''^3 y'''^2 + 12y''^7 \qquad = 0 \end{aligned}$$

such that $\forall t \in \mathbb{R}$,

$$|y(t) - f(t)| \leqslant \varepsilon(t).$$

# Universal differential algebraic equation (Rubel)




**Theorem (Rubel, 1981)**

There exists a **fixed** $k$ and nontrivial polynomial $p$ such that for any $f \in C^0(\mathbb{R})$ and $\varepsilon \in C^0(\mathbb{R}, \mathbb{R}_{>0})$, there exists a solution $y : \mathbb{R} \to \mathbb{R}$ to

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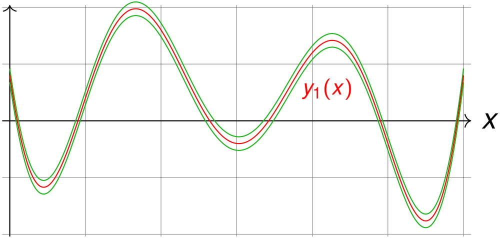


**Open Problem**

Can we have unicity of the solution with initial conditions ?

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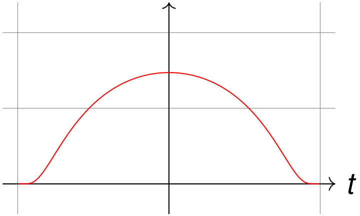
# Rubel's ("disappointing") proof in one slide

- Take $f(t) = e^{\frac{-1}{1-t^2}}$ for $-1 < t < 1$ and $f(t) = 0$ otherwise.

$$\text{It satisfies} \quad (1-t^2)^2 f''(t) + 2tf'(t) = 0.$$



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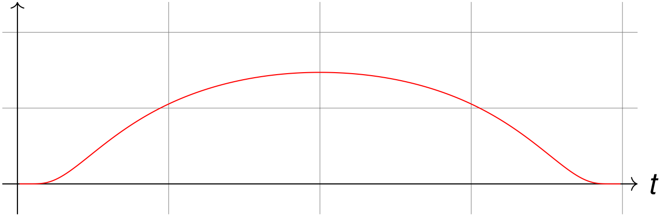

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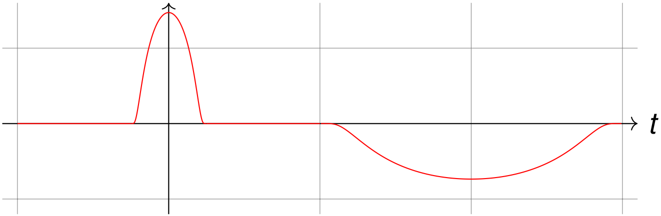
- Can glue together arbitrary many such pieces
  $\rightsquigarrow$crucial (and tricky) part of the proof



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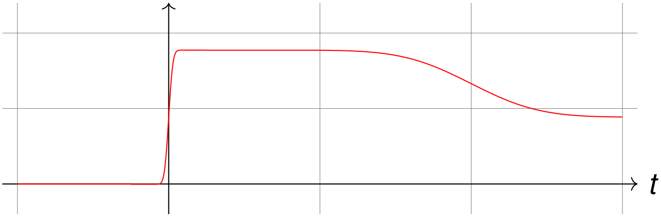
- Can arrange so that $\int f$ is solution : piecewise pseudo-linear



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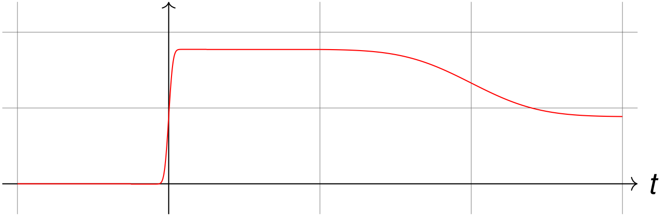


Conclusion : Rubel's equation allows any piecewise pseudo-linear functions, and those are **dense in $C^0$**

# The problem with Rubel's DAE

The solution $y$ is not unique, **even with added initial conditions** :

$$p(y, y', \ldots, y^{(k)}) = 0, \quad y(0) = \alpha_0, y'(0) = \alpha_1, \ldots, y^{(k)}(0) = \alpha_k$$

In fact, this is fundamental for Rubel's proof to work !

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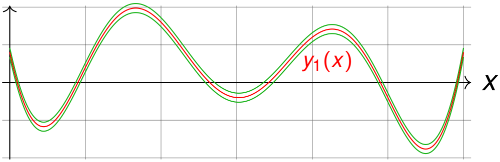
In fact, this is fundamental for Rubel's proof to work !

- Rubel's statement : this DAE is universal
- More realistic interpretation : this DAE allows almost anything

**Open Problem (Rubel, 1981)**

Is there a universal ODE $y' = p(y)$ ?
Note : explicit polynomial ODE $\Rightarrow$ unique solution

# Universal explicit ordinary differential equation




## Main result

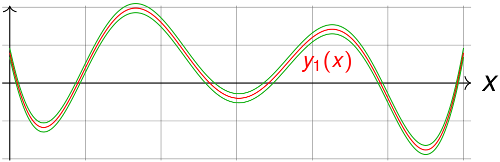
There exists a **fixed** (vector of) polynomial $p$ such that for any $f \in C^0(\mathbb{R})$ and $\varepsilon \in C^0(\mathbb{R}, \mathbb{R}_{>0})$, there exists $\alpha \in \mathbb{R}^d$ such that

$$y(0) = \alpha, \qquad y'(t) = p(y(t))$$

has a **unique solution** $y : \mathbb{R} \to \mathbb{R}^d$ and $\forall t \in \mathbb{R}$,

$$|y_1(t) - f(t)| \leqslant \varepsilon(t).$$

# Universal explicit ordinary differential equation




Notes :

- **system** of ODEs,
- $y$ must be analytic,
- we need $d \approx 300$.

