Sparse resultants in differential and difference algebra: an overview

Wei Li

Academy of Mathematics and Systems Science,
Chinese Academy of Sciences

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CUNY, New York

Jointly with X.S. Gao, C.M. Yuan

Main Problem

Let

$$\begin{cases} f_0(\mathbf{c}_0; y_1, \ldots, y_n) = 0 \\ f_1(\mathbf{c}_1; y_1, \ldots, y_n) = 0 \\ \qquad\vdots \\ f_n(\mathbf{c}_n; y_1, \ldots, y_n) = 0 \end{cases} \tag{1}$$

be a system of algebraic (differential, difference) equations.

To find conditions, $\mathbf{R}(\mathbf{c}_0, \ldots, \mathbf{c}_n)$, on the coefficients of the f_i s.t.

$$(1) \text{ has a “solution” } \Longleftrightarrow^{1} \mathbf{R}(\mathbf{c}_0, \ldots, \mathbf{c}_n) = 0.$$

[1] In the sense of Zariski (Kolchin, Cohn) closure.

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Existence/Properties/Algorithms of such $\mathbf{R}$?

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Outline

- Motivation: algebraic sparse resultants
- Sparse differential resultants
- Sparse difference resultants
- Summary and problems

Sylvester Resultant (Sylvester 1883)

Two univariate polynomials $f = a_l x^l + a_{l-1}x^{l-1} + \cdots + a_0$ and $g = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0$. **The resultant of f and g** is

$$\mathrm{Res}(f,g) = \begin{vmatrix} a_l & a_{l-1} & a_{l-2} & \cdots & a_0 & & \\ & a_l & a_{l-1} & a_{l-2} & \cdots & a_0 & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & a_l & a_{l-1} & a_{l-2} & \cdots & a_0 \\ b_m & b_{m-1} & b_{m-2} & \cdots & b_0 & & \\ & b_m & b_{m-1} & b_{m-2} & \cdots & b_0 & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & b_m & b_{m-1} & b_{m-2} & \cdots & b_0 \end{vmatrix}.$$

Property: $\mathrm{Res}(f,g) = 0 \iff f(x) = g(x) = 0$ has a solution.

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Macaulay resultant (1902): $n+1$ polynomials in n variables.

Algebraic Sparse Poly System

Given $F := (f_1, \ldots, f_n) \subset \mathbb{C}[x_1, \ldots, x_n]$ with

$$f_i = \sum_{\alpha \in \mathcal{A}_i} c_{i,\alpha} x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

BKK-bound (Bernshtein, Khovanskii, Kushnirenko)

The number of isolated roots of F in $(\mathbb{C}^*)^n$ is bounded by the mixed volume of $\mathrm{NP}(f_i)$.

Example

$f_1 = a_0 + a_1 x_1 + a_2 x_1^n x_2^n, \quad f_2 = b_0 + b_1 x_2 + b_2 x_1^n x_2^n$

Bézout-bound: $(2n)^2$

BKK-bound: $2n$

Work on Algebraic Sparse Resultant

- Gelfand et al (1991, 1994) introduced the sparse resultant.
- Sturmfels (1993, 1994) proved basic properties.

  i) The $f_i = 0$ have common solutions in $(\mathbb{C}^*)^n$ iff $\mathrm{Res}(f_0, \ldots, f_n) = 0$.

  ii) (BKK degree) $\deg(\mathrm{Res}, \mathrm{Coeff}(f_i)) = \mathcal{MV}((\mathrm{NP}(f_k))_{k \neq i})$.
- Canny-Emiris (1993, 1995, 2000): matrix formulas and efficient algorithms.
- D'Andrea (2002): $\mathrm{Res} = \det(A)/\det(B)$.

Sparse Differential Resultants

$\mathcal{E}$: a fixed universal δ-field over a base field (such as $\mathbb{Q}, \mathbb{Q}(x)$).

$\mathcal{E}^{\wedge} := \{a \in \mathcal{E} : a^{(k)} \neq 0, k \geq 0\}$.