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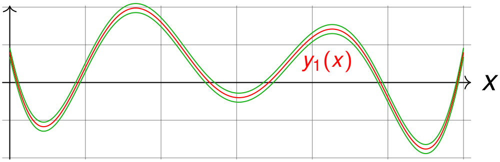


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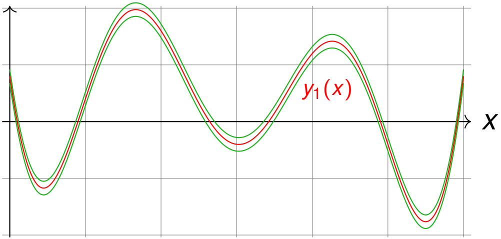
Futhermore, $\alpha$ is computable † from $f$ and $\varepsilon$.

†. This statement can be made precise with the theory of Computable Analysis.

# Universal DAE, again but better




## Corollary of main result

There exists a **fixed** $k$ and nontrivial polynomial $p$ such that for any $f \in C^0(\mathbb{R})$ and $\varepsilon \in C^0(\mathbb{R}, \mathbb{R}_{>0})$, there exists $\alpha_0, \dots, \alpha_k \in \mathbb{R}$ such that

$$p(y, y', \dots, y^{(k)}) = 0, \quad y(0) = \alpha_0, y'(0) = \alpha_1, \dots, y^{(k)}(0) = \alpha_k$$

has a **unique analytic solution** $y : \mathbb{R} \to \mathbb{R}$ and $\forall t \in \mathbb{R}$,

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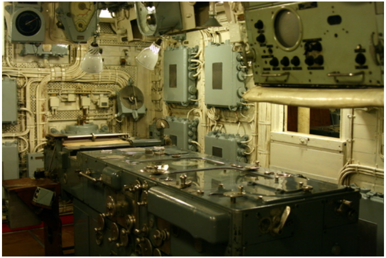
# Some motivation

Polynomial ODEs correspond to analog computers :

Differential Analyser



British Navy mecanical computer

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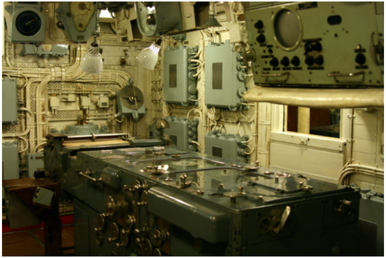
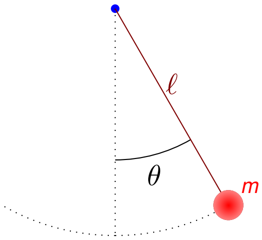


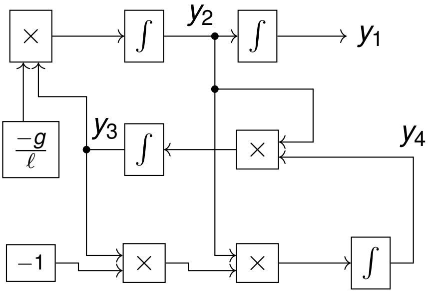
British Navy mecanical computer

- They are **equivalent** to Turing machines !
- One can **characterize P** with pODEs (ICALP 2016)

Take away : polynomial ODEs are a natural programming language.

# Example of differential equation




General Purpose Analog Computer (GPAC)
Shannon's model of the Differential Analyser

$$\ddot{\theta} + \frac{g}{\ell}\sin(\theta) = 0$$

$$\begin{cases} y_1' = y_2 \\ y_2' = -\frac{g}{\ell} y_3 \\ y_3' = y_2 y_4 \\ y_4' = -y_2 y_3 \end{cases} \Leftrightarrow \begin{cases} y_1 = \theta \\ y_2 = \dot{\theta} \\ y_3 = \sin(\theta) \\ y_4 = \cos(\theta) \end{cases}$$

# A brief stop

Before I can explain the proof, you need to know more of polynomial ODEs and what I mean by programming with ODEs.

# Generable functions (total, univariate)

**Definition**

$f : \mathbb{R} \to \mathbb{R}$ is generable if there exists $d, p$ and $y_0$ such that the solution $y$ to

$$y(0) = y_0, \qquad y'(x) = p(y(x))$$

satisfies $f(x) = y_1(x)$ for all $x \in \mathbb{R}$.

**Types**

- $d \in \mathbb{N}$ : dimension
- $\mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{R}$ : field
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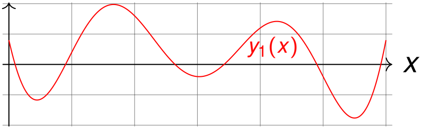


Note : existence and unicity of $y$ by Cauchy-Lipschitz theorem.

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Example : $f(x)=x$ ▶ identity

$$y(0)=0,\qquad y'=1\quad\rightsquigarrow\quad y(x)=x$$

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Example : $f(x) = x^2$ ► squaring

$$\begin{array}{llll} y_1(0) = 0, & y_1' = 2y_2 & \leadsto & y_1(x) = x^2 \\ y_2(0) = 0, & y_2' = 1 & \leadsto & y_2(x) = x \end{array}$$

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Example : $f(x)=x^n$ ▶ $n^{th}$ power

$$\begin{array}{lll}
y_1(0)=0, & y_1'=ny_2 & \rightsquigarrow\; y_1(x)=x^n\\
y_2(0)=0, & y_2'=(n-1)y_3 & \rightsquigarrow\; y_2(x)=x^{n-1}\\
\ldots & \ldots & \quad\ldots\\
y_n(0)=0, & y_n=1 & \rightsquigarrow\; y_n(x)=x
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Example : $f(x)=\exp(x)$ ► exponential

$$y(0)=1,\qquad y'=y\quad\rightsquigarrow\quad y(x)=\exp(x)$$

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Example : $f(x) = \sin(x)$ or $f(x) = \cos(x)$ ▶ sine/cosine

$$\begin{aligned} y_1(0) &= 0, & y_1' &= y_2 & &\leadsto & y_1(x) &= \sin(x) \\ y_2(0) &= 1, & y_2' &= -y_1 & &\leadsto & y_2(x) &= \cos(x) \end{aligned}$$

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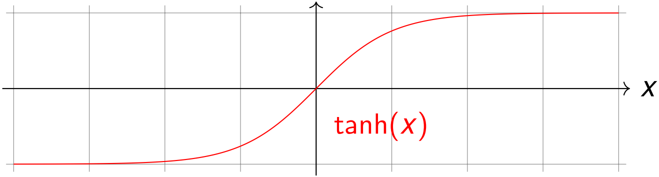
Example : $f(x)=\tanh(x)$ ▶ hyperbolic tangent

$$y(0)=0,\qquad y'=1-y^2 \quad\rightsquigarrow\quad y(x)=\tanh(x)$$



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Example : $f(x) = \frac{1}{1+x^2}$ ▸ rational function

$$f'(x) = \frac{-2x}{(1+x^2)^2} = -2xf(x)^2$$

$$\begin{array}{lll} y_1(0) = 1, & y_1' = -2y_2y_1^2 & \rightsquigarrow \quad y_1(x) = \frac{1}{1+x^2} \\ y_2(0) = 0, & y_2' = 1 & \rightsquigarrow \quad y_2(x) = x \end{array}$$

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Example : $f = g \pm h$ ▶ sum/difference

$$(g \pm h)' = g' \pm h'$$

*assume* :

$$\begin{array}{lll} z(0) = z_0, & z' = p(z) & \rightsquigarrow \quad z_1 = g \\ w(0) = w_0, & w' = q(w) & \rightsquigarrow \quad w_1 = h \end{array}$$

*then* :

$$y(0) = z_{0,1} + w_{0,1}, \qquad y' = p_1(z) \pm q_1(w) \quad \rightsquigarrow \quad y = z_1 \pm w_1$$

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Example : $f=gh$ ▶ product

$$(gh)'=g'h+gh'$$

*assume* :

$$\begin{aligned} z(0)&=z_0, & z'&=p(z) &\rightsquigarrow\quad z_1&=g\\ w(0)&=w_0, & w'&=q(w) &\rightsquigarrow\quad w_1&=h \end{aligned}$$

*then* :

$$y(0)=z_{0,1}w_{0,1},\qquad y'=p_1(z)w_1+z_1q_1(w)\quad\rightsquigarrow\quad y=z_1w_1$$

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satisfies $f(x)=y_1(x)$ for all $x\in\mathbb{R}$.

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- $y_0\in\mathbb{K}^d, y:\mathbb{R}\to\mathbb{R}^d$

Example : $f=\frac{1}{g}$ ▶ inverse

$$f'=\frac{-g'}{g^2}=-g'f^2$$

*assume* :

$$z(0)=z_0,\qquad z'=p(z)\qquad\leadsto\qquad z_1=g$$

*then* :

$$y(0)=\frac{1}{z_{0,1}},\qquad y'=-p_1(z)y^2\qquad\leadsto\qquad y=\frac{1}{z_1}$$

# Generable functions (total, univariate)

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Example : $f=\int g$ ▶ integral

*assume :*

$$z(0)=z_0,\qquad z'=p(z)\quad\rightsquigarrow\quad z_1=g$$

*then :*

$$y(0)=0,\qquad y'=z_1\quad\rightsquigarrow\quad y=\int z_1$$

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Example : $f = g'$ ▶ derivative

$$f' = g'' = (p_1(z))' = \nabla p_1(z) \cdot z'$$

*assume* :

$$z(0) = z_0, \qquad z' = p(z) \qquad \rightsquigarrow \qquad z_1 = g$$

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Example : $f=g\circ h$ ▶ composition

$$(z\circ h)'=(z'\circ h)h'=p(z\circ h)h'$$

*assume* :

$$\begin{aligned} z(0)&=z_0, & z'&=p(z) &\leadsto\quad z_1&=g\\ w(0)&=w_0, & w'&=q(w) &\leadsto\quad w_1&=h\end{aligned}$$

*then* :

$$y(0)=z(w_0),\qquad y'=p(y)z_1\quad\leadsto\quad y=z\circ h$$

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Is this coefficient in $\mathbb{K}$ ?

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*then* :

$$y(0) = z(w_0), \qquad y' = p(y)z_1 \quad \rightsquigarrow \quad y = z \circ h$$

Is this coefficient in $\mathbb{K}$? Fields with this property are called generable.

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Example : $f'=\tanh\circ f$ ▸ Non-polynomial differential equation

$$f''=(\tanh'\circ f)f'=(1-(\tanh\circ f)^2)f'$$

$$\begin{array}{lll} y_1(0)=f(0), & y_1'=y_2 & \rightsquigarrow\ y_1(x)=f(x) \\ y_2(0)=\tanh(f(0)), & y_2'=(1-y_2^2)y_2 & \rightsquigarrow\ y_2(x)=\tanh(f(x)) \end{array}$$

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Example : $f(0) = f_0, f' = g \circ f$ ▶ Initial Value Problem (IVP)

$$f' = g'' = (p_1(z))' = \nabla p_1(z) \cdot z'$$

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# Generable functions : a first summary

Nice theory for the class of total and univariate generable functions :

- analytic
- contains polynomials, sin, cos, tanh, exp
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Limitations :

- total functions
- univariate

# Generable functions (generalization)

## Definition

$f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ is generable if $X$ is open **connected** and $\exists d, p, x_0, y_0, y$ such that

$$y(x_0) = y_0, \qquad J_y(x) = p(y(x))$$

and $f(x) = y_1(x)$ for all $x \in X$.