Differential Resultants

- Let $P_i(\mathbf{u}_i; y_1, \ldots, y_n)$ be a generic δ-poly of order s_i and degree r_i with coefficients $\mathbf{u}_i$ for $i = 0, \ldots, n$;
- $\mathcal{Z}_0 = \{(\mathbf{c}_0, \ldots, \mathbf{c}_n) \big| P_0(\mathbf{c}_0; y_1, \ldots, y_n) = \cdots = P_n(\mathbf{c}_n; y_1, \ldots, y_n) = 0$ has a solution in $\mathcal{E}^n\}$.

Definition(Gao-Li-Yuan, 2013) Let $\mathcal{Z}$ be the Kolchin closure of $\mathcal{Z}_0$. By GDIT, there exists an irr. δ-poly $\mathbf{R} \in \mathbb{Q}\{\mathbf{u}_0, \ldots, \mathbf{u}_n\}$ s.t.

$$\mathcal{Z} = \mathbb{V}\big(\mathrm{sat}(\mathbf{R})\big).$$

R is defined as the **differential resultant** of P_i's.

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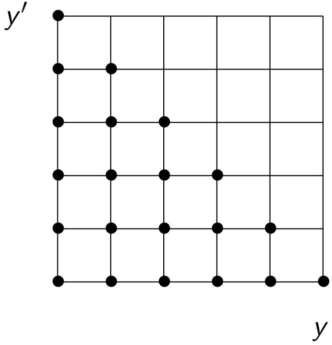
Note: differential resultant for $n = 1$ was studied by Ritt (1932).

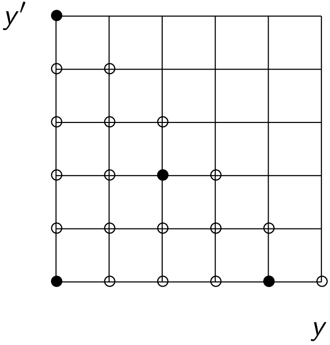
Sparse Differential Polynomials

- **Sparse Differential Polynomials**: with fixed monomials

  Most differential polynomials in practice are sparse.



A dense δ-Poly of order one and degree five



A sparse δ-Poly $f = * + *y^4 + *y'^5 + *y^2y'^2$

Sparse differential resultant (Li-Yuan-Gao, 2015)

Let $\mathcal{A}_i = \{M_{i0}, \ldots, M_{il_i}\}\,(i = 0, \ldots, n)$ be sets of Laurent δ-monomials in $\mathbf{y} = (y_1, \ldots, y_n)$, and

$$P_i(\mathbf{u}_i; \mathbf{y}) = \sum_{k=0}^{l_i} u_{ik} M_{ik} \ (i = 0, \ldots, n) \tag{2}$$

be generic Laurent δ-polynomials defined over $\mathcal{A}_i$. Let

$$Z_0 = \big\{(\mathbf{c}_0, \ldots, \mathbf{c}_n) | P_0(\mathbf{c}_0; \mathbf{y}) = \cdots = P_0(\mathbf{c}_n; \mathbf{y}) = 0 \text{ has a solution in}(\mathcal{E}^\wedge)^n\big\}$$

and Z be the Kolchin closure of Z_0 in $\mathbb{P}^{l_0} \times \cdots \times \mathbb{P}^{l_n}$.

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Definition. If Z is of codimension 1, then $\exists$ $\mathbf{R} \in \mathbb{Q}\{\mathbf{u}_0 \ldots, \mathbf{u}_n\}$ s.t.

$$Z = \mathbb{V}\big(\text{sat}(\mathbf{R})\big).$$

$\mathbf{R}$: the **sparse differential resultant**, denoted by $\text{Res}_{\mathcal{A}_0,\ldots,\mathcal{A}_n}$.