## Types

- $n \in \mathbb{N}$ : input dimension
- $d \in \mathbb{N}$ : dimension
- $p \in \mathbb{K}^{d \times d}[\mathbb{R}^d]$ : polynomial matrix
- $x_0 \in \mathbb{K}^n$
- $y_0 \in \mathbb{K}^d$, $y : X \to \mathbb{R}^d$

$J_y(x)$ = Jacobian matrix of $y$ at $x$

Notes :

- Partial differential equation !
- Unicity of solution $y$...
- ... **but not existence** (ie you have to show it exists)

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Example : $f(x_1, x_2) = x_1 x_2^2 \quad (n = 2, d = 3)$ ▶ monomial

$$y(0,0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad J_y = \begin{pmatrix} y_3^2 & 3y_2y_3 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \rightsquigarrow \quad y(x) = \begin{pmatrix} x_1 x_2^2 \\ x_1 \\ x_2 \end{pmatrix}$$

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$$\begin{array}{llll} y_1(0,0) = 0, & \partial_{x_1} y_1 = y_3^2, & \partial_{x_2} y_1 = 3y_2y_3 & \rightsquigarrow \quad y_1(x) = x_1x_2^2 \\ y_2(0,0) = 0, & \partial_{x_1} y_2 = 1, & \partial_{x_2} y_2 = 0 & \rightsquigarrow \quad y_2(x) = x_1 \\ y_3(0,0) = 0, & \partial_{x_1} y_3 = 0, & \partial_{x_2} y_3 = 1 & \rightsquigarrow \quad y_3(x) = x_2 \end{array}$$

This is tedious !

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Last example : $f(x) = \frac{1}{x}$ for $x \in (0, \infty)$ ▶ inverse function

$$y(1) = 1, \qquad \partial_x y = -y^2 \quad \rightsquigarrow \quad y(x) = \frac{1}{x}$$

# Generable functions : summary

Nice theory for the class of multivariate generable functions (over connected domains) :

- analytic
- contains polynomials, $\sin$, $\cos$, $\tanh$, $\exp$, ...
- stable under $\pm, \times, /, \circ$ and Initial Value Problems (IVP)
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**Exercice :** are all analytic functions generable ? No
Riemann $\Gamma$ and $\zeta$ are not generable.

# Why is this useful ?

Writing polynomial ODEs by hand is hard.

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Using generable functions, we can build complicated **multivariate partial functions** using other operations, and we know they are solutions to polynomial ODEs **by construction**.

**Example (almost rounding function)**

There exists a generable function round such that for any $n \in \mathbb{Z}$, $x \in \mathbb{R}$, $\lambda > 2$ and $\mu \geqslant 0$ :

- if $x \in \left[n - \frac{1}{2}, n + \frac{1}{2}\right]$ then $|\operatorname{round}(x, \mu, \lambda) - n| \leqslant \frac{1}{2}$,
- if $x \in \left[n - \frac{1}{2} + \frac{1}{\lambda}, n + \frac{1}{2} - \frac{1}{\lambda}\right]$ then $|\operatorname{round}(x, \mu, \lambda) - n| \leqslant e^{-\mu}$.

# Reminder of the result

**Main result (reminder)**

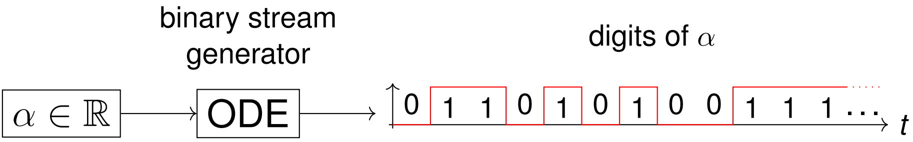
There exists a **fixed** (vector of) polynomial $p$ such that for any $f \in C^0(\mathbb{R})$ and $\varepsilon \in C^0(\mathbb{R}, \mathbb{R}_{>0})$, there exists $\alpha \in \mathbb{R}^d$ such that

$$y(0) = \alpha, \qquad y'(t) = p(y(t))$$

has a **unique solution** $y : \mathbb{R} \to \mathbb{R}^d$ and $\forall t \in \mathbb{R}$,

$$|y_1(t) - f(t)| \leqslant \varepsilon(t).$$

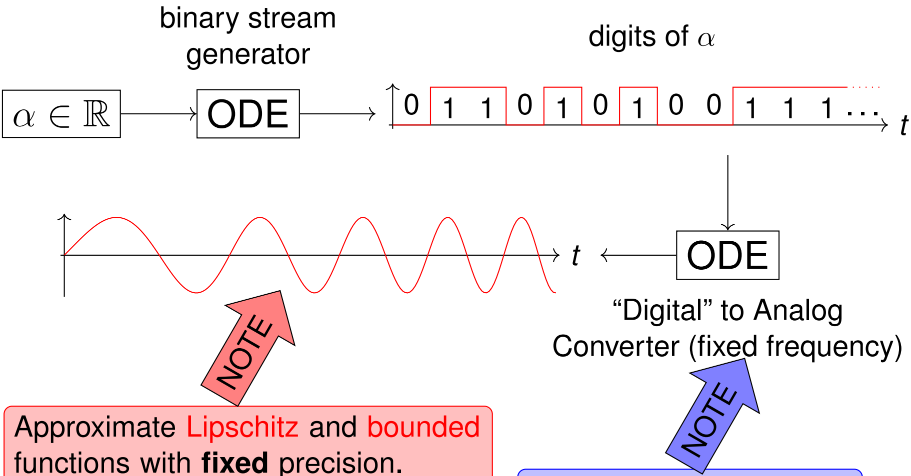
# A simplified proof




NOTE

This is the ideal curve, the real one is an approximation of it.

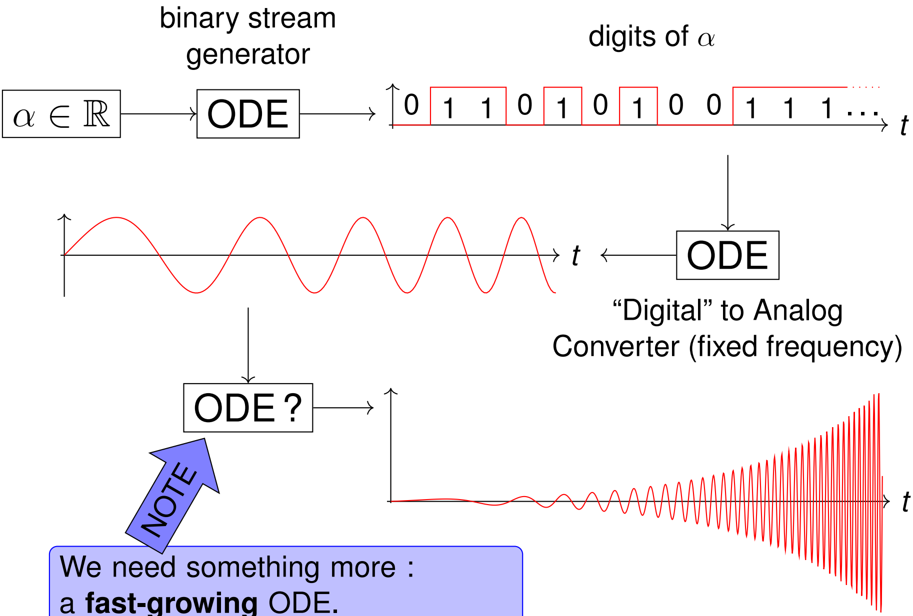
# A simplified proof




Approximate Lipschitz and bounded functions with **fixed** precision.

That’s the trickiest part.

# A simplified proof




We need something more :
a **fast-growing** ODE.

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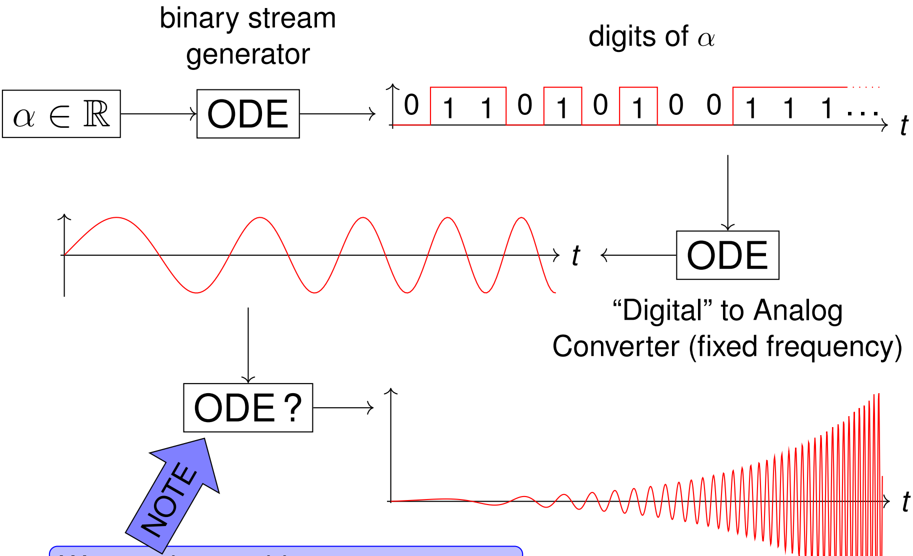
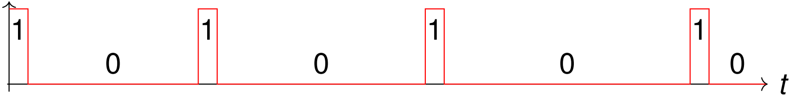



We need something more :
an **arbitrarily** **fast-growing** ODE.

# A less simplified proof

binary stream generator : digits of $\alpha \in \mathbb{R}$




$$f(\alpha, \mu, \lambda, t) = \frac{1}{2} + \frac{1}{2}\tanh(\mu \sin(2\alpha\pi 4^{\text{round}(t-1/4,\lambda)} + 4\pi/3))$$

It's horrible, but generable

round is the mysterious rounding function...

# A less simplified proof

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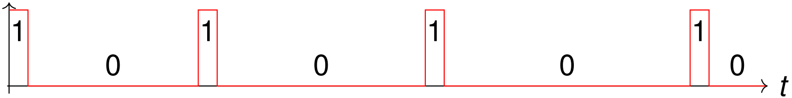


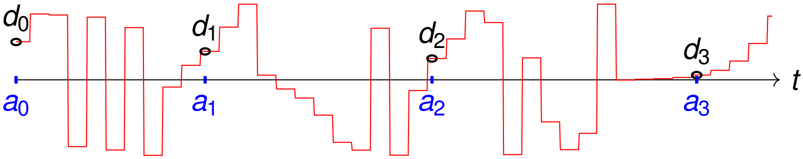


dyadic stream generator : $d_i = m_i 2^{-d_i}$, $a_i = 9i + \sum_{j<i} d_j$

$$f(\alpha, \gamma, t) = \sin(2\alpha\pi 2^{\text{round}(t-1/4,\gamma)}))$$

round is the mysterious rounding function...