Criterion for the existence of sparse diff resultant

Assume $M_{ik}/M_{i0} = \prod_{j=1}^{n}\prod_{l=0}^{s_i}(y_j^{(l)})^{t_{ikjl}}$ ($t_{ikjl} \in \mathbb{Z}$). Set

$$\beta_{ik} = \Big(\sum_{l=0}^{s_i} t_{ikjl}x_1^l, \ldots, \sum_{l=0}^{s_i} t_{ikjl}x_n^l\Big) \in \mathbb{Z}[x_1, \ldots, x_n]^n.$$

Let

$$\mathbf{M} = \begin{pmatrix} \sum_{k=0}^{l_0} u_{0k}\beta_{0k} \\ \sum_{k=0}^{l_1} u_{1k}\beta_{1k} \\ \vdots \\ \sum_{k=0}^{l_n} u_{nk}\beta_{nk} \end{pmatrix} \in \mathbb{Z}[\mathbf{u}; x_1, \ldots, x_n]^{(n+1)\times n}.$$

Theorem. $\mathrm{Res}_{\mathcal{A}_0,\ldots,\mathcal{A}_n}$ exists $\Longleftrightarrow \mathrm{rank}(\mathbf{M}) = n$

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$\Longleftrightarrow$ There exist k_i $(1 \le k_i \le l_i)$ s.t. $\mathrm{rank}(\mathbf{M}_{k_0,\ldots,k_n}) = n$, where the i-th row of $\mathbf{M}_{k_0,\ldots,k_n}$ is $\beta_{i-1,k_{i-1}}$.

Example

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$\mathcal{A}_0 = \{1, y_1 y_2\}$,

$\mathcal{A}_1 = \{1, y_1' y_2'\}$,

$\mathcal{A}_2 = \{1, y_1' y_2\}$.

$\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$ form a Laurent δ-essential system for $\mathbf{M} = \begin{pmatrix} u_{01} & u_{01} \\ u_{11}x_1 & u_{11}x_2 \\ u_{21}x_1 & u_{21} \end{pmatrix}$ (or $\mathbf{M}_{1,1,1} = \begin{pmatrix} 1 & 1 \\ x_1 & x_2 \\ x_1 & 1 \end{pmatrix}$) has rank 2.

Using differential characteristic method, we have

$$\mathrm{Res} = -u_{11} u_{20}^2 u_{01}^2 - u_{01} u_{00} u_{21}^2 u_{10} + u_{01} u_{11} u_{20} u_{21} u_{00}' - u_{11} u_{20} u_{00} u_{21} u_{01}'.$$

Necessary/sufficient conditions for the existence of solutions

Differential resultant:
If $P_0(\mathbf{c}_0; \mathbf{y}) = \cdots = P_n(\mathbf{c}_n; \mathbf{y}) = 0$ has a solution in $\mathcal{E}^n$, then $\mathbf{R}(\mathbf{c}_0, \ldots, \mathbf{c}_n) = 0$;

Conversely, if $\mathbf{R}(\mathbf{c}_0, \ldots, \mathbf{c}_n) = 0$ and $\mathrm{S}_{\mathbf{R}}(\mathbf{c}_0, \ldots, \mathbf{c}_n) \neq 0$, then the system has a solution in $\mathcal{E}^n$.

Sparse differential resultant:
If $P_0(\mathbf{c}_0; \mathbf{y}) = \cdots = P_n(\mathbf{c}_n; \mathbf{y}) = 0$ has a solution in $(\mathcal{E}^\wedge)^n$, then $\mathrm{Res}_{\mathcal{A}_0,\ldots,\mathcal{A}_n}(\mathbf{c}_0, \ldots, \mathbf{c}_n) = 0$;

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Order and Differential homogeneity

$G = \{g_1, \ldots, g_n\}$: differential polynomials in $y_1, \ldots, y_n$.

Jacobi number: $\mathrm{Jac}(G) = \max_{\sigma \in S_n} \sum_{i=1}^{n} \mathrm{ord}(g_i, y_{\sigma(i)})$.

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Proposition (Order and Differential homogeneity).