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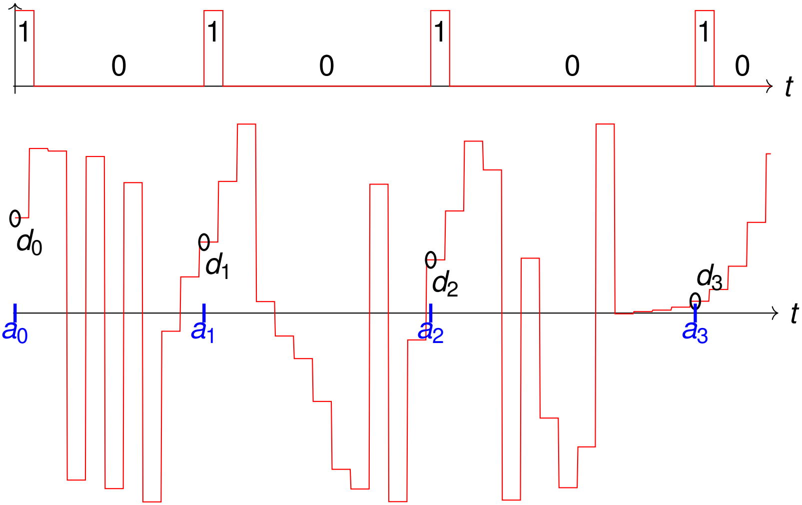

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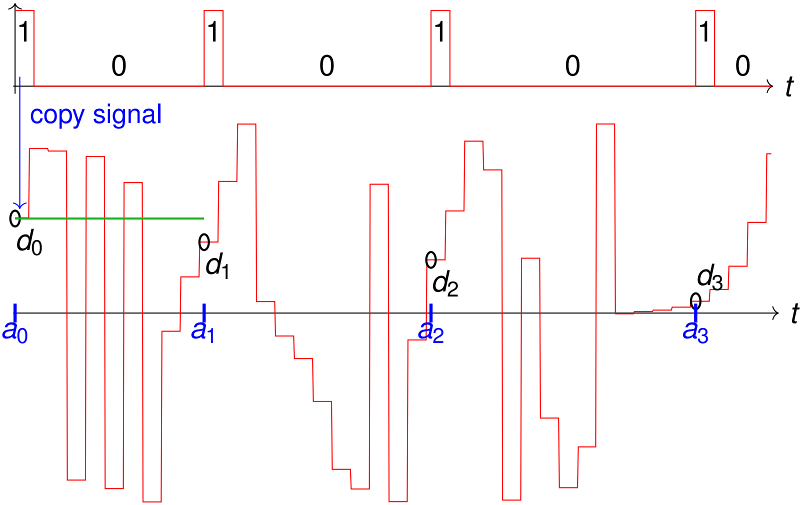

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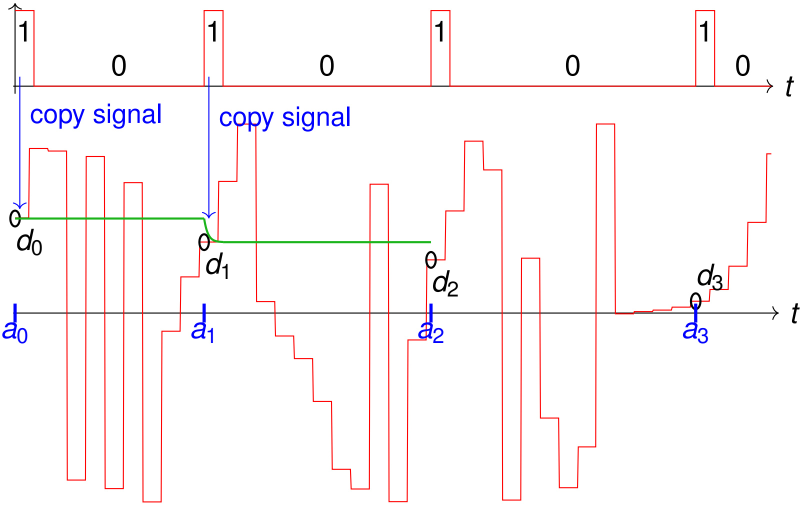

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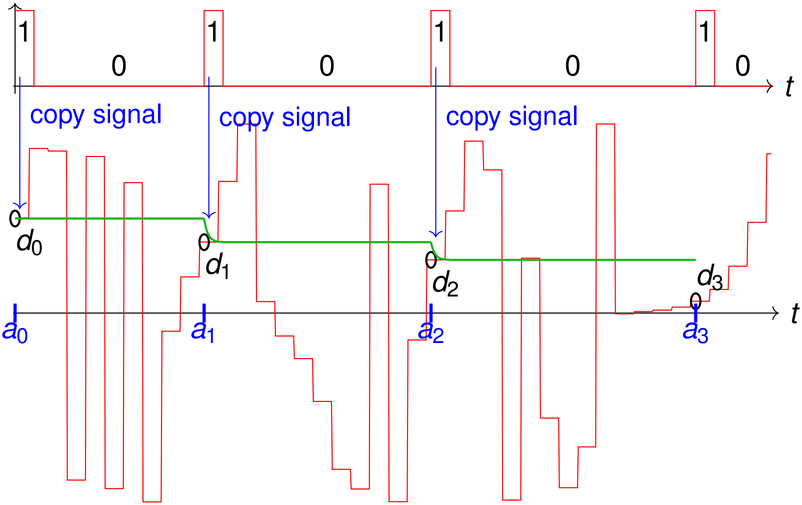