- The δ-resultant $\mathbf{R}(\mathbf{u}_0, \ldots, \mathbf{u}_n)$ is δ-homogeneous in each $\mathbf{u}_i$ and $\mathrm{ord}(\mathbf{R}, \mathbf{u}_i) = \sum_{j \neq i} \mathrm{ord}(P_j)$.
- $\mathrm{Res}_{\mathcal{A}_0, \ldots, \mathcal{A}_n}$ is δ-homogeneous in each $\mathbf{u}_i$ and

$$\mathrm{ord}(\mathbf{Res}_{\mathcal{A}_0, \ldots, \mathcal{A}_n}, \mathbf{u}_i) \leq \mathbf{Jac}(P_{\hat{i}}),$$

where $P_{\hat{i}} = \{P_0, \ldots, P_n\} \backslash \{P_i\}$.

Poisson-Type Product Formula

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When 1) Any n of the $\mathcal{A}_i$ diff independent and

2) $\mathbf{e}_j \in \mathrm{Span}_{\mathbb{Z}}\{\alpha_{ij} - \alpha_{i0}\}$,

$\mathrm{Res}_{\mathcal{A}_0,\ldots,\mathcal{A}_n} = A \prod_{\tau=1}^{t_0} \left(\frac{\mathbf{P_0}(\eta_\tau)}{\mathbf{M_{00}}(\eta_\tau)}\right)^{(\mathbf{h_0})}$, and η_τ lies on $P_1, \ldots, P_n$.

Degree bounds

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$\deg(\mathrm{Res}_{\mathcal{A}_0,\ldots,\mathcal{A}_n}, \mathbf{u}_i) \leq \prod_{i=0}^{n} (\deg(P_i) + 1)^{J_i+1}$.

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Example. $P_0 = u_{00} + u_{01}y + u_{02}y' + u_{03}y^2 + u_{04}yy' + u_{05}(y')^2$;
$P_1 = u_{10} + u_{11}y + u_{12}y' + u_{13}y^2 + u_{14}yy' + u_{15}(y')^2$.

Bézout-type bound: $\deg(\mathbf{R}) \leq (2+1)^4 = 81$.
BKK-type bound: $\deg(\mathbf{R}) \leq 20$.

Representation of resultants

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Sparse Differential Resultant: $\text{Res}_{\mathcal{A}_0,\ldots,\mathcal{A}_n} = \sum_{i=0}^{n} \sum_{j=0}^{J_i} G_{ij} \delta^j P_i$ and G_{ij} also has a bounded degree $(m+1)\deg(\text{Res})$.

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Algebraic Resultant: $\text{Res}(A(x), B(x)) = A(x)T(x) + B(x)W(x)$, with $\deg(T) \leq \deg(B), \deg(W) \leq \deg(A)$.

Differential Resultant: $\mathbf{R}(\mathbf{u}_0, \ldots, \mathbf{u}_n) = \sum_{i=0}^{n} \sum_{j=0}^{s-s_i} h_{ij}\delta^j P_i$, and the h_{ij} have degrees at most $(sn+n)^2 d^{sn+n} + d(sn+n)$.

Sparse Differential Resultant: $\text{Res}_{\mathcal{A}_0,\ldots,\mathcal{A}_n} = \sum_{i=0}^{n} \sum_{j=0}^{J_i} G_{ij}\delta^j P_i$ and G_{ij} also has a bounded degree $(m+1)\text{deg}(\text{Res})$.

Note: If the (sparse) differential resultant of $f_0, \ldots, f_n$ is nonzero, then we have

$$1 = \sum_{i=0}^{n} \sum_{j=0}^{J_i} G_{ij}\delta^j f_i.$$

Single-exponential computational algorithm

- A **single exponential algorithm** to compute the sparse diff resultant with complexity: $O((J+n)^{O(lJ)}m^{O(lJ2)})$.
  Here J:**Jacobi number**; $l=\sum_i l_i$: **size** of the system.

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Unsolved: Matrix representation for (sparse) differential resultant?

Progress on matrix representation:

- Differential resultant for two univariate δ-polynomials of order one (Zhang-Yuan-Gao, 2014);
- Linear sparse differential resultant for some special cases (Rueda, 2016).