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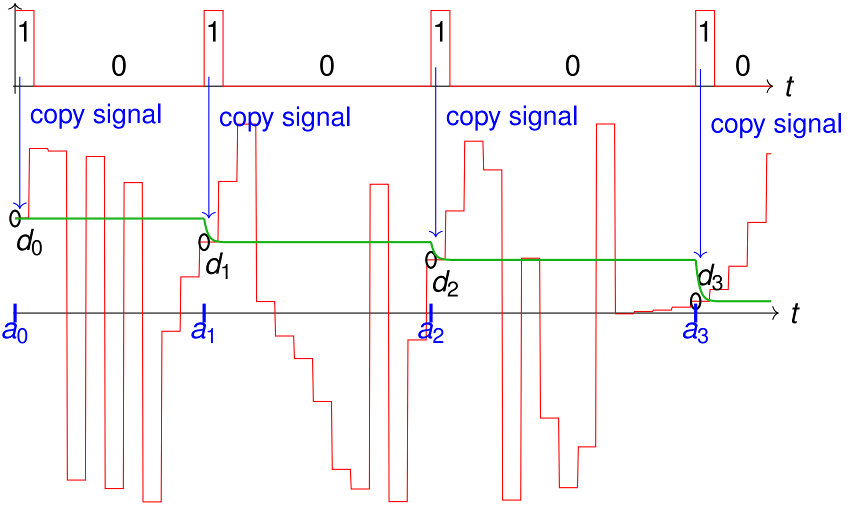

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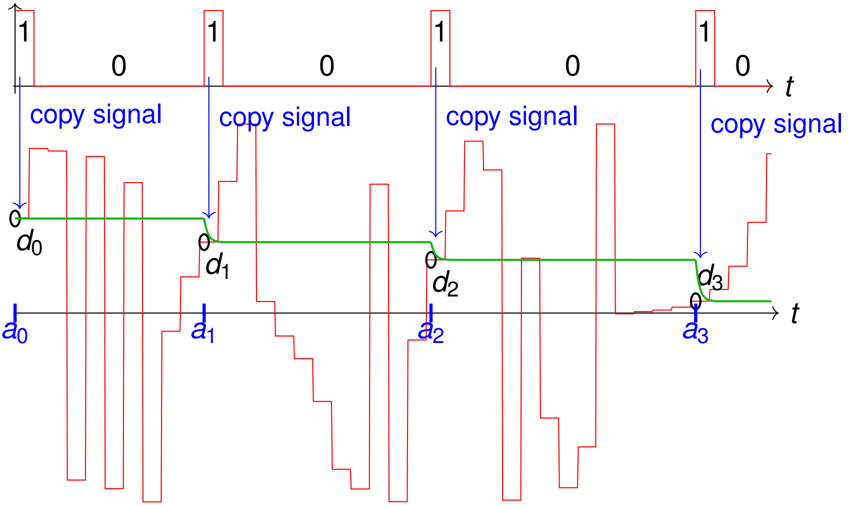
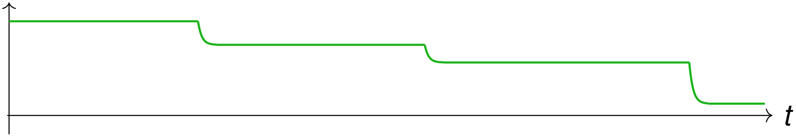



This copy operation is the “non-trivial” part.

# A less simplified proof




We can do **almost piecewise constant functions...**

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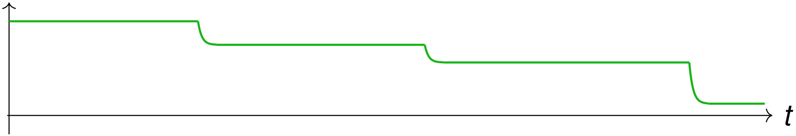


We can do **almost piecewise constant functions...**

- ...that are bounded by 1...
- ...and have super slow changing frequency.

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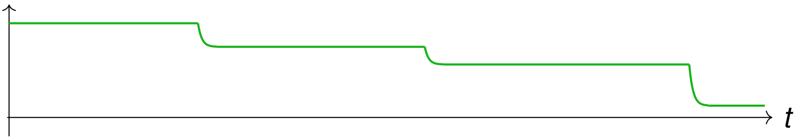


We can do **almost piecewise constant functions...**

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How do we go to arbitrarily large and growing functions ? Can a polynomial ODE even have arbitrary growth ?

# An old question on growth

Building a fast-growing ODE, that exists over $\mathbb{R}$ :

$$y_1' = y_1 \qquad \rightsquigarrow \qquad y_1(t) = \exp(t)$$

# An old question on growth

Building a fast-growing ODE, that exists over $\mathbb{R}$ :

$$
\begin{aligned}
y_1' &= y_1 & \rightsquigarrow \quad & y_1(t) = \exp(t) \\
y_2' &= y_1 y_2 & \rightsquigarrow \quad & y_1(t) = \exp(\exp(t))
\end{aligned}
$$

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\dots & & \dots \\
y_n' = y_1 \cdots y_n & \rightsquigarrow & y_n(t) = \exp(\cdots \exp(t) \cdots) := e_n(t)
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\end{array}
$$

**Conjecture (Emil Borel, 1899)**

With $n$ variables, cannot do better than $\mathcal{O}_t(e_n(At^k))$.

# An old question on growth

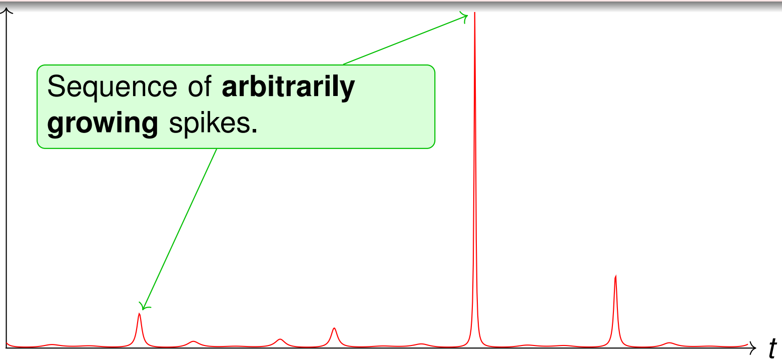
$e_n(t) = \exp(\cdots \exp(t) \cdots)$  ($n$ compositions)

**Conjecture (Emil Borel, 1899)**

With $n$ variables, cannot do better than $\mathcal{O}_t(e_n(At^k))$.

**Counter-example (Vijayaraghavan, 1932)**

$$\frac{1}{2 - \cos(t) - \cos(\alpha t)}$$



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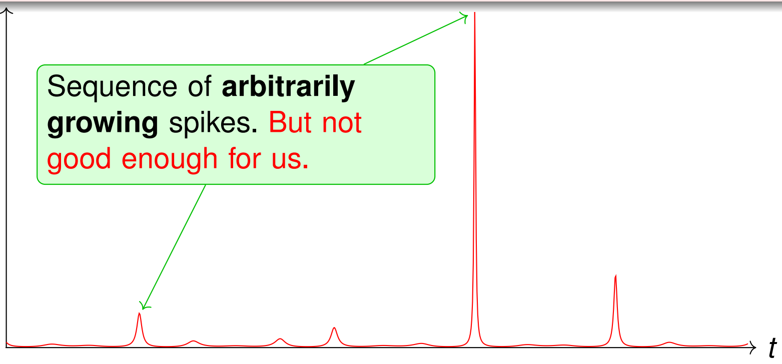

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**Theorem (In the paper)**

There exists a polynomial $p : \mathbb{R}^d \to \mathbb{R}^d$ such that for any continuous function $f : \mathbb{R}_{\geqslant 0} \to \mathbb{R}$, we can find $\alpha \in \mathbb{R}^d$ such that

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Note : both results require $\alpha$ to be **transcendental**. Conjecture still open for **rational** (or algebraic) coefficients.

# Proof gem : iteration with differential equations

Assume f is generable, can we **iterate** $f$ with an ODE ?
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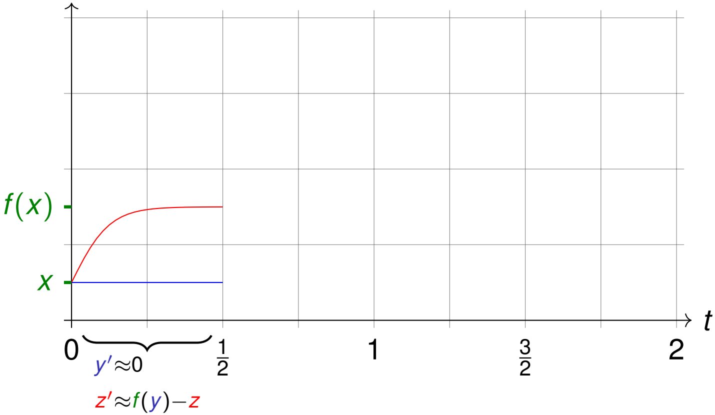

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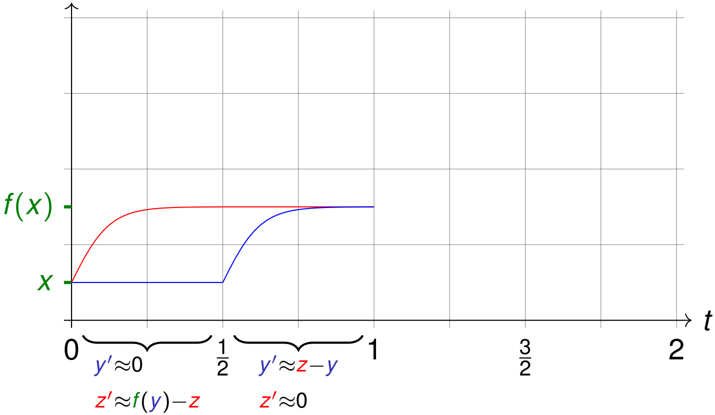

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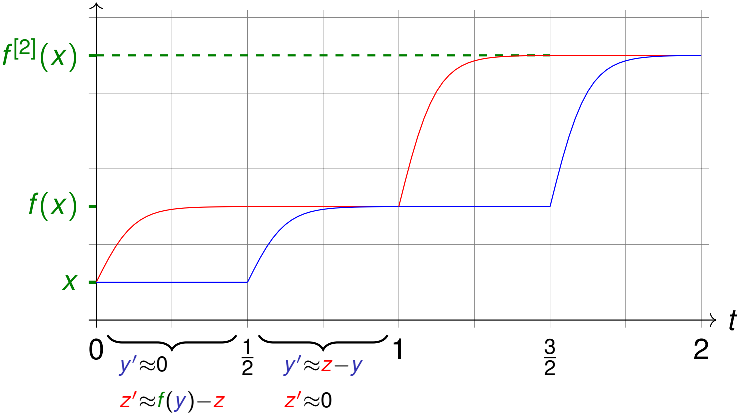

# Main result, remark and end

## Main result (reminder)

There exists a **fixed** (vector of) polynomial $p$ such that for any $f \in C^0(\mathbb{R})$ and $\varepsilon \in C^0(\mathbb{R}, \mathbb{R}_{>0})$, there exists $\alpha \in \mathbb{R}^d$ such that

$$y(0) = \alpha, \qquad y'(t) = p(y(t))$$

has a **unique solution** $y : \mathbb{R} \to \mathbb{R}^d$ and $\forall t \in \mathbb{R}$,

$$|y_1(t) - f(t)| \leqslant \varepsilon(t).$$

Futhermore, $\alpha$ is computable from $f$ and $\varepsilon$.

Remarks :

- if $f$ and $\varepsilon$ are computable then $\alpha$ is computable
- if $f$ or $\varepsilon$ is **not computable** then $\alpha$ is **not computable**
- in all cases $\alpha$ is a horrible transcendental number