Sparse Difference Resultant (Li-Yuan-Gao, 2016)

Comparison of Difference and Differential Resultant:

	Difference Case	Differential Case
Definition	$[P_0, \ldots, P_n] \cap \mathbb{Q}\{\mathbf{u}_0, \ldots, \mathbf{u}_n\}$ $= \text{sat}(\mathbf{R}, R_1, \ldots, R_m)$	sat($\mathbf{R}$)
Essential Criterion	$\mathbf{M}_P \in \mathbb{Z}[x]^{(n+1)\times n}$	$\mathbf{M}_P \in \mathbb{Z}[x_1, \ldots, x_n]^{(n+1)\times n}$
Matrix Formula	$\mathbf{R} = \det(M)/\det(M_0)$	**?**
Conditions for ∃ solutions	Only necessary conditions (non-zero solutions)	Necessary and sufficient (non-poly solutions)
Homogeneity Property	Transformally homogenous $\big(f(\lambda Y) = M(\lambda) f(Y)\big)$	Differentially homogenous $\big(f(\lambda Y) = \lambda^m f(Y)\big)$
degree	**"="** BKK number	BKK bound
Order	Jacobi bound (Dense: $s - s_i$)	The same

Example (extended)

$$\mathcal{A}_0 = \{1, y_1 y_2\},$$
$$\mathcal{A}_1 = \{1, y_1^{(1)} y_2^{(1)}\},$$
$$\mathcal{A}_2 = \{1, y_1^{(1)} y_2\}.$$

The **sparse differential resultant** is

$$\mathrm{Res} = -u_{11} u_{20}^2 u_{01}^2 - u_{01} u_{00} u_{21}^2 u_{10} + u_{01} u_{11} u_{20} u_{21} u_{00}' - u_{11} u_{20} u_{00} u_{21} u_{01}'.$$

While the **sparse difference resultant** is

$$\mathrm{Res} = u_{00}^{(1)} u_{11} - u_{01}^{(1)} u_{10}.$$

Summary

A theory of sparse differential/difference resultants is developed and a single exponential computational algorithm is given.

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Problems for further study:

- Matrix representation for (sparse) differential resultants;
- In the definition of sparse difference resultant, is $m = 0$?
- Resultant theory for a system of partial differential polynomials?

References:

- W. Li, C.M. Yuan and X.S. Gao. Sparse Differential Resultant for Laurent Differential Polynomials. *Found. Comput. Math.*, 15:451-517, 2015.
- W. Li, C.M. Yuan and X.S. Gao. Sparse Difference Resultant. *J. Symb. Comput.*, 68, 169-203, 2015.
- W. Li, X.S. Gao and C.M. Yuan. Sparse Differential Resultant. *Proc. ISSAC 2011*, San Jose, CA, USA, 225-232, 2011.
- Z.Y. Zhang, C.M. Yuan, and X.S. Gao, Matrix formulae of differential resultant for first order generic ordinary differential polynomials., *Computer Mathematics*, 479-503, 2014.
- S. L. Rueda, Differential elimination by differential specialization of Sylvester style matrices, *Adv. Appl. Math.*, 72, 4–37, 2016.

References:

- W. Li, C.M. Yuan and X.S. Gao. Sparse Differential Resultant for Laurent Differential Polynomials. *Found. Comput. Math.*, 15:451-517, 2015.
- W. Li, C.M. Yuan and X.S. Gao. Sparse Difference Resultant. *J. Symb. Comput.*, 68, 169-203, 2015.
- W. Li, X.S. Gao and C.M. Yuan. Sparse Differential Resultant. *Proc. ISSAC 2011*, San Jose, CA, USA, 225-232, 2011.
- Z.Y. Zhang, C.M. Yuan, and X.S. Gao, Matrix formulae of differential resultant for first order generic ordinary differential polynomials., *Computer Mathematics*, 479-503, 2014.
- S. L. Rueda, Differential elimination by differential specialization of Sylvester style matrices, *Adv. Appl. Math.*, 72, 4–37, 2016.

Thanks